ARCHIVUM MATHEMATICUM (BRNO)
Tomus 53 (2017), 35–48

GENERALIZED KÄHLERIAN MANIFOLDS AND TRANSFORMATION OF GENERALIZED CONTACT STRUCTURES

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Abstract. The aim of this paper is two-fold. First, new generalized Kähler manifolds are constructed starting from both classical almost contact metric and almost Kählerian manifolds. Second, the transformation construction on classical Riemannian manifolds is extended to the generalized geometry setting.

1. Introduction

Let M be an even dimensional smooth manifold together with its generalized tangent bundle $TM \oplus T^*M$. A generalized complex structure on M is given by a bundle isomorphism $J\colon TM \oplus T^*M \to TM \oplus T^*M$ which preserves the natural inner product on $TM \oplus T^*M$, satisfies $J^2 = -\,\mathrm{id}$ and some integrability condition (see Section 2.4 below for more details). A generalized Kähler structure on M is a pair of commuting generalized complex structures that are compatible, in the sense, that they define a positive definite metric on $TM \oplus T^*M$. Such a structure can also be equivalently defined as a quadruple (g, b, J_+, J_-), where g is a Riemannian metric, b is a two-form and $J_\pm$ are almost Hermitian structures on (M, g) satisfying some torsion condition. In other words, a generalized Kähler structure on M can be viewed as a bi-Hermitian structure satisfying some torsion condition. Generalized Kähler structures on smooth manifolds were introduced and studied by Gualtieri in [14]. They form a large class of geometric structures which includes classical Kählerian structures. Furthermore, they hold remarkable properties of Kähler geometry such as the Hodge decomposition. In fact, bi-Hermitian structures already appeared in the setting of the (2, 2) supersymmetric sigma model [11]. In the last decade, generalized Kähler geometry has received a lot of attention. Examples of Kähler structures appearing in the literature were constructed from various

2010 *Mathematics Subject Classification*: primary 53C10; secondary 53C15, 53C18, 53D25.
Key words and phrases: product manifolds, trans-Sasakian manifolds, generalized Kählerian manifolds, generalized contact structures, transformation of generalized almost contact structures, generalized almost complex structures.
Received March 21, 2016, revised February 2017. Editor J. Slovák.
DOI: 10.5817/AM2017-1-35

approaches: by exploiting symmetry [15], by a reduction procedure [9, 10, 18] by means of deformation theory [12, 13] as well as twistors [1].

One of the goals of this paper is to explore other ways for constructing generalized Kähler structures. Results in our paper can be divided in two parts. In the first part, we construct generalized Kähler structures starting from classical odd-dimensional almost contact metric manifolds or even-dimensional almost Kählerian manifolds and using the $\mathcal{D}$-homothetic bi-warping construction (see [3, 6]). In the second part, we extend the $\mathcal{D}$-homothetic transformation construction to generalized Riemannian manifolds. The $\mathcal{D}$-homothetic bi-warping [3, 6] is a construction in classical Riemannian manifolds generalizing the warped product and defined as follows. Let (M_1, g_1) be a Riemannian manifold together with two smooth functions f and h on M_1. Let $(M_2, \varphi_2, \xi_2, \eta_2, g_2)$ be an almost contact metric manifold. The $\mathcal{D}$-homothetic bi-warping $M_1 \times_{(f,h)} M_2$ is the product manifold $M_1 \times M_2$ with the metric $g = g_1 + f^2 g_2 + f^2(h^2 - 1)\eta_2 \otimes \eta_2$.

The paper is organized as follows. In Section 2, we review basic definitions and results that are needed to state and prove our results. In Section 3, we state and proof our first construction result (Theorem 3.2). This result shows how to build generalized Kählerian manifolds out of a β-Kenmotsu manifold and using the $\mathcal{D}$-homothetic bi-warping framework. In Section 4, generalized Kählerian structures are constructed from classical Kählerian structures using the warped product metric and a $\mathcal{D}$-homothetic bi-warping. In Section 5, we explore an extension of $\mathcal{D}$-homothetic deformation to generalized geometry.

2. Preliminaries

2.1. **Preliminaries on contact metric geometry.**

Throughout this paper, all manifolds are connected and smooth. We will briefly review the basic ingredients that are needed here, for more details on these classical structures, we refer the reader to references [4, 5, 8, 27].

An $(2n+1)$-dimensional Riemannian manifold (M, g) is said to be an almost contact metric manifold if there exists a triple (φ, ξ, η) consisting of a $(1,1)$ tensor field φ, a vector field ξ (called the Reeb vector field) and a 1-form η such that

$$\eta(\xi) = 1\,,\ \varphi^2(X) = -X + \eta(X)\xi\,,\ \ g(\varphi X, \varphi Y) = g(X,Y) - \eta(X)\eta(Y)\,, \tag{2.1}$$

for any vector fields X, Y on M. For any almost contact metric manifold, we have:

$$\varphi\xi = 0 \quad \text{and} \quad \eta \circ \varphi = 0\,.$$

The fundamental 2-form ω of an almost contact metric manifold (M, φ, ξ, η) is defined by $\omega(X,Y) = g(X, \varphi Y)$. We say that (φ, ξ, η) is normal if

$$N_\varphi(X,Y) = [\varphi,\varphi](X,Y) + 2d\eta\ (X,Y)\xi = 0\,, \tag{2.2}$$

where $[\varphi, \varphi]$ is the Nijenhuis tensor of φ and d denotes the exterior derivative.

