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EXISTENCE AND SHARP ASYMPTOTIC BEHAVIOR OF POSITIVE DECREASING SOLUTIONS OF A CLASS OF DIFFERENTIAL SYSTEMS WITH POWER-TYPE NONLINEARITIES

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Abstract. The system of nonlinear differential equations

(A) $\quad x' + p_1(t)x^{\alpha_1} + q_1(t)y^{\beta_1} = 0\,, \qquad y' + p_2(t)x^{\alpha_2} + q_2(t)y^{\beta_2} = 0\,,$

is under consideration, where α_i and β_i are positive constants and $p_i(t)$ and $q_i(t)$ are positive continuous functions on $[a,\infty)$. There are three types of different asymptotic behavior at infinity of positive solutions $(x(t),y(t))$ of (A). The aim of this paper is to establish criteria for the existence of solutions of these three types by means of fixed point techniques. Special emphasis is placed on those solutions with both components decreasing to zero as $t\to\infty$, which can be analyzed in detail in the framework of regular variation.

1. Introduction

This paper is devoted to the asymptotic analysis of positive solutions of the system of nonlinear differential equations

(A) $$x' + p_1(t)x^{\alpha_1} + q_1(t)y^{\beta_1} = 0\,, \qquad y' + p_2(t)x^{\alpha_2} + q_2(t)y^{\beta_2} = 0\,,$$

under the assumptions

(a) α_i and β_i, $i=1,2$, are positive constants;

(b) $p_i(t)$ and $q_i(t)$, $i=1,2$, are positive continuous functions on $[a,\infty)$, $a>0$.

By a positive solution of (A) we mean a vector function $(x(t),y(t))$ both components of which are positive and satisfy the system (A) in a neighborhood of infinity, say for $t\ge T$. It is clear that both components of a positive solution of (A) are decreasing for $t\ge T$, so that $x(t)$ and $y(t)$ satisfy

(1.1) $$\text{(I)}\ \lim_{t\to\infty} x(t) = \text{const} > 0 \quad \text{or} \quad \text{(II)}\ \lim_{t\to\infty} x(t) = 0,$$

(1.2) $$\text{(I)}\ \lim_{t\to\infty} y(t) = \text{const} > 0 \quad \text{or} \quad \text{(II)}\ \lim_{t\to\infty} y(t) = 0\,.$$

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Noting that the asymptotic behavior of $(x(t), y(t))$ is determined by the combination of (1.1) and (1.2), we see that the set of possible positive solutions is essentially classified into the following three types according to their asymptotic behavior as $t \to \infty$:

(i) The subclass of solutions of type (I,I) which consists of positive solutions $(x(t), y(t))$ such that

$$\lim_{t\to\infty} x(t) = \text{const} > 0 \quad \text{and} \quad \lim_{t\to\infty} y(t) = \text{const} > 0;$$

(ii) The subclass of solutions of type (I,II) which consists of positive solutions $(x(t), y(t))$ such that

$$\lim_{t\to\infty} x(t) = \text{const} > 0 \quad \text{and} \quad \lim_{t\to\infty} y(t) = 0\,;$$

(iii) The subclass of solutions of type (II,II) which consists of positive solutions $(x(t), y(t))$ such that

$$\lim_{t\to\infty} x(t) = 0 \quad \text{and} \quad \lim_{t\to\infty} y(t) = 0\,.$$

Solutions of type (II,II) are often referred to as *strongly decreasing solutions* of (A).

The present work was motivated by the observation that little is known about the qualitative properties of nonlinear differential systems of the form (A) and aims at acquiring as detailed information as possible about the precise asymptotic behavior of positive solutions, with special emphasis on type-(II,II) solutions, of (A). We begin with the study of solutions of type (I,I) in Section 2 and continue to study solutions of types (I,II) and (II,II) in Sections 3 and 4, respectively. Type-(I,I) solutions are easy to analyze and their existence can be completely characterized by means of fixed point techniques with no restriction on the values of the exponents α_i and β_i, $i = 1, 2$. However, in Sections 3 and 4, because of the difficulty in dealing successfully with the components of solutions decreasing to zero as $t \to \infty$, we have to require that some of α_i and β_i to be less than 1, and at the final stage of the analysis of type-(II,II) solutions we have to make extensive use of theory of regular variation (in the sense of Karamata) in order to determine their order of decay explicitly and accurately. For the reader's convenience the definition and some basic properties of regularly varying functions will be summarized in the appendix.

2. Solutions of type (I,I)

We start with the study of positive solutions of type (I,I) of system (A), that is, those solutions both components of which decrease to finite positive constants as $t \to \infty$. Such solutions are the simplest of all possible positive solutions of (A) in the sense that the situation for their existence can be characterized for any values of the exponents α_i and β_i, $i = 1, 2$.

Theorem 2.1. *System* (A) *has positive solutions of type* (I,I) *if and only if*

$$\int_a^\infty p_i(t)\,dt < \infty\,, \qquad \int_a^\infty q_i(t)\,dt < \infty\,, \qquad i = 1, 2\,, \tag{2.1}$$

in which case there exists a positive solution $(x(t), y(t))$ *such that*

$$\lim_{t\to\infty} x(t) = c\,, \qquad \lim_{t\to\infty} y(t) = d\,, \tag{2.2}$$

for any given constants $c > 0$ *and* $d > 0$.

Proof. Suppose that (A) has a solution $(x(t), y(t))$ on $[T, \infty)$ satisfying (2.2). Then, integrating (A) from t to ∞, we have

$$\begin{aligned} x(t) &= c + \int_t^\infty \big(p_1(s)x(s)^{\alpha_1} + q_1(s)y(s)^{\beta_1}\big)\, ds,\\ y(t) &= d + \int_t^\infty \big(p_2(s)x^{\alpha_2} + q_2(s)y(s)^{\beta_2}\big)\, ds, \end{aligned} \tag{2.3}$$

for $t \geq T$. This combined with (2.2) implies that

$$\int_T^\infty p_i(s)\, ds < \infty\,, \qquad \int_T^\infty q_i(s)\, ds < \infty\,, \quad i = 1, 2\,,$$

confirming the validity of (2.1).

Suppose conversely that (2.1) holds. Let $c > 0$ and $d > 0$ be given arbitrarily. Choose $T > a$ so that

$$\begin{aligned} &\int_T^\infty p_1(s)\, ds \leq \frac{c^{1-\alpha_1}}{2^{1+\alpha_1}}\,, \qquad \int_T^\infty q_1(s)\, ds \leq \frac{c}{2^{1+\beta_1} d^{\beta_1}}\,,\\ &\int_T^\infty p_2(s)\, ds \leq \frac{d}{2^{1+\alpha_2} c^{\alpha_2}}\,, \quad \int_T^\infty q_2(s)\, ds \leq \frac{c^{1-\beta_2}}{2^{1+\beta_2}}\,, \end{aligned} \tag{2.4}$$

and define the set

$$\mathcal{U} = \big\{(x, y) \in C[T, \infty)^2 : c \leq x(t) \leq 2c,\ d \leq y(t) \leq 2d,\ t \geq T\big\}\,, \tag{2.5}$$

and the integral operators

$$\begin{aligned} \mathcal{F}(x, y)(t) &= c + \int_t^\infty \big(p_1(s)x(s)^{\alpha_1} + q_1(s)y(s)^{\beta_1}\big)\, ds\,, \quad t \geq T\,,\\ \mathcal{G}(x, y)(t) &= d + \int_t^\infty \big(p_2(s)x^{\alpha_2} + q_2(s)y(s)^{\beta_2}\big)\, ds\,, \quad t \geq T\,. \end{aligned} \tag{2.6}$$

Consider the mapping $\Phi\colon \mathcal{U} \to C[T, \infty)^2$ defined by

$$\Phi(x, y)(t) = \big(\mathcal{F}(x, y)(t), \mathcal{G}(x, y)(t)\big)\,, \quad t \geq T\,. \tag{2.7}$$

It can be shown that Φ is a continuous map on $\mathcal{U}$ and sends $\mathcal{U}$ into a relatively compact subset of $C[T, \infty)^2$, so that the Schauder-Tychonoff fixed point theorem (cf. [2, Chapter I]) is applicable to Φ.

(i) $\Phi(\mathcal{U}) \subset \mathcal{U}$. Using (2.6) and (2.4), we easily see that if $(x, y) \in \mathcal{U}$, then

$$c \leq \mathcal{F}(x, y)(t) \leq 2c, \qquad d \leq \mathcal{G}(x, y) \leq 2d\,, \quad t \geq T\,,$$

which implies that $\Phi(x, y) = (\mathcal{F}(x, y), \mathcal{G}(x, y)) \in \mathcal{U}$.

(ii) $\Phi(\mathcal{U})$ is relatively compact. From the inclusion $\Phi(\mathcal{U}) \subset \mathcal{U}$ it follows that $\Phi(\mathcal{U})$ is uniformly bounded on $[T,\infty)$. The inequalities

$$0 \geq (\mathcal{F}(x,y))'(t) \geq -(2c)^{\alpha_1} p_1(t) - (2d)^{\beta_1} q_1(t)\,,$$

$$0 \geq (\mathcal{G}(x,y))'(t) \geq -(2c)^{\alpha_2} p_2(t) - (2d)^{\beta_2} q_2(t)\,,$$

holding for all $t \geq T$ and for all $(x,y) \in \mathcal{U}$ ensure that $\Phi(\mathcal{U})$ is equicontinuous on $[T,\infty)$. The relative compactness of $\Phi(\mathcal{U})$ then follows from the Arzela-Ascoli lemma.

(iii) Φ is continuous. Let $\{(x_n,y_n)\}$ be a sequence in $\mathcal{U}$ converging, as $n\to\infty$, to $(x,y) \in \mathcal{U}$ in $C[T,\infty)^2$, which means that $x_n(t) \to x(t)$ and $y_n(t) \to y(t)$, as $n \to \infty$, uniformly on any compact subinterval of $[T,\infty)$. To prove the continuity of Φ it suffices to verify that as $n\to\infty$

$$\mathcal{F}(x_n,y_n)(t) \to \mathcal{F}(x,y)(t)\,, \qquad \mathcal{G}(x_n,y_n)(t) \to \mathcal{G}(x,y)(t)$$

uniformly on compact subintervals of $[T,\infty)$. But this is an immediate consequence of the Lebesgue dominated convergence theorem applied to the right-hand sides of the following inequalities

$$\big|\mathcal{F}(x_n,y_n)(t) - \mathcal{F}(x,y)(t)\big| \leq \int_t^\infty \big(p_1(s)|x_n(s)^{\alpha_1} - x(s)^{\alpha_1}| + q_1(s)|y_n(s)^{\beta_1} - y(s)^{\beta_1}|\big)\,ds\,,$$

$$\big|\mathcal{G}(x_n,y_n)(t) - \mathcal{G}(x,y)(t)\big| \leq \int_t^\infty \big(p_2(s)|x_n(s)^{\alpha_2} - x(s)^{\alpha_2}| + q_2(s)|y_n(s)^{\beta_2} - y(s)^{\beta_2}|\big)\,ds\,.$$

Therefore, there exists $(x,y) \in \mathcal{U}$ such that $(x,y) = \Phi(x,y) = (\mathcal{F}(x,y), \mathcal{G}(x,y))$, which is equivalent to the system of integral equations (2.3). This shows that $(x(t),y(t))$ is a solution of system (A) satisfying (2.2). This completes the proof of Theorem 2.1. □

Remark 2.2. The conclusion of Theorem 2.1 remains valid if we replace the assumption of positivity of $p_i(t)$ and $q_i(t)$, $i=1,2$, by the conditions

$$p_i(t)q_i(t) > 0\,, \quad i = 1,2\,,$$

for $t \geq a$ and integral conditions (2.1) by

$$\int_a^\infty |p_i(t)|dt < \infty\,, \quad \int_a^\infty |q_i(t)|\,dt < \infty\,, \qquad i = 1,2\,. \tag{2.8}$$

Also, if $p_i(t)$ and $q_i(t)$ are not necessarily of the same sign, but they satisfy

$$p_1(t)p_2(t)q_1(t)q_2(t) \neq 0$$

for $t \geq a$, then the sufficiency part of Theorem 2.1 still remains true.

3. Solutions of type (I,II)

We now turn to discuss the existence of solutions of type (I,II) for system (A), that is, positive solutions $(x(t), y(t))$ such that $\lim_{t\to\infty} x(t) = \text{const} > 0$ and $\lim_{t\to\infty} y(t) = 0$. Solutions of this type are obtained by solving the system of integral equations

$$\begin{aligned} x(t) &= c + \int_t^\infty \big(p_1(s)x(s)^{\alpha_1} + q_1(s)y(s)^{\beta_1}\big)\, ds\,, \\ y(t) &= \int_t^\infty \big(p_2(s)x(s)^{\alpha_2} + q_2(s)y(s)^{\beta_2}\big)\, ds\,, \end{aligned} \tag{3.1}$$

on $[T, \infty)$, where $c > 0$ and $T > a$ are positive constants. Our primary concern is the possibility of finding explicit asymptotic formulas for the y-component of type-(I,II) solutions. It is expected that the decay order of the y-component of the solutions of (A) in question may depend on either of the terms $p_2(t)x^{\alpha_2}$ and $q_2(t)y^{\beta_2}$ which is dominant over the other in a certain sense.

The first result describes the effect generated by the term $p_2(t)x^{\alpha_2}$.

Theorem 3.1. *Suppose that*

$$\int_a^\infty p_1(t)\, dt < \infty, \qquad \int_a^\infty p_2(t)\, dt < \infty\,, \tag{3.2}$$

$$\int_a^\infty q_1(t)\Big(\int_t^\infty p_2(s)\, ds\Big)^{\beta_1} dt < \infty\,, \tag{3.3}$$

and

$$\lim_{t\to\infty} \frac{q_2(t)}{p_2(t)}\Big(\int_t^\infty p_2(s)\, ds\Big)^{\beta_2} = 0\,. \tag{3.4}$$

Then, for any constant $c > 0$, system (A) *has a positive solution $(x(t), y(t))$ of type* (I,II) *such that*

$$x(t) \sim c\,, \qquad y(t) \sim c^{\alpha_2} \int_t^\infty p_2(s)\, ds\,, \qquad t \to \infty\,. \tag{3.5}$$

Here and throughout the symbol $\sim$ is used to denote the asymptotic equivalence between two positive functions

$$f(t) \sim g(t), \quad t \to \infty \qquad \Longleftrightarrow \qquad \lim_{t\to\infty} \frac{g(t)}{f(t)} = 1\,.$$

Proof. Let a constant $c > 0$ be given arbitrarily. Define

$$\pi(t) = \int_t^\infty p_2(s)\, ds\,, \tag{3.6}$$

which, in view of the second condition in (3.2), satisfies $\pi(t) \to 0$ as $t \to \infty$. Choose $T > a$ so large that the following inequalities hold:

$$\int_T^\infty p_2(s)\, ds \le \frac{c^{1-\alpha_1}}{2^{1+\alpha_1}}\,, \quad \int_T^\infty q_1(s)\pi(s)^{\beta_1} ds \le \frac{c^{(1-\alpha_1)\beta_1}}{2^{(1+\alpha_1)\beta_1+1}}\,, \tag{3.7}$$

and

$$\frac{q_2(t)}{p_2(t)}\pi(t)^{\beta_2} \le \frac{c^{\alpha_2(1-\beta_2)}}{2^{(1+\alpha_2)\beta_2}} \qquad \text{for} \quad t \ge T\,. \tag{3.8}$$

Such a choice of T is possible because of (3.2)–(3.4).