Suppose there are smooth functions α and β on an almost contact metric manifold $(M, \varphi, \xi, \eta, g)$ such that:

$$(\nabla\varphi)(X,Y) = \alpha\big(g(X,Y)\xi - \eta(Y)X\big) + \beta\big(g(\varphi X, Y)\xi - \eta(Y)\varphi X\big) \tag{2.3}$$

where $(\nabla\varphi)(X,Y) = \nabla_X\varphi Y - \varphi(\nabla_X Y)$, $\quad X, Y \in \Gamma(TM)$ and ∇ is the Levi-Civita connection with respect to the metric g. Then M is called a **trans-Sasakian manifold of type** (α,β) (cf. [4], [5], [17]). Equivalently a trans-Sasakian manifold of type (α,β) is an a *normal* almost contact metric manifold (M,φ,ξ,η,g) whose fundamental 2-form ω satisfies:

$$d\omega = \beta(\omega\wedge\eta)\,, \quad d\eta = \alpha\omega\,, \quad \varphi^*(\delta\omega) = 0\,, \tag{2.4}$$

where $\delta\omega$ is the coderivative of ω given by:

$$\delta\omega(X) = -\sum_{i=1}^{2n}\big((\nabla_{e_i}\omega)(e_i,X) + (\nabla_{\varphi e_i}\omega)(\varphi e_i,X)\big) - (\nabla_\xi\varphi)(\xi,X)\,,$$

with $\{e_i\}$ is an orthonormal frame on M.
In particular, we have the definitions:

- M is said to be β**-Kenmotsu** when $\alpha = 0$.
- M is said to be α**-Sasakian** when $\beta = 0$.
- If $\alpha = \beta = 0$ then M is called a **cosymplectic or coKählerian manifold**.

The following result was proved by J.C. Marrero:

Proposition 2.1 ([19]). *A trans-Sasakian manifold of dimension ≥ 5 is either α-Sasakian, β-Kenmotsu or co-Kählerian.*

2.2. Hermitian manifolds.

An almost complex manifold (M,J) equipped with a Hermitian metric g is called an almost Hermitian manifold. Thus, we have:

$$J^2 = -1, \qquad g(JX,JY) = g(X,Y)\,. \tag{2.5}$$

An almost complex structure J is said to be integrable if its Nijenhuis tensor $[J,J]$ vanishes with

$$[J,J](X,Y) = [JX,JY] - [X,Y] - J[X,JY] - J[JX,Y]\,.$$

For an almost Hermitian manifold (M,J,g), its fundamental Kähler form Ω is given by:

$$\Omega(X,Y) = g(X,JY)\,.$$

We say that (M,J,g) is almost Kähler if $d\Omega = 0$. An almost Kähler manifold with integrable J is called a Kähler manifold, and thus is characterized by the conditions: $d\Omega = 0$ and $N_J = 0$. One can prove that both these conditions combined are equivalent with the single condition:

$$\nabla J = 0\,.$$

For more background on almost complex structure manifolds, we refer the reader to [27].

2.3. **Warped product metrics.**

Let $(M^{'}, g^{'})$ and (M, g) be two Riemannian manifolds and let f be a function on $M^{'}$. Then the Riemannian metric $\tilde{g} = g^{'} + fg$ on $M^{'} \times M$ is called a *warped product metric*. We use the notation $M^{'} \times_f M$ for the product manifold. In Riemaniann geometry, the notion of warped product of metrics is known to produce very interesting metrics.

Let $(M, \varphi, \xi, \eta, g)$ be an almost contact metric manifold with $\dim M = 2n+1$. The equation $\eta = 0$ defines a $2n$-dimensionl distribution $\mathcal{D}$ on M. By an $2n$-homothetic deformation or $\mathcal{D}$-homothetic deformation [24] we mean a change of structure tensors of the form:

$$\overline{\varphi} = \varphi, \quad \overline{\eta} = a\eta, \quad \overline{\xi} = \frac{1}{a}\xi\,, \quad \overline{g} = ag + a(a-1)\eta \otimes \eta, \tag{2.6}$$

where a is a positive constant. If $(M, \varphi, \xi, \eta, g)$ is a contact metric structure with contact form η, then $(M, \overline{\varphi}, \overline{\xi}, \overline{\eta}, \overline{g})$ is also a contact metric structure [24].

The idea works equally well for almost contact metric structures. In this direction, Blair [6] introduced the notion of $\mathcal{D}$-homothetic warped metric on $\tilde{M} = M' \times M$ where M' is a Riemannian manifolds and M is an almost contact metric manifold by:

$$\tilde{g} = g' + fg + f(f-1)\eta \otimes \eta\,, \tag{2.7}$$

where f is a positive function on M'. Recently, Beldjilali and Belkhelfa introduced a generalization of $\mathcal{D}$-homothetic warped metric on $\tilde{M} = M' \times M$ as follows [3]:

$$\tilde{g} = g' + f^2 g + f^2(h^2-1)\eta \otimes \eta\,, \tag{2.8}$$

where f and h be two smooth functions on M' and $fh \neq 0$ everywhere, this metric $\tilde{g}$ is called a $\mathcal{D}$-homothetic bi-warping metric.
In particular, if $h = \pm 1$ then we recover a warped product metric and if $h = \pm f$ we get a $\mathcal{D}$-homothetically warped metric.

2.4. **Preliminaries on generalized geometry.**

In this section we briefly recall basic notions and results from generalized geometry. For more details, we refer the reader to references [14], [22], [26].

Let M be a m-dimensional smooth manifold, the space of sections of the vector bundle $TM \oplus T^*M \longrightarrow M$ is endowed with the following **R**-bilinear operations.