Let us define the set $\mathcal{V}$ by

$$\mathcal{V} = \big\{(x,y) \in C[T,\infty)^2 : c \le x(t) \le 2c,\ c^{\alpha_2}\pi(t) \le y(t) \le 2^{\alpha_2+1}c^{\alpha_2}\pi(t),\ t \ge T\big\}\,, \tag{3.9}$$

and the integral operators $\mathcal{F}(x,y)$ and $\mathcal{G}(x,y)$ by

$$\begin{aligned} \mathcal{F}(x,y)(t) &= c + \int_t^\infty \big(p_1(s)x(s)^{\alpha_1} + q_1(s)y(s)^{\beta_1}\big)\,ds\,, \quad t \ge T\,,\\ \mathcal{G}(x,y)(t) &= \int_t^\infty \big(p_2(s)x(s)^{\alpha_2} + q_2(s)y(s)^{\beta_2}\big)\,ds\,, \qquad t \ge T\,. \end{aligned} \tag{3.10}$$

Finally define the mapping $\Phi\colon \mathcal{V} \to C[T,\infty)^2$ by

$$\Phi(x,y)(t) = \big(\mathcal{F}(x,y)(t), \mathcal{G}(x,y)(t)\big)\,, \quad t \ge T\,. \tag{3.11}$$

Let $(x(t),y(t)) \in \mathcal{V}$. Then, using (3.7) we see that $c \le \mathcal{F}(x,y)(t) \le 2c$ for $t \ge T$. On the other hand, in view of (3.8) we find that

$$\begin{aligned} p_2(t)x(t)^{\alpha_2} + q_2(t)y(t)^{\beta_2} &= p_2(t)x(t)^{\alpha_2}\Big(1 + \frac{q_2(t)y(t)^{\beta_2}}{p_2(t)x(t)^{\alpha_2}}\Big)\\ &\le p_2(t)x(t)^{\alpha_2}\Big(1 + \frac{(2^{\alpha_2+1}c^{\alpha_2})^{\beta_2}q_2(t)\pi(t)^{\beta_2}}{c^{\alpha_2}p_2(t)}\Big) \le 2p_2(t)x(t)^{\alpha_2}\,, \end{aligned}$$

and hence that

$$c^{\alpha_2}\pi(t) \le \mathcal{G}(x,y)(t) \le 2\int_t^\infty p_2(s)x(s)^{\alpha_2}\,ds \le 2(2c)^{\alpha_2}\pi(t)\,, \quad t \ge T\,.$$

It follows therefore that $\Phi(x,y) \in \mathcal{V}$, which implies that Φ is a self-map of $\mathcal{V}$.

Since as in the proof of Theorem 2.1 it can be shown that Φ is continuous and sends $\mathcal{V}$ into a relatively compact subset of $C[T,\infty)^2$, there exists $(x,y) \in \mathcal{V}$ such that $(x,y) = \Phi(x,y) = (\mathcal{F}(x,y), \mathcal{G}(x,y))$, which is equivalent to the system of integral equations (3.1). This shows that $(x(t),y(t))$ provides a solution of system (A) on $[T,\infty)$. It is clear that $x(t) \sim c$ as $t \to \infty$. Since

$$p_2(t)x(t)^{\alpha_2} + q_2(t)y(t)^{\beta_2} \sim p_2(t)x(t)^{\alpha_2} \sim c^{\alpha_2}p_2(t)\,, \quad t \to \infty\,,$$

from the second equation of (3.1) we conclude that $y(t) \sim c^{\alpha_2}\pi(t)$ as $t \to \infty$. This completes the proof. □

The term $q_2(t)y^{\beta_2}$ may determine the behavior of the second component of the solutions as the following theorem shows.

Theorem 3.2. *Assume that* $0 < \beta_2 < 1$ *and suppose that*

$$\int_a^\infty p_1(t\,dt < \infty, \qquad \int_a^\infty q_2(t)\,dt < \infty\,, \tag{3.12}$$

$$\int_a^\infty q_1(t)\Big(\int_t^\infty q_2(s)\,ds\Big)^{\frac{\beta_1}{1-\beta_2}}\,dt < \infty\,, \tag{3.13}$$

and

$$\lim_{t\to\infty}\frac{p_2(t)}{q_2(t)}\Big(\int_t^\infty q_2(s)ds\Big)^{-\frac{\beta_2}{1-\beta_2}} = 0\,. \tag{3.14}$$

Then, for any constant $c > 0$*, system* (A) *has a positive solution* $(x(t), y(t))$ *of type* (I,II) *such that*

$$x(t) \sim c\,, \qquad y(t) \sim \Big((1-\beta_2)\int_t^\infty q_2(s)\,ds\Big)^{\frac{1}{1-\beta_2}}\,, \qquad t\to\infty\,. \tag{3.15}$$

Proof. Let a constant $c > 0$ be given arbitrarily. Define

$$\eta(t) = \Big((1-\beta_2)\int_t^\infty q_2(s)\,ds\Big)^{\frac{1}{1-\beta_2}}\,, \qquad t \ge a\,. \tag{3.16}$$

It is clear that $\eta(t)$ satisfies

$$\int_t^\infty q_2(s)\eta(s)^{\beta_2}\,ds = \eta(t)\,, \qquad t \ge a\,, \tag{3.17}$$

from which it follows trivially that

$$\frac{1}{2}\eta(t) \le \int_t^\infty q_2(s)\eta(s)^{\beta_2}\,ds \le 2\eta(t)\,, \qquad t \ge a\,. \tag{3.18}$$

Let m and M be positive constants such that

$$m \le 2^{-\frac{1}{1-\beta_2}}\,, \qquad M \ge 4^{\frac{1}{1-\beta_2}}\,, \tag{3.19}$$

and choose $T > a$ so large that the following inequalities are satisfied (cf. (3.12)–(3.14)):

$$\int_T^\infty p_1(s)\,ds \le \frac{c^{1-\alpha_1}}{2^{1+\alpha_1}}\,, \qquad \int_T^\infty q_1(s)\eta(s)^{\beta_1}\,ds \le \frac{c}{2M^{\beta_1}}\,, \tag{3.20}$$

and

$$\frac{p_2(t)}{q_2(t)}\eta(t)^{-\beta_2} \le \frac{m^{\beta_2}}{(2c)^{\alpha_2}}\,, \qquad t \ge T\,. \tag{3.21}$$

Using the same integral operators $\mathcal{F}(x,y)$, $\mathcal{G}(x,y)$ as in (3.10) we define the mapping Φ by (3.11) and let it act on the set

$$\mathcal{V} = \big\{(x,y) \in C[T,\infty)^2 : c \le x(t) \le 2c,\ m\eta(t) \le y(t) \le M\eta(t),\ t \ge T\big\}\,. \tag{3.22}$$

Let $(x,y) \in \mathcal{V}$. Using (3.20) one easily sees that $c \le \mathcal{F}(x,y)(t) \le 2c$ for $t \ge T$. As for $\mathcal{G}(x,y)$, using (3.19) and (3.21) one finds that

$$\begin{aligned}\mathcal{G}(x,y)(t) &\ge \int_t^\infty q_2(s)y(s)^{\beta_2}\,ds \ge m^{\beta_2}\int_t^\infty q_2(s)\eta(s)^{\beta_2}\,ds\\ &\ge \frac{1}{2}m^{\beta_2}\eta(t) \ge m\eta(t)\,, \quad t \ge T\,,\end{aligned}$$

and

$$\begin{aligned}\mathcal{G}(x,y)(t) &\le \int_t^\infty q_2(s)y(s)^{\beta_2}\Big(1+\frac{p_2(s)x(s)^{\alpha_2}}{q_2(s)y(s)^{\beta_2}}\Big)\,ds\\ &\le \int_t^\infty q_2(s)y(s)^{\beta_2}\Big(1+\frac{(2c)^{\alpha_2}p_2(s)}{m^{\beta_2}q_2(s)\eta(s)^{\beta_2}}\Big)\,ds \le 2\int_t^\infty q_2(s)y(s)^{\beta_2}ds\\ &\le 2M^{\beta_2}\int_t^\infty q_2(s)\eta(s)^{\beta_2}ds \le 4M^{\beta_2}\eta(t) \le M\eta(t)\,, \quad t \ge T\,.\end{aligned}$$

Thus, $m\eta(t) \le \mathcal{G}(x,y)(t) \le M\eta(t)$ for $t \ge T$, and it is concluded that $\Phi(x,y) \in \mathcal{V}$, that is, Φ maps $\mathcal{V}$ into itself. □

The continuity of Φ and the relative compactness of $\Phi(\mathcal{V})$ are proved in a routine manner, and so the Schauder-Tychonoff theorem applied to Φ ensures the existence of a fixed point $(x,y) \in \mathcal{V}$ of Φ, which gives birth to a type-(I,II) solution $(x(t),y(t))$ of system (A). It remains to prove that $y(t) \sim \eta(t)$ as $t \to \infty$. This can be done with the help of the following generalized L'Hospital's rule (see e.g. [4]).

Lemma 3.3. *Let $f(t), g(t) \in C^1[T,\infty)$ and suppose that*

$$\lim_{t\to\infty} f(t) = \lim_{t\to\infty} g(t) = \infty \qquad \text{and} \quad g'(t) > 0 \quad \text{for all large } t\,,$$

or

$$\lim_{t\to\infty} f(t) = \lim_{t\to\infty} g(t) = 0 \qquad \text{and} \quad g'(t) < 0 \quad \text{for all large } t\,.$$

Then,

$$\liminf_{t\to\infty}\frac{f'(t)}{g'(t)} \le \liminf_{t\to\infty}\frac{f(t)}{g(t)}\,, \qquad \limsup_{t\to\infty}\frac{f(t)}{g(t)} \le \limsup_{t\to\infty}\frac{f'(t)}{g'(t)}\,.$$

To complete the proof of Theorem 3.2, we note that the second component of the solution $(x(t),y(t))$ obtained above satisfies

$$y(t) = \int_t^\infty \big(p_2(s)x(s)^{\alpha_2} + q_2(s)y(s)^{\beta_2}\big)\,ds\,.$$

Consider the function $z(t)$ given by

$$z(t) = \int_t^\infty \big(p_2(s)x(s)^{\alpha_2} + q_2(s)\eta(s)^{\beta_2}\big)\,ds\,,$$

and put

$$l = \liminf_{t\to\infty}\frac{y(t)}{z(t)}\,, \qquad L = \limsup_{t\to\infty}\frac{y(t)}{z(t)}\,.$$

First we apply Lemma 3.3 to l. Using the relations

$$p_2(t)x(t)^{\alpha_2}+q_2(t)y(t)^{\beta_2} \sim q_2(t)y(t)^{\beta_2}\,, \qquad p_2(t)x(t)^{\alpha_2}+q_2(t)\eta(t)^{\beta_2} \sim q_2(t)\eta(t)^{\beta_2}\,,$$

as $t \to \infty$, we obtain

$$\begin{aligned} l &\geq \liminf_{t\to\infty} \frac{y'(t)}{z'(t)} = \liminf_{t\to\infty} \frac{p_2(t)x(t)^{\alpha_2}+q_2(t)y(t)^{\beta_2}}{p_2(t)x(t)^{\alpha_2}+q_2(t)\eta(t)^{\beta_2}} \\ &= \liminf_{t\to\infty} \frac{q_2(t)y(t)^{\beta_2}}{q_2(t)\eta(t)^{\beta_2}} = \Big(\liminf_{t\to\infty} \frac{y(t)}{\eta(t)}\Big)^{\beta_2} = \Big(\liminf_{t\to\infty} \frac{y(t)}{z(t)}\Big)^{\beta_2} = l^{\beta_2}\,, \end{aligned}$$

where we have used the fact that $z(t) \sim \eta(t)$ as $t \to \infty$. Since $\beta_2 < 1$, the inequality $l \geq l^{\beta_2}$ thus obtained implies that $l \geq 1$. Likewise, Lemma 3.3 applied to L leads to the inequality $L \leq 1$ from which it follows that $l = L = 1$, that is,

$$\lim_{t\to\infty} \frac{y(t)}{z(t)} = 1 \quad \Longrightarrow \quad y(t) \sim z(t) \sim \eta(t), \quad t \to \infty\,.$$

This establishes the desired asymptotic formula (3.15) for the y-component of the solution $(x(t), y(t))$. This completes the proof.

Remarks 3.4. An inspection of the proofs of the above results shows that Theorems 3.1 and 3.2 remain valid even if the coefficients $q_1(t)$ and $q_2(t)$ (resp. $q_1(t)$ and $p_2(t)$) are negative or sign-changing functions satisfying "smallness conditions" (3.3) and (3.4) (resp. (3.13) and (3.14)). However, in such a case positive solutions need not to be decreasing and the structure of the solution set for (A) is not so simple as described in the introduction.

4. Solutions of type (II,II)

In this section we focus our attention on type-(II,II) solutions of system (A), that is, those solutions both components of which decrease to zero as $t \to \infty$. We show that two kinds of criteria for the existence of solutions of this type can be established by regarding (A) as a small perturbation of the simplest diagonal system

$$x' + p_1(t)x^{\alpha_1} = 0\,, \qquad y' + q_2(t)y^{\beta_2} = 0\,, \tag{4.1}$$

or of the cyclic system

$$x' + q_1(t)y^{\beta_1} = 0\,, \qquad y' + p_2(t)x^{\alpha_2} = 0\,. \tag{4.2}$$

4.1. **Perturbations of the diagonal system.**

Throughout his subsection we limit ourselves to the case where $0 < \alpha_1 < 1$ and $0 < \beta_2 < 1$. In this case the diagonal system (4.1) has a type-(II,II) solution if and only if

$$\int_a^\infty p_1(t)\,dt < \infty \qquad \text{and} \qquad \int_a^\infty q_2(t)\,dt < \infty\,, \tag{4.3}$$

and its unique solution is given by $(x(t), y(t)) = (\xi(t), \eta(t))$, where

$$\xi(t) = \Big((1-\alpha_1)\int_t^\infty p_1(s)ds\Big)^{\frac{1}{1-\alpha_1}}, \quad \eta(t) = \Big((1-\beta_2)\int_t^\infty q_2(s)ds\Big)^{\frac{1}{1-\beta_2}}. \tag{4.4}$$

System (A) in which the terms $p_1(t)x^{\alpha_1}$ and $q_2(t)y^{\beta_2}$ are dominant over the terms $q_1(t)y^{\beta_1}$ and $p_2(t)x^{\alpha_2}$, respectively, in a certain sense may be considered as a small perturbation of the diagonal system (4.1). The following result exhibits an example of such differential systems which possess type-(II,II) solutions behaving like (4.4) as $t \to \infty$.