- A symmetric, non-degenerate and bilinear form $\langle -, - \rangle$ is defined by:

$$\langle X + \alpha, Y + \beta \rangle := \frac{1}{2}(\iota_X \beta + \iota_Y \alpha)\,.$$

- The Courant bracket $[-, -]_c$ is a skew-symmetric bracket,

$$[X + \alpha, Y + \beta]_c := [X, Y] + \mathcal{L}_X \beta - \mathcal{L}_Y \alpha - \frac{1}{2} d(\iota_X \beta - \iota_Y \alpha)\,,$$

where X, $Y \in TM$ and α, $\beta \in T^*M$.

A subbundle is Courant involutive if the space of sections of the subbundle is closed under the Courant bracket. Let M be an even dimensional smooth manifold:

Definition 2.2 ([14]). A generalized almost complex structure on an even dimensional manifold M is an endomorphism $\mathcal{J}$ of the direct sum $TM \oplus T^*M$ which satisfies two conditions,

$$\mathcal{J} + \mathcal{J}^* = 0\,, \qquad \mathcal{J}^2 = -\operatorname{id}\,,$$

where $\mathcal{J}^*$ is defined by $\langle \mathcal{J}A, B\rangle = \langle A, \mathcal{J}^*B\rangle$ for any A, $B \in \Gamma(TM \oplus T^*M)$.

Definition 2.3 ([14]).

- A *generalized Riemannian metric* on M is a positive definite metric on $TM \oplus T^*M$.
- A generalized Kähler structure is a pair $(\mathcal{J}_1, \mathcal{J}_2)$ of commuting generalized complex structures such that $G = -\mathcal{J}_1\mathcal{J}_2$ is a generalized Riemannian metric.
- Let B be a smooth 2-form. Then the invertible bundle map given by exponentiating B,

$$\mathrm{e}^B := \begin{pmatrix} 1 & 0 \\ B & 1 \end{pmatrix} : X + \alpha \mapsto X + \alpha + i_X B$$

is called a B-field transformation.

Lemma 2.4 ([14, 23]). *A generalized Kähler metric is uniquely determined by a Riemannian metric g together with a 2-form b as follows:*

$$G(g,b) = \begin{pmatrix} -g^{-1}b & g^{-1} \\ g - bg^{-1}b & bg^{-1} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix} \begin{pmatrix} 0 & g^{-1} \\ g & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -b & 1 \end{pmatrix}$$

Let C_+ be a positive definite subbundle of $TM \oplus T^*M$ and C_- a negative definite subbundle with respect to the inner product which are given by:

$$C_\pm = \{X \pm g(X,\cdot) + b(X,\cdot) /\ X \in TM\}\,.$$

The projection from $C_\pm$ to TM, $\mathcal{J}_1$ induces two almost complex structures $J_\pm$ on TM. If both (g, J_+) and (g, J_-) are Hermitian structures, $(g, J_\pm)$ is called a bi-Hermitian structure.

Theorem 2.5 ([14]). *A generalized Kähler structure $(\mathcal{J}_1, \mathcal{J}_2)$ is equivalent to bi-Hermitian structure $(g, b, J_\pm)$ which satisfies the following condition:*

$$d\omega_\pm\big(J_\pm X, J_\pm Y, J_\pm Z\big) = \pm db(X,Y,Z)\,,$$

where $\omega_\pm = g(X, J_\pm Y)$, and for all vector fields X, Y, Z.

Proposition 2.6 ([14]). *There is a one to-one correspondence between generalized Riemannian metrics G on M and pairs (g, b), where g is a classical generalized Riemannian metric and b a differential 2-form on M.*

Let M be an odd dimensional smooth manifold:

Definition 2.7. A generalized almost contact structure on an odd dimensional manifold M is a triple $(\Phi, E_\pm)$ where Φ is an endomorphism of $TM \oplus T^*M$, and E_+ and E_- are sections of $TM \oplus T^*M$ which satisfy

$$\Phi + \Phi^* = 0 \tag{2.9}$$

$$\Phi \circ \Phi = -\operatorname{Id} + E_+ \otimes E_- + E_- \otimes E_+ \tag{2.10}$$

$$\langle E_\pm, E_\pm\rangle = 0\,, \quad 2\langle E_+, E_-\rangle = 1\,. \tag{2.11}$$

It immediately follows that $\Phi(E_\pm) = 0$. In fact, such a Φ has three eigenvalues: $\lambda = 0, \pm\sqrt{-1}$. Moreover, its kernel is the rank 2 complex vector bundle spanned by $E_\pm$. If we denote by $E^{(1,0)}$ the $\sqrt{-1}$ eigenbundle and $E^{(0,1)}$ the $-\sqrt{-1}$ eigenbundle then we have [23]:

$$E^{(1,0)} = \{X + \alpha - \sqrt{-1}\Phi(X+\alpha) \mid \langle E_\pm, X+\alpha\rangle = 0\},$$

$$E^{(0,1)} = \{X + \alpha + \sqrt{-1}\Phi(X+\alpha) \mid \langle E_\pm, X+\alpha\rangle = 0\}.$$

The following complex vector bundles are maximal isotropic:

$$L^+ = L_{E_+} \oplus E^{(1,0)} \quad \text{and} \quad L^- = L_{E_-} \oplus E^{(1,0)}.$$

When either L^+ or L^- is involutive then $(\Phi, E_\pm)$ is simply called a generalized contact structure. If both $L^\pm$ are involutive, we say that $(\Phi, E_\pm)$ is said to be a strong generalized contact structure.