Theorem 4.1. *Let $0 < \alpha_1 < 1$ and $0 < \beta_2 < 1$ and let (4.3) hold. Suppose that*

$$\lim_{t\to\infty} \frac{q_1(t)\eta(t)^{\beta_1}}{p_1(t)\xi(t)^{\alpha_1}} = 0 \qquad \text{and} \qquad \lim_{t\to\infty} \frac{p_2(t)\xi(t)^{\alpha_2}}{q_2(t)\eta(t)^{\beta_2}} = 0\,. \tag{4.5}$$

Then, system (A) possesses solutions $(x(t), y(t))$ of type (II,II) all of which enjoy the unique asymptotic behavior

$$x(t) \sim \xi(t)\,, \qquad y(t) \sim \eta(t)\,, \qquad t \to \infty\,, \tag{4.6}$$

where $\xi(t)$ and $\eta(t)$ are given by (4.4).

Proof. Choose positive constants h, H, k and K such that

$$0 < h < 1\,, \qquad 0 < k < 1\,, \qquad H \ge 2^{\frac{1}{1-\alpha_1}}\,, \qquad K \ge 2^{\frac{1}{1-\beta_2}}\,. \tag{4.7}$$

Choose $T > a$ so large that

$$\frac{q_1(t)\eta(t)^{\beta_1}}{p_1(t)\xi(t)^{\alpha_1}} \le \frac{h^{\alpha_1}}{K^{\beta_1}}\,, \qquad \frac{p_2(t)\xi(t)^{\alpha_2}}{q_2(t)\eta(t)^{\beta_2}} \le \frac{k^{\beta_2}}{H^{\alpha_2}}\,, \qquad t \ge T\,, \tag{4.8}$$

which is possible by (4.5). Let $\mathcal{W}$ denote the set

$$\mathcal{W} = \{(x,y) \in C[T,\infty)^2 : h\xi(t) \le x(t) \le H\xi(t), k\eta(t) \le y(t) \le K\eta(t), t \ge T\}\,. \tag{4.9}$$

Consider the integral operators

$$\mathcal{F}(x,y)(t) = \int_t^\infty \big(p_1(s)x(s)^{\alpha_1} + q_1(s)y(s)^{\beta_1}\big)\,ds\,, \quad t \ge T\,, \qquad \mathcal{G}(x,y)(t) = \int_t^\infty \big(p_2(s)x(s)^{\alpha_2} + q_2(s)y(s)^{\beta_2}\big)\,ds\,, \quad t \ge T\,, \tag{4.10}$$

and define the mapping $\Phi\colon \mathcal{W} \to C[T,\infty)^2$ by

$$\Phi(x,y)(t) = \big(\mathcal{F}(x,y)(t), \mathcal{G}(x,y)(t)\big)\,, \quad t \ge T\,. \tag{4.11}$$

Let $(x,y) \in \mathcal{W}$. Then, using (4.7) and (4.8) we see that

$$\mathcal{F}(x,y)(t) \ge \int_t^\infty p_1(s)x(s)^{\alpha_1}ds \ge h^{\alpha_1}\int_t^\infty p_1(s)\xi(s)^{\alpha_1}ds = h^{\alpha_1}\xi(t) \ge h\xi(t)\,,$$

and

$$\begin{aligned}\mathcal{F}(x,y)(t) &= \int_t^\infty \big(p_1(s)x(s)^{\alpha_1} + q_1(s)y(s)^{\beta_1}\big)\,ds\\ &= \int_t^\infty p_1(s)x(s)^{\alpha_1}\Big(1+\frac{q_1(s)y(s)^{\beta_1}}{p_1(s)x(s)^{\alpha_1}}\Big)\,ds\\ &\le \int_t^\infty p_1(s)x(s)^{\alpha_1}\Big(1+\frac{K^{\beta_1}q_1(s)\eta(s)^{\beta_1}}{h^{\alpha_1}p_1(s)\xi(s)^{\alpha_1}}\Big)\,ds\\ &\le 2\int_t^\infty p_1(s)x(s)^{\alpha_1}\,ds \le 2H^{\alpha_1}\int_t^\infty p_1(s)\xi(s)^{\alpha_1}ds = 2H^{\alpha_1}\xi(t) \le H\xi(t)\,,\end{aligned}$$

for $t \ge T$. Thus, $h\xi(t) \le \mathcal{F}(x,y)(t) \le H\xi(t)$, $t \ge T$. Similarly, we obtain $k\eta(t) \le \mathcal{G}(x,y)(t) \le K\eta(t)$ for $t \ge T$. This shows that Φ is a self-map on $\mathcal{W}$.

It is clear that the set $\Phi(\mathcal{W})$ is uniformly bounded on $[T,\infty)$. This set is equicontinuous on $[T,\infty)$ because of the inequalities

$$0 \ge (\mathcal{F}(x,y))'(t) \ge -H^{\alpha_1}p_1(t)\xi(t)^{\alpha_1} - K^{\beta_1}q_1(t)\eta(t)^{\beta_1}\,,$$

$$0 \ge (\mathcal{G}(x,y))'(t) \ge -H^{\alpha_2}p_2(s)\xi(t)^{\alpha_2} - K^{\beta_2}q_2(t)\eta(t)^{\beta_2}\,,$$

holding for $t \ge T$ and for all $(x,y) \in \mathcal{W}$. Then the Arzela-Ascoli lemma guarantees that $\Phi(\mathcal{W})$ is relatively compact in $C[T,\infty)^2$.

Since, for any sequence $\{(x_n(t),y_n(t))\}$ in $\mathcal{W}$ converging to $(x(t),y(t))$ uniformly on compact subintervals of $[T,\infty)$, it holds that

$$|\mathcal{F}(x_n,y_n)(t) - \mathcal{F}(x,y)(t)| \le \int_t^\infty \big(p_1(s)|x_n(s)^{\alpha_1} - x(s)^{\alpha_1}| + q_1(s)|y(s)_n^{\beta_1} - y(s)^{\beta_1}|\big)\,ds$$

$$|\mathcal{G}(x_n,y_n)(t) - \mathcal{G}(x,y)(t)| \le \int_t^\infty \big(p_2(s)|x_n(s)^{\alpha_2} - x(s)^{\alpha_2}| + q_2(s)|y(s)_n^{\beta_2} - y(s)^{\beta_2}|\big)\,ds$$

for $t \ge T$, applying the Lebesgue dominated convergence theorem to the right-hand sides of the above inequalities, we conclude that $\Phi(x_n,y_n)(t) \to \Phi(x,y)(t)$ uniformly on any compact subinterval of $[T,\infty)$, verifying the continuity of Φ.