Definition 2.8. A generalized almost contact metric structure on M^{2n+1} is a quadruple $(\Phi, E_\pm, G)$, where $(\Phi, E_\pm)$ is a generalized almost contact structure and G is a generalized Riemannian metric such that:

$$-\Phi G \Phi = G - E_+ \otimes E_+ - E_- \otimes E_-. \tag{2.12}$$

3. From trans-Sasakian to generalized Kählerian structures

Let $(M^{2n+1}, \varphi, \xi, \eta, g)$ be a trans-Saskian manifold of type (α, β) and I be an open interval of $\mathbf{R}$. Given two functions f, $h\colon I \to \mathbf{R}$ with $fh \neq 0$ everywhere, we get two hermitian structures on the product $\widetilde{M}^{2n+2} = M \times I$. These are defined by (see [3]):

$$\tilde{g} = f^2 g + f^2(h^2 - 1)\eta \otimes \eta + dt^2, \tag{3.1}$$

$$\tilde{J}_\pm = \pm\varphi + fh\eta \otimes \partial t - \frac{1}{fh}\, dt \otimes \xi. \tag{3.2}$$

Proposition 3.1. *These two pairs $(\tilde{g}, \tilde{J}_\pm)$ form a bi-Hermitian structure on $\widetilde{M}^{2n+2}$.*

Proof. The proof is straightforward. It is simply obtained by using (2.5). □

Now, our main goal is to find a pair (f, h) of functions defined on some fixed interval $I \subset \mathbf{R}$ for which, there exists a 2-form $\tilde{b}$ such that $(\tilde{g}, \tilde{b}, \tilde{J}_+, \tilde{J}_-)$ defines a generalized Kählerian structure. We need to write down the fundamental 2-form $\tilde{\omega}_\pm$ of $(\tilde{g}, \tilde{J}_\pm)$. This is defined by:

$$\tilde{\omega}_\pm\Big(\big(X, a\frac{\partial}{\partial t}\big), \big(Y, b\frac{\partial}{\partial t}\big)\Big) = \tilde{g}\Big(\big(X, a\frac{\partial}{\partial t}\big), \tilde{J}_\pm\big(Y, b\frac{\partial}{\partial t}\big)\Big),$$

for all $\big(X, a\frac{\partial}{\partial t}\big), \big(Y, b\frac{\partial}{\partial t}\big)$ vector fields on $\widetilde{M}$. Thus, we obtain:

$$\tilde{\omega}_\pm = \pm f^2\omega + 2fh\, dt \wedge \eta. \tag{3.3}$$

There follows:

$$d\tilde{\omega}_{\pm} = \pm 2ff'dt \wedge \omega \pm f^2 d\omega - 2fh\, dt \wedge d\eta\,. \tag{3.4}$$

Using (2.4) we get:

$$d\tilde{\omega}_{\pm} = \pm 2ff'dt \wedge \omega \pm \beta f^2 \omega \wedge \eta - 2\alpha fh\, dt \wedge \omega\,. \tag{3.5}$$

Thus

$$\begin{aligned} d\tilde{\omega}_{\pm}(\tilde{J}_{\pm}\cdot, \tilde{J}_{\pm}\cdot, \tilde{J}_{\pm}\cdot) &= \pm 2ff'\big(fh\eta\big) \wedge \omega \pm \beta f^2 \omega \wedge \big(-\frac{1}{fh}dt\big) - 2\alpha fh\big(fh\eta\big) \wedge \omega \\ &= \pm f\Big(2ff'h\eta - \frac{\beta}{h}\, dt\Big) \wedge \omega - 2\alpha f^2 h^2 \eta \wedge \omega\,. \end{aligned}$$

Assume $\alpha = 0$, then we get:

$$d\tilde{\omega}_{\pm}(\tilde{J}_{\pm}\cdot, \tilde{J}_{\pm}\cdot, \tilde{J}_{\pm}\cdot) = \pm f\Big(2ff'h\eta - \frac{\beta}{h}\, dt\Big) \wedge \omega\,.$$

This can be re-written as:

$$d\tilde{\omega}_{\pm}(\tilde{J}_{\pm}\cdot, \tilde{J}_{\pm}\cdot, \tilde{J}_{\pm}\cdot) = \pm d\tilde{b}(\cdot, \cdot, \cdot)$$

where $\tilde{b}$ is the 2-form given by:

$$\tilde{b} = \frac{2}{\beta} f^2 f' h\, \omega$$

and the following system must be satisfied:

$$\begin{cases} \frac{2}{\beta}\Big(f^2 f' h\Big)' = -\beta \frac{f}{h} \\ d\beta = 0\,. \end{cases} \tag{3.6}$$

Suppose β is constant, then the last condition is trivial. If in addition, we fix a bounded function f such that $f^2 f' \neq 0$ then there exist two functions h satisfying the system (3.6). More precisely, one gets:

$$h(t) = \pm \frac{\sqrt{4c - \beta^2 f^4}}{2f^2 f'}\,, \tag{3.7}$$

where c is a sufficiently large constant such that $4c - \beta^2 f^4 \geq 0$ on the interval I. There follows our first main theorem:

Theorem 3.2. *Let $(M^{2n+1}, \varphi, \xi, \eta, g)$ be a β-Kenmotsu manifold where β is constant and let I be an open interval of $\mathbf{R}$. Given a bounded function $f\colon I \to \mathbf{R}$ such that $ff' \neq 0$ and a sufficiently large constant c, we consider the positive function h defined as in Equation (3.7) along with the bi-Hermitian structure $(\tilde{g}, \tilde{J}_{\pm})$ on $\widetilde{M} = M \times I$ defined as in (3.1) and (3.2). Let $\tilde{b}$ be the 2-form given by:*

$$\tilde{b} = \sqrt{4c - \beta^2 f^4}\; \omega\,.$$

Then $(\tilde{g}, \tilde{b}, \tilde{J}_+, \tilde{J}_-)$ defines a generalized Kählerian structure on $\widetilde{M}$.