Consequently, Φ has a fixed point $(x,y) \in \mathcal{W}$, which satisfies the system of integral equations

$$\begin{aligned} x(t) &= \int_t^\infty \big(p_1(s)x(s)^{\alpha_1} + q_1(s)y(s)^{\beta_1}\big)\,ds\,, \quad t \ge T\,,\\ y(t) &= \int_t^\infty \big(p_2(s)x(s)^{\alpha_2} + q_2(s)y(s)^{\beta_2}\big)\,ds\,, \quad t \ge T\,. \end{aligned} \tag{4.12}$$

It follows therefore that $(x(t),y(t))$ is a type-(II,II) solution of system (A).

It remains to confirm that $(x(t), y(t))$ has the asymptotic behavior (4.6). For this purpose define the functions

$$(4.13)\qquad \begin{aligned} u(t) &= \int_t^\infty \big(p_1(s)\xi(s)^{\alpha_1} + q_1(s)\eta(s)^{\beta_1}\big)\, ds\,, \\ v(t) &= \int_t^\infty \big(p_2(s)\xi(s)^{\alpha_2} + q_2(s)\eta(s)^{\beta_2}\big)\, ds\,, \end{aligned}$$

and apply Lemma 3.3 to the following inferior and superior limits:

$$\liminf_{t\to\infty} \frac{x(t)}{u(t)}\,, \qquad \limsup_{t\to\infty} \frac{x(t)}{u(t)}\,, \qquad \liminf_{t\to\infty} \frac{y(t)}{v(t)}\,, \qquad \limsup_{t\to\infty} \frac{y(t)}{v(t)}\,.$$

Proceeding exactly as in the final part of the proof of Theorem 3.2 and using the relations

$$p_1(t)\xi(t)^{\alpha_1} + q_1(t)\eta(t)^{\beta_1} \sim p_1(t)\xi(t)^{\alpha_1}\,, \quad p_2(t)\xi(t)^{\alpha_2} + q_2(t)\eta(t)^{\beta_2} \sim q_2(t)\eta(t)^{\beta_2}\,,$$

one concludes that

$$\lim_{t\to\infty} \frac{x(t)}{u(t)} = 1 \qquad \text{and} \qquad \lim_{t\to\infty} \frac{y(t)}{v(t)} = 1$$

from which it follows that

$$x(t) \sim u(t) \sim \xi(t) \quad \text{and} \quad y(t) \sim v(t) \sim \eta(t)\,, \qquad t \to \infty\,.$$

The details may be omitted. This completes the proof. □

Remark 4.2. Let us consider the problem in the framework of regular variation. Assume that the functions $p_i(t)$ and $q_i(t)$ are regularly varying functions of indices λ_i and μ_i, i.e., $p_i \in \mathrm{RV}(\lambda_i)$ and $q_i \in \mathrm{RV}(\mu_i)$, $i = 1, 2$. For the definition of regularly varying function see the appendix at the end of the paper.

We note that condition (4.3) implies that $\lambda_1 \le -1$ and $\mu_2 \le -1$, and that the functions $\xi(t)$ and $\eta(t)$ defined by (4.4) are regularly varying:

$$(4.14)\qquad \xi \in \mathrm{RV}\Big(\frac{\lambda_1 + 1}{1 - \alpha_1}\Big)\,, \qquad \eta \in \mathrm{RV}\Big(\frac{\mu_2 + 1}{1 - \beta_2}\Big)\,.$$

For simplicity we use ρ and σ to denote the regularity indices of $\xi(t)$ and $\eta(t)$:

$$(4.15)\qquad \rho = \frac{\lambda_1 + 1}{1 - \alpha_1}\,, \qquad \sigma = \frac{\mu_2 + 1}{1 - \beta_2}\,.$$

Since

$$\frac{q_1(t)\eta(t)^{\beta_1}}{p_1(t)\xi(t)^{\alpha_1}} \in \mathrm{RV}(\mu_1 - \lambda_1 + \beta_1\sigma - \alpha_1\rho) = \mathrm{RV}(\mu_1 + 1 + \beta_1\sigma - \rho)\,,$$

$$\frac{p_2(t)\xi(t)^{\alpha_2}}{q_2(t)\eta(t)^{\beta_2}} \in \mathrm{RV}(\lambda_2 - \mu_2 + \alpha_2\rho - \beta_2\sigma) = \mathrm{RV}(\lambda_2 + 1 + \alpha_2\rho - \sigma)\,,$$

condition (4.5) is satisfied if

$$\mu_1 + 1 + \beta_1\sigma - \rho < 0 \quad \text{and} \quad \lambda_2 + 1 + \alpha_2\rho - \sigma < 0\,.$$

Using above observations we are able to give a criterion for system (A) to have regularly varying solutions of type (II,II).

Corollary 4.3. *Assume that $p_i \in RV(\lambda_i)$ and $q_i \in RV(\mu_i)$, $i = 1, 2$. Suppose that $p_1(t)$ and $q_2(t)$ satisfy condition* (4.3). *Suppose moreover that*

$$\mu_1 + 1 + \beta_1\sigma < \rho \qquad \textit{and} \qquad \lambda_2 + 1 + \alpha_2\rho < \sigma\,. \tag{4.16}$$

Then, system (A) *possesses regularly varying solutions $(x(t), y(t))$ of index (ρ, σ), all of which behave like $x(t) \sim \xi(t)$ and $y(t) \sim \eta(t)$ as $t \to \infty$.*

Remark 4.4. Corollary 4.3 is slightly weaker than Theorem 4.1 in that (4.16) implies (4.5) but not conversely, but has a merit that the criterion (4.16) is easy to check because it depends only on the regularity indices of $p_i(t)$, $q_i(t)$ and the exponents α_i, β_i. If $\lambda_1 = \mu_2 = -1$, for example, then (4.16) reduces to $\mu_1 < -1$ and $\lambda_2 < -1$, which ensures the existence of slowly varying solutions for (A) enjoying one and the same asymptotic behavior $x(t) \sim \xi(t)$ and $y(t) \sim \eta(t)$ as $t \to \infty$.

Example 4.5. Consider the system of differential equations

$$\begin{aligned} x' &= t^{-2}(\log t)^2 x^{\frac{1}{3}} + t^{\gamma}\exp\big(-\sqrt{\log t}\big)y^{\frac{7}{5}}\,,\\ y' &= t^{\delta}\exp\big(\sqrt{\log t}\big)x^{\frac{5}{3}} + t^{-3}(\log t)^4 y^{\frac{1}{5}}\,, \end{aligned} \tag{4.17}$$

which is a special case of (A) with $\alpha_1 = \frac{1}{3}$, $\beta_1 = \frac{7}{5}$, $\alpha_2 = \frac{5}{3}$, $\beta_2 = \frac{1}{5}$, and

$$p_1(t) = t^{-2}(\log t)^2\,, \qquad q_1(t) = t^{\gamma}\exp\big(-\sqrt{\log t}\big)\,,$$

$$p_2(t) = t^{\delta}\exp\big(\sqrt{\log t}\big)\,, \qquad q_2(t) = t^{-3}(\log t)^4\,.$$

Clearly, the above functions are regularly varying of indices $\lambda_1 = 2$, $\mu_1 = \gamma$, $\lambda_2 = \delta$ and $\mu_2 = -3$. A simple calculation yields

$$\Big((1-\alpha_1)\int_t^\infty p_1(s)ds\Big)^{\frac{1}{1-\alpha_1}} = \Big(\frac{2}{3}\Big)^{\frac{3}{2}} t^{-\frac{3}{2}}(\log t)^3 \in \mathrm{RV}\Big(-\frac{3}{2}\Big)\,, \qquad \rho = -\frac{3}{2}\,,$$

and

$$\Big((1-\beta_2)\int_t^\infty q_2(s)\,ds\Big)^{\frac{1}{1-\beta_2}} = \Big(\frac{2}{5}\Big)^{\frac{5}{4}} t^{-\frac{5}{2}}(\log t)^5 \in \mathrm{RV}\Big(-\frac{5}{2}\Big), \qquad \sigma = -\frac{5}{2}\,.$$