Remark 3.3. In the above proposition we could also pick the negative function h defined as in Equation (3.7) then the sign of the 2-form $\tilde{b}$ changes.

Corollary 3.4. *Given a β-Kenmotsu manifold $(M^{2n+1},\varphi,\xi,\eta,g)$ where β is constant, there is a two-parameter family of generalized coKähler structures on $M\times\mathbf{R}^2$.*

Proof. Applying Theorem 3.2, we get a generalized Kähler structure $(\mathcal{J}_1,\mathcal{J}_2)$ on $\widetilde{M}=M\times\mathbf{R}$. We identify $G=-\mathcal{J}_1\circ\mathcal{J}_2$ with its corresponding pair (g,b), where g is a Riemannian metric and b is a 2-form. Using the standard classical almost contact structure $(\varphi_{\mathbf{R}}=0,\frac{\partial}{\partial t},dt)$ on $\mathbf{R}$, we get a generalized coKähler structure on $\overline{M}=\widetilde{M}\times\mathbf{R}$, with $\overline{\Phi}=\mathcal{J}_1$ and $\overline{G}\equiv(g+dt^2,b)$, $E_+=(0,dt)$ and $E_-=(\frac{\partial}{\partial t},0)$. □

From Propositions 1 and 2 in [20], it follows that, for any 3-dimensional normal almost contact metric manifold (M,φ,ξ,η,g), we have:

$$(\nabla_X\varphi)Y=\frac{1}{2}\operatorname{tr}_g(\varphi\nabla\xi)\Big(g(X,Y)\xi-\eta(Y)X\Big)+\frac{1}{2}\operatorname{div}\xi\Big(g(\varphi X,Y)\xi-\eta(Y)\varphi X\Big),$$

where ∇ is the Levi-Civita connection on M i.e. (M^3,φ,ξ,η,g) is a trans-Sasakian manifold of type $\left(\frac{1}{2}\operatorname{tr}_g(\varphi\nabla\xi),\frac{1}{2}\operatorname{div}\xi\right)$, (see [7]).

Using Theorem (3.2), we immediately get:

Proposition 3.5. *Any connected normal almost contact metric manifold (M^3,φ,ξ,η,g), such that, $\operatorname{tr}_g(\varphi\nabla\xi)=0$ and $\operatorname{div}\xi$ is a constant, gives rise to a multi-parameter family of generalized Kählerian structures on $M\times\mathbf{R}$, where ∇ is the Levi-Civita connection on M.*

4. From almost Kählerian to generalized Kählerian

Let (M'^{2n},J',g',ω') be an almost Kählerian manifold. We define an almost contact metric structure $(\overline{\varphi},\overline{\eta},\overline{\xi},\overline{g})$ on $\overline{M}=M'\times\mathbf{R}$ by:

$$\overline{\varphi}=J',\quad \overline{\eta}=dr,\quad \overline{\xi}=\frac{\partial}{\partial r},\quad \overline{g}=f^2g'+dr^2,$$

where $f=f(r)$ is a function on $\mathbf{R}$.

On $\widetilde{M}^{2n+2}=\overline{M}\times\mathbf{R}=M'\times\mathbf{R}\times\mathbf{R}$, we define a complex structures and a metric by:

$$\tilde{J}_\pm=\pm\varphi\pm hk dr\otimes\frac{\partial}{\partial t}\mp\frac{1}{hk}dt\otimes\frac{\partial}{\partial r},$$
$$\tilde{g}=f^2h^2g'+h^2k^2dr^2+dt^2,$$

where $h=h(t)$ and $k=k(t)$ are two functions on $\mathbf{R}$. Then $(\tilde{g},J_\pm)$ is a bi-Hermitian structure and the fundamental 2-form $\omega_\pm$ is:

$$\omega_\pm\Big(\big(X,a\frac{\partial}{\partial r},b\frac{\partial}{\partial t}\big),\big(Y,a'\frac{\partial}{\partial r},b'\frac{\partial}{\partial t}\big)\Big)=\tilde{g}\Big(\big(X,a\frac{\partial}{\partial r},b\frac{\partial}{\partial t}\big),J_\pm\big(Y,a'\frac{\partial}{\partial r},b'\frac{\partial}{\partial t}\big)\Big),$$

thus, we obtain:

$$\omega_\pm=\pm f^2h^2\omega'-2hkdr\wedge dt.$$

There immediately follows that:

$$d\omega_\pm=\pm2f^2hh'dt\wedge\omega'\pm2ff'h^2dr\wedge\omega'.$$

Thus

$$d\omega_\pm(\tilde{J}_\pm\cdot,\tilde{J}_\pm\cdot,\tilde{J}_\pm\cdot) = \pm 2f^2hh'(\pm hkdr)\wedge\omega' \pm 2ff'h^2(\mp\frac{1}{hk}dt)\wedge\omega'$$

$$= \big(2f^2h^2h'kdr - 2ff'\frac{h}{k}dt\big)\wedge\omega'\,. \tag{4.1}$$

Directly, we see that $d\omega_\pm$ is exact if and only if:

$$(*):\begin{cases} d(ff') = 2f^2\,,\\ d(h^2h'k) = -2\frac{h}{k}\,,\end{cases} \quad \text{or} \quad (**):\begin{cases} d(-ff') = 2f^2\,,\\ d(h^2h'k) = 2\frac{h}{k}\,.\end{cases} \tag{4.2}$$

First from $(*)$ we obtain the following two ODEs:

$$f'^2 + ff'' - 2f^2 = 0\,, \tag{4.3}$$

$$2k^2h'^2 + hh''k^2 + k'kh'h + 2 = 0\,. \tag{4.4}$$

The solution $f(r)$ of the first ODE (4.3) is:

$$f(r) = \pm\frac{1}{\sqrt{2}}\sqrt{a\mathrm{e}^{-2r} - b\mathrm{e}^{2r}}\,,$$

where a and b are two constants. For the second ODE we observe that any function $k(t)$ which satisfies:

$$k(t) = \pm\frac{\sqrt{c - h(t)^4}}{h'(t)h(t)^2}\,,$$

with $c > 0$ is a solution of the differential equation (4.4). Under these conditions the equation (4.1) gives:

$$d\omega_\pm(\tilde{J}_\pm\cdot,\tilde{J}_\pm\cdot,\tilde{J}_\pm\cdot) = d\Big(\frac{1}{2}\sqrt{c-h^4}(ae^{-2r} + be^{2r})\omega'\Big)\,.$$

Secondly, from $(**)$ we obtain the following two ODEs:

$$f'^2 + ff'' + 2f^2 = 0, \tag{4.5}$$

$$2k^2h'^2 + hh''k^2 + k'kh'h - 2 = 0\,. \tag{4.6}$$

The solution $f(r)$ of the first ODE (4.5) is:

$$f(r) = \pm\sqrt{a\sin(2r) + b\cos(2r)}\,,$$

where a and b are two constants. For the second ODE we observe that any function $k(t)$ which satisfies:

$$k(t) = \pm\frac{1}{h'(t)}$$

is a solution of the differential equation (4.6). Under these conditions the equation (4.1) gives:

$$d\omega_\pm(\tilde{J}_\pm\cdot,\tilde{J}_\pm\cdot,\tilde{J}_\pm\cdot) = d\Big(h^2\big(a\cos(2r) - b\sin(2r)\big)\omega'\Big)\,.$$

Therefore $\big(\tilde{g}, \frac{1}{2}\sqrt{c-h^4}(a\mathrm{e}^{-2r}+b\mathrm{e}^{2r})\omega', \tilde{J}_\pm\big)$ and $\big(\tilde{g}, \big(h^2\big(a\cos(2r) - b\sin(2r)\big)\omega', \tilde{J}_\pm\big)$ are two 1-parameter families of generalized Kählerian structures.

Remark 4.1. In particular, when $a=-1$, $b=0$ and $h=r$, we get $\big(\tilde{g}, \frac{-1}{2}\sqrt{c-r^4}\mathrm{e}^{-2r}\omega', \tilde{J}_{\pm}\big)$ and $\big(\tilde{g}, -r^2\cos(2r)\omega', \tilde{J}_{\pm}\big)$ two generalized Kählerian structures, where the second one is given by Sekiya (see [23]).

5. Transformation of generalized almost contact manifolds

Let $(M_i, G_i, \Phi_i, E^i_{\pm})$, $i=1,2$ be two generalized almost contact metric manifolds. For each $i=1,2$, we identify G_i with the pair (g_i, b_i) of a Riemannian metric g_i and 2-form b_i associated to it. Write:

$$E^i_{\pm} = (\xi^i_{\pm}, \eta^i_{\pm}) \qquad \text{and} \qquad \Phi_i = \begin{pmatrix} \varphi_i & \pi_i^{\sharp} \\ \omega_i^{\flat} & -\varphi_i * \end{pmatrix}.$$

Without loss of generality, we can assume that $b_1 \neq 0$ since otherwise, we can replace $(G_1, \Phi_1, E^1_{\pm})$ by $(\mathrm{e}^{b_1}G_1\mathrm{e}^{-b_1}, \mathrm{e}^{b_1}\Phi_1\mathrm{e}^{-b_1}, \mathrm{e}^{b_1}E^1_{\pm})$ on M_1.

Under the above notations, a transformation of $(G_2, \Phi_2, E^2_{\pm})$ is given by:

$$\overline{G_2} \equiv (f^2 g_2, f^2 b_2)\,, \quad \overline{E^2_{\pm}} = (\frac{1}{f}\xi^2_{\pm},\ f\eta^2_{\pm}) \quad \text{and} \quad \overline{\Phi_2} = \begin{pmatrix} \varphi_2 & \frac{1}{f^2}\pi_2^{\sharp} \\ f^2\omega_2^{\flat} & -\varphi_2^* \end{pmatrix}, \tag{5.1}$$

where f is a constant.

Remark 5.1. Let $\mathcal{D}_1 = \Gamma(TM_2)$ and $\mathcal{D}_2 = \Gamma(T^*M_2)$. Note that, the transformation (5.1) is $\mathcal{D}_1$-homothetic and $\mathcal{D}_2$-homothetic, because:

$$\overline{G_2}(A,A) := \langle \overline{G_2}A, A\rangle = \frac{1}{2}\Big(\frac{1}{f^2}|\alpha|^2 - f^2|b_2X|^2 + f^2|X|^2\Big),$$

where $A = X + \alpha \in \Gamma(TM_2 \oplus T^*M_2)$, and $|\ \ |$ means the norm on M_2.
If $A \in \mathcal{D}_1$, i.e. $\alpha = 0$, we have:

$$\overline{G_2}(A,A) = f^2 G_2(A,A)\,,$$

and if $A \in \mathcal{D}_2$, i.e. $X=0$, we have:

$$\overline{G_2}(A,A) = \frac{1}{f^2}G_2(A,A)\,,$$

where,

$$G_2(A,A) := \langle G_2A, A\rangle = \frac{1}{2}\big(|\alpha|^2 - |b_2X|^2 + |X|^2\big)\,.$$

We can call this transformation a $\mathcal{D}$-homothetic deformation with $\mathcal{D} = \{\mathcal{D}_1, \mathcal{D}_2\}$.