It is easily checked that condition (4.16) is satisfied if $\mu_1 = \gamma < 1$ and $\lambda_2\delta < -1$. Therefore, from Corollary 4.3 we conclude that if $\gamma < 1$ and $\delta < -1$, then system (4.17) possesses type-(II,II) regularly varying solutions of index $(-\frac{3}{2}, -\frac{5}{2})$, and any such solution $(x(t), y(t))$ enjoys the unique asymptotic behavior

$$x(t) \sim \Big(\frac{2}{3}\Big)^{\frac{3}{2}} t^{-\frac{3}{2}}(\log t)^3\,, \qquad y(t) \sim \Big(\frac{2}{5}\Big)^{\frac{5}{4}} t^{-\frac{5}{2}}(\log t)^5\,, \qquad t \to \infty\,.$$

4.2. **Perturbations of the cyclic system.**

Recently the present authors [5] considered system (4.2) under the assumption $\alpha_2\beta_1 < 1$ in the framework of regular variation and characterized the existence of its type-(II,II) solutions which are regularly varying of negative indices.

Proposition 4.6. *Assume that*

$$0 < \alpha_2\beta_1 < 1\,, \quad \textit{and} \quad p_2 \in \mathrm{RV}(\lambda_2)\,, \qquad q_1 \in \mathrm{RV}(\mu_1)\,. \tag{4.18}$$

Then, system (4.2) *has regularly varying solutions of negative indices* (ρ, σ) *if and only if*

$$\mu_1 + 1 + \beta_1(\lambda_2 + 1) < 0\,, \qquad \alpha_2(\mu_1 + 1) + \lambda_2 + 1 < 0\,, \tag{4.19}$$

in which case ρ *and* σ *are uniquely determined by*

$$\rho = \frac{\mu_1 + 1 + \beta_1(\lambda_2 + 1)}{1 - \alpha_2\beta_1}\,, \qquad \sigma = \frac{\alpha_2(\mu_1 + 1) + \lambda_2 + 1}{1 - \alpha_2\beta_1}\,, \tag{4.20}$$

and any such solution $(x(t), y(t))$ *enjoys one and the same asymptotic behavior*

$$\begin{aligned} x(t) &\sim \Big[\frac{t^{1+\beta_1} q_1(t) p_2(t)^{\beta_1}}{(-\rho)(-\sigma)^{\beta_1}}\Big]^{\frac{1}{1-\alpha_2\beta_1}}\,, \\ y(t) &\sim \Big[\frac{t^{1+\alpha_2} q_1(t)^{\alpha_2} p_2(t)}{(-\rho)^{\alpha_2}(-\sigma)}\Big]^{\frac{1}{1-\alpha_2\beta_1}}\,, \quad t \to \infty\,. \end{aligned} \tag{4.21}$$

Our purpose here is to utilize this result to find criteria ensuring the existence of regularly varying solutions of negative indices for system (A) with regularly varying coefficients $p_i(t)$ and $q_i(t)$ which can be viewed as a small perturbation of the cyclic system (4.2).

In what follows it is assumed that α_i and β_i, $i = 1, 2$, are positive constants and $p_i \in \mathrm{RV}(\lambda_i)$ and $q_i \in \mathrm{RV}(\mu)$, $i = 1, 2$, and use is made of the functions $X(t)$ and $Y(t)$ defined by

$$X(t) = \Big[\frac{t^{1+\beta_1} q_1(t) p_2(t)^{\beta_1}}{(-\rho)(-\sigma)^{\beta_1}}\Big]^{\frac{1}{1-\alpha_2\beta_1}}\,, \quad Y(t) = \Big[\frac{t^{1+\alpha_2} q_1(t)^{\alpha_2} p_2(t)}{(-\rho)^{\alpha_2}(-\sigma)}\Big]^{\frac{1}{1-\alpha_2\beta_1}}\,. \tag{4.22}$$

It is elementary to check that $X(t) \in \mathrm{RV}(\rho)$ and $Y(t) \in \mathrm{RV}(\sigma)$ satisfy the cyclic system of asymptotic relations

$$\int_t^\infty q_1(s) Y(s)^{\beta_1} ds \sim X(t)\,, \qquad \int_t^\infty p_2(s) X(s)^{\alpha_2}\, ds \sim Y(t), \quad t \to \infty\,. \tag{4.23}$$

We now state and prove the main result of this subsection.

Theorem 4.7. *Suppose that* (4.18) *and* (4.19) *hold. Suppose moreover that*

$$\lambda_1 + 1 < (1 - \alpha_1)\rho\,, \qquad \mu_2 + 1 < (1 - \beta_2)\sigma\,, \tag{4.24}$$

where ρ *and* σ *are given by* (4.20)*. Then, system* (A) *possesses strongly decreasing regularly varying solutions* $(x(t), y(t))$ *of index* (ρ, σ)*, all of which enjoy one and the same asymptotic behavior*

$$x(t) \sim X(t)\,, \quad y(t) \sim Y(t)\,, \qquad t \to \infty\,. \tag{4.25}$$

Proof. Let h, H, k, K denote the constants

$$h = 2^{-\frac{1+\beta_1}{1-\alpha_2\beta_1}}, \quad H = 4^{\frac{1+\beta_1}{1-\alpha_2\beta_1}}\,, \quad k = 2^{-\frac{1+\alpha_2}{1-\alpha_2\beta_1}}, \quad K = 4^{\frac{1+\alpha_2}{1-\alpha_2\beta_1}}\,. \tag{4.26}$$

Consider the functions $p_1(t)X(t)^{\alpha_1}/q_1(t)Y(t)^{\beta_1}$ and $q_2(t)Y(t)^{\beta_2}/p_2(t)X(t)^{\alpha_2}$. Since these are regularly varying functions of indices

$$\lambda_1 - \mu_1 + \alpha_1\rho - \beta_1\sigma = \lambda_1 + 1 - (1 - \alpha_1)\rho$$

and

$$\mu_2 - \lambda_2 + \beta_2\sigma - \alpha_2\rho = \mu_2 + 1 - (1-\beta_2)\sigma\,,$$

both of which are negative by condition (4.24). Therefore, these two functions tend to zero as $t \to \infty$, so that there exists $T > a$ such that

$$\frac{p_1(t)X(t)^{\alpha_1}}{q_1(t)Y(t)^{\beta_1}} \le \frac{k^{\beta_1}}{H^{\alpha_1}} \quad \text{and} \quad \frac{q_2(t)Y(t)^{\beta_2}}{p_2(t)X(t)^{\alpha_2}} \le \frac{h^{\alpha_2}}{K^{\beta_2}} \quad \text{for} \quad t \ge T\,. \tag{4.27}$$

Furthermore, because of (4.23) one can choose $T > a$ so large that in addition to (4.27) the following inequalities are satisfied for $t \ge T$:

$$\begin{aligned} &\frac{1}{2}X(t) \le \int_t^\infty q_1(s)Y(s)^{\beta_1} ds \le 2X(t)\,,\\ &\frac{1}{2}Y(t) \le \int_t^\infty p_2(s)X(s)^{\alpha_2} ds \le 2Y(t)\,. \end{aligned} \tag{4.28}$$

Now we define the set $\mathcal{W}$ by (4.9), the integral operators $\mathcal{F}$ and $\mathcal{G}$ by (4.10), and the mapping $\Phi : \mathcal{W} \to C[T,\infty)^2$ by (4.11). It can be shown that Φ is a continuous self-map on $\mathcal{W}$ and sends $\mathcal{W}$ into a relatively compact subset of $C[T,\infty)^2$.