Proposition 5.2. *Under the above notations* (5.1)*, the transformation structure* $(\overline{G_2}, \overline{\Phi_2}, \overline{E^2_{\pm}})$*, is a generalized almost contact metric structure on* M_2*.*

Proof. Let $(M_2, G_2, \Phi_2, E^2_{\pm})$ be a generalized almost contact manifold which satisfies the conditions (2.9), (2.10) and (2.11). By simple calculations, we have:

$$\langle E^2_{\pm}, E^2_{\pm}\rangle = \eta^2_{\pm}(\xi^2_{\pm}) = 0\,, \tag{5.2}$$

$$\langle E^2_{+}, E^2_{-}\rangle = \frac{1}{2}[\eta^2_{+}(\xi^2_{-}) + \eta^2_{-}(\xi^2_{+})] = \frac{1}{2}\,, \tag{5.3}$$

(5.4)

$$\left(\Phi_2\circ\Phi_2=-I+E_+^2\otimes E_-^2+E_-^2\otimes E_+^2\right)\Rightarrow\begin{cases}\varphi_2^2+\pi_2^\sharp(\omega_2^\flat)=-I+\eta_+^2\otimes\xi_-^2+\eta_-^2\otimes\xi_+^2,\\ \omega_2^\flat(\varphi_2)-\varphi_2^*(\omega_2^\flat)=\eta_+^2\otimes\eta_-^2+\eta_-^2\otimes\eta_+^2,\\ \varphi_2(\pi_2^\sharp)-\pi_2^\sharp(\varphi_2^*)=\xi_+^2\otimes\xi_-^2+\xi_-^2\otimes\xi_+^2\\ \omega_2^\flat(\pi_2^\sharp)+(\varphi_2^*)^2=-I+\xi_+^2\otimes\eta_-^2+\xi_+^2\otimes\eta_+^2\end{cases}$$

By definition of transformation structure in (5.1) we have:

$$\begin{aligned}\langle\overline{\Phi_2}(X+\alpha),Y+\beta\rangle&=\frac{1}{2}\Big[(f^2\omega_2^\flat(X)-\varphi_2^*(\alpha))(Y)+\beta(\varphi_2(X)+\frac{1}{f^2}\pi_2^\sharp(\alpha))\Big]\\&=\frac{-1}{2}\Big[(f^2\omega_2^\flat(Y)(X)-\varphi_2^*\beta(X)+\alpha(\varphi_2(Y)+\frac{1}{f^2}\alpha(\pi_2^\sharp(\beta))\Big]\\&=-\langle X+\alpha,\overline{\Phi_2}(Y+\beta)\rangle\,,\end{aligned}$$

where $X+\alpha,Y+\beta\in\Gamma(TM\oplus T^*M)$, then $\overline{\Phi_2}+\overline{\Phi_2}^*=0$.
From (5.2) and (5.3) we obtain:

$$\langle\overline{E_\pm^2},\overline{E_\pm^2}\rangle=0\,,\quad\text{and}\quad\langle\overline{E_+^2},\overline{E_-^2}\rangle=\frac{1}{2}\,,$$

Moreover, we have:

$$\overline{\Phi_2}\circ\overline{\Phi_2}=\begin{pmatrix}\varphi_2^2+\pi_2^\sharp(\omega_2^\flat) & \frac{1}{f^2}\varphi_2(\pi_2^\sharp)-\frac{1}{f^2}\pi_2^\sharp(\varphi_2^*)\\ f^2\omega_2^\flat(\varphi_2)-f^2\varphi_2^*(\omega_2^\flat) & \omega_2^\flat(\pi_2^\sharp)+(\varphi_2^*)^2\end{pmatrix},$$

and

$$-I+\overline{E_+^2}\otimes\overline{E_-^2}+\overline{E_-^2}\otimes\overline{E_+^2}=\begin{pmatrix}-I+\eta_+^2\otimes\xi_-^2+\eta_-^2\otimes\xi_+^2 & \frac{1}{f^2}[\xi_+^2\otimes\xi_-^2+\xi_-^2\otimes\xi_+^2]\\ f^2[\eta_+^2\otimes\eta_-^2+\eta_-^2\otimes\eta_+^2] & -I+\xi_+^2\otimes\eta_-^2+\xi_+^2\otimes\eta_+^2\end{pmatrix}.$$

Using (5.4) we get:

$$\overline{\Phi_2}\circ\overline{\Phi_2}=-I+\overline{E_+^2}\otimes\overline{E_-^2}+\overline{E_-^2}\otimes\overline{E_+^2}\,.$$

Thus, $(\overline{\Phi_2},\overline{E_\pm^2})$ is a generalized almost contact structure on M_2.
Since $(M_2,G_2,\Phi_2,E_\pm^2)$ is a generalized almost contact metric structure, and from (2.12) we have:

$$-\Phi_2G_2\Phi_2=G_2-E_+^2\otimes E_+^2-E_-^2\otimes E_-^2\,.\tag{5.5}$$

By a computation, we obtain:

$$(5.6)\quad \begin{cases} \varphi_2 g_2^{-1} b_2 \varphi_2 - \varphi_2 g_2^{-1}\omega_2^\flat - \pi_2^\sharp g_2\varphi_2 + \pi_2^\sharp b_2 g_2^{-1} b_2\varphi_2 - \pi_2^\sharp b_2 g_2^{-1}\omega_2^\flat \\ \qquad\qquad = -g_2^{-1}b_2 - \eta_+^2\otimes\xi_+^2 - \eta_-^2\otimes\xi_-^2 \\ \omega_2^\flat g_2^{-1} b_2\varphi_2 - \omega_2^\flat g_2^{-1}\omega_2^\flat + \varphi_2^* g_2\varphi_2 - \varphi_2^* b_2 g_2^{-1} b_2\varphi_2 + \varphi_2^* b_2 g_2^{-1}\omega_2^\flat \\ \qquad\qquad = g_2 - b_2 g_2^{-1} b_2 - \eta_+^2\otimes\eta_+^2 - \eta_-^2\otimes\eta_-^2 \\ \varphi_2 g_2^{-1} b_2 \pi_2^\sharp + \varphi_2 g_2^{-1}\varphi_2^* - \pi_2^\sharp g_2\pi_2^\sharp + \pi_2^\sharp b_2 g_2^{-1} b_2\pi_2^\sharp + \pi_2^\sharp b_2 g_2^{-1}\varphi_2^* \\ \qquad\qquad = g_2^{-1} - \xi_+^2\otimes\xi_+^2 - \xi_-^2\otimes\xi_-^2 \\ \omega_2^\flat g_2^{-1} b_2\pi_2^\sharp + \omega_2^\flat g_2^{-1}\varphi_2^* + \varphi_2^* g_2\pi_2^\sharp - \varphi_2^* b_2 g_2^{-1} b_2\pi_2^\sharp - \varphi_2^* b_2 g_2^{-1}\varphi_2^* \\ \qquad\qquad = b_2 g_2^{-1} - \xi_+^2\otimes\eta_+^2 - \xi_-^2\otimes\eta_-^2 \end{cases}$$

A generalized Riemannian metric $\overline{G_2}$ of transformation structure is given by:

$$\begin{aligned} \overline{G_2} &= \begin{pmatrix} 1 & 0 \\ f^2 b_2 & 1 \end{pmatrix} \begin{pmatrix} 0 & \frac{1}{f^2} g_2^{-1} \\ f^2 g_2 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -f^2 b_2 & 1 \end{pmatrix} \\ &= \begin{pmatrix} -g_2^{-1} b_2 & \frac{1}{f^2} g_2^{-1} \\ -f^2 b_2 g_2^{-1} b_2 + f^2 g_2 & b_2 g_2^{-1} \end{pmatrix}. \end{aligned}$$

Thus

$$(5.7)\qquad -\overline{\Phi_2}\quad\overline{G_2}\quad\overline{\Phi_2} = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where

$$A = \varphi_2 g_2^{-1} b_2\varphi_2 - \varphi_2 g_2^{-1}\omega_2^\flat + \pi_2^\sharp b_2 g_2^{-1} b_2\varphi_2 - \pi_2^\sharp g_2\varphi_2 - \pi_2^\sharp b_2 g_2^{-1}\omega_2^\flat$$

$$B = \frac{1}{f^2}[\varphi_2 g_2^{-1} b_2\pi_2^\sharp + \varphi_2 g_2^{-1}\varphi_2^* + \pi_2^\sharp b_2 g_2^{-1} b_2\pi_2^\sharp - \pi_2^\sharp g_2\pi_2^\sharp + \pi_2^\sharp b_2 g_2^{-1}\varphi_2^*]$$

$$C = f^2[\omega_2^\flat g_2^{-1} b_2\varphi_2 - \omega_2^\flat g_2^{-1}\omega_2^\flat - \varphi_2^* b_2 g_2^{-1} b_2\varphi_2 + \varphi_2^* g_2\varphi_2 + \varphi_2^* b_2 g_2^{-1}\omega_2^\flat]$$

$$D = \omega_2^\flat g_2^{-1} b_2\pi_2^\sharp + \omega_2^\flat g_2^{-1}\varphi_2^* - \varphi_2^* b_2 g_2^{-1} b_2\pi_2^\sharp + \varphi_2^* g_2\pi_2^\sharp - \varphi_2^* b_2 g_2^{-1}\varphi_2^*$$

by the other side:

$$\overline{G_2} - \overline{E_+^2}\otimes\overline{E_+^2} - \overline{E_-^2}\otimes\overline{E_-^2} =$$
$$\begin{pmatrix} -g_2^{-1}b_2 - \eta_+^2\otimes\xi_+^2 - \eta_-^2\otimes\xi_-^2 & \frac{1}{f^2}[g_2^{-1} - \xi_+^2\otimes\xi_+^2 - \xi_-^2\otimes\xi_-^2] \\ f^2[-b_2 g_2^{-1} b_2 + g_2 - \eta_+^2\otimes\eta_+^2 - \eta_-^2\otimes\eta_-^2] & b_2 g_2^{-1} - \xi_+^2\otimes\eta_+^2 - \xi_-^2\otimes\eta_-^2 \end{pmatrix}.$$

Using (5.6) we get:

$$-\overline{\Phi_2}\quad\overline{G_2}\quad\overline{\Phi_2} = \overline{G_2} - \overline{E_+^2}\otimes\overline{E_+^2} - \overline{E_-^2}\otimes\overline{E_-^2} \quad\text{and}\quad \overline{G_2}(\overline{E_\pm^2}) = \overline{E_\mp^2}.$$

Therefore, $(M_2, \overline{G_2}, \overline{\Phi_2}, \overline{E_\pm^2})$ is a generalized almost contact metric structure. □

We will discuss the following question elsewhere: is the class of generalized Sasakian structures invariant under $\mathcal{D}$-homothetic deformation ?

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