Let $(x,y) \in \mathcal{W}$. Using (4.8), (4.10), (4.26)–(4.28), we compute:

$$\begin{aligned} \mathcal{F}(x,y)(t) &= \int_t^\infty q_1(s)y(s)^{\beta_1}\Big(1 + \frac{p_1(s)x(s)^{\alpha_1}}{q_1(s)y(s)^{\beta_1}}\Big)\, ds \\ &\le K^{\beta_1}\int_t^\infty q_1(s)Y(s)^{\beta_1}\Big(1 + \frac{H^{\alpha_1}}{k^{\beta_1}}\frac{p_1(s)X(s)^{\alpha_1}}{q_1(s)Y(s)^{\beta_1}}\Big)\, ds \\ &\le 2K^{\beta_1}\int_t^\infty q_1(s)Y(s)^{\beta_1} ds \le 4K^{\beta_1}X(t) = HX(t)\,, \quad t \ge T\,, \end{aligned}$$

and

$$\mathcal{F}(x,y)(t) \ge \int_t^\infty q_1(s)y(s)^{\beta_1}\, ds \ge \frac{1}{2}k^{\beta_1}X(t) = hX(t)\,, \quad t \ge T\,,$$

which implies that $hX(t) \le \mathcal{F}(x,y)(t) \le HX(t)$ for $t \ge T$. Since it can be shown similarly that $kY(t) \le \mathcal{G}(x,y)(t) \le KY(t)$ for $t \ge T$, we conclude that $\Phi(x,y) \in \mathcal{W}$, that is, Φ maps $\mathcal{W}$ into itself. Furthermore, proceeding exactly as in the proof of Theorem 4.1, we can verify the continuity of Φ and the relative compactness of $\Phi(\mathcal{W})$. Therefore, the Schauder-Tychonoff fixed point theorem ensures the existence of $(x,y) \in \mathcal{W}$ such that $(x,y) = \Phi(x,y) = (\mathcal{F}(x,y), \mathcal{G}(x,y))$, which means that $(x(t),y(t))$ satisfies the system of integral equations (4.12) for $t \ge T$, and hence gives a type-(II,II) solution of system (A) on $[t,\infty)$.

To complete the proof we have to prove that $(x(t),y(t))$ is regularly varying of index (ρ,σ). Define $u(t)$ and $v(t)$ by (4.13) by using $X(t)$ and $Y(t)$ given in (4.22). Since

$$\begin{aligned} &p_1(t)X(t)^{\alpha_1} + q_1(t)Y(t)^{\alpha_1} \sim q_1(t)Y(t)^{\beta_1}\,,\\ &p_2(t)X(t)^{\alpha_2} + q_2(t)Y(t)^{\beta_2} \sim p_2(t)X(t)^{\alpha_2}\,, \end{aligned} \tag{4.29}$$

as $t \to \infty$, $u(t)$ and $v(t)$ satisfy

$$
\begin{aligned}
u(t) &\sim \int_t^\infty q_1(s)Y(s)^{\beta_1}ds \sim X(t)\,,\\
v(t) &\sim \int_t^\infty p_2(s)X(s)^{\alpha_2}ds \sim Y(t)\,, \qquad t \to \infty\,.
\end{aligned} \tag{4.30}
$$

Define the finite positive constants l, L, m and M by

$$
\begin{aligned}
l &= \liminf_{t\to\infty} \frac{x(t)}{u(t)}\,, \quad L = \limsup_{t\to\infty} \frac{x(t)}{u(t)}\,,\\
m &= \liminf_{t\to\infty} \frac{y(t)}{v(t)}, \quad M = \limsup_{t\to\infty} \frac{y(t)}{v(t)}.
\end{aligned} \tag{4.31}
$$

We apply Lemma 3.3 to L. Using (4.30) and (4.31), we find that

$$
\begin{aligned}
L &\le \limsup_{t\to\infty} \frac{x'(t)}{u'(t)} = \limsup_{t\to\infty} \frac{p_1(t)x(t)^{\alpha_1} + q_1(t)y(t)^{\beta_1}}{p_1(t)X(t)^{\alpha_1} + q_1(t)Y(t)^{\beta_1}}\\
&= \limsup_{t\to\infty} \frac{q_1(t)y(t)^{\beta_1}}{q_1(t)Y(t)^{\beta_1}} = \Big(\limsup_{t\to\infty} \frac{y(t)}{Y(t)}\Big)^{\beta_1} = \Big(\limsup_{t\to\infty} \frac{y(t)}{v(t)}\Big)^{\beta_1} = M^{\beta_1}\,.
\end{aligned}
$$

Likewise, applying Lemma 3.3 to M, we see that $M \le L^{\alpha_2}$. From these inequalities, we obtain

$$
L \le L^{\alpha_2\beta_1} \quad \text{and} \quad M \le M^{\alpha_2\beta_1} \quad \Longrightarrow \quad L \le 1 \quad \text{and} \quad M \le 1.
$$

On the other hand, from Lemma 3.3 applied to l and m it follows that $l \ge 1$ and $m \ge 1$, which, combined with the above, leads to the conclusion that

$$
l = L = m = M = 1 \quad \Longrightarrow \quad \lim_{t\to\infty} \frac{x(t)}{u(t)} = \lim_{t\to\infty} \frac{y(t)}{v(t)} = 1\,.
$$

It follows that $x(t) \sim u(t) \sim X(t)$ and $y(t) \sim v(t) \sim Y(t)$ as $t \to \infty$. This completes the proof. □

Remark 4.8. In Theorem 4.1 it is essential that $\alpha_2\beta_1 < 1$, that is, the principal (cyclic) part of (A) must be sublinear. However, the exponents α_1 and β_2 may be greater than 1, in which case (A) involves the superlinear terms $p_1(t)x^{\alpha_1}$ and $q_2(t)y^{\beta_2}$. Notice that Theorem 4.7 deals only with regularly varying solutions of negative indices. It would be of interest to prove a variant of Theorem 4.7 which ensures the existence of strongly decreasing slowly varying solutions for system (A).

Related results on the existence of positive solutions with specific asymptotic behavior for a class of nonlinear differential systems which includes system (4.2) and which could be combined with perturbation techniques to produce some new results for system (A) can be found in [3].

Example 4.9. The system of differential equations

$$\begin{cases} x' + f(t)x^{\alpha} + 2t^{\beta-3}\exp\big(\exp(\beta+1)\sqrt{\log t}\big)y^{\beta} = 0\,, \\ \\ y' + t^{2(\gamma-1)}\exp\big(-(\gamma+1)\sqrt{\log t}\big)x^{\gamma} + g(t)y^{\delta} = 0\,, \end{cases}$$

is under consideration, where α, β, γ and δ are positive constants, and $f(t)$ and $g(t)$ are continuous regularly varying functions on $[a,\infty)$, $a>1$. This system is a special case of (A) with $\alpha_1=\alpha$, $\beta_1=\beta$, $\alpha_2=\gamma$, $\beta_2=\delta$, and

$$p_1(t) = f(t)\,, \quad q_1(t) = 2t^{\beta-3}\exp\big(\exp(\beta+1)\sqrt{\log t}\big)\,,$$

$$p_2(t) = t^{2(\gamma-1)}\exp\big(-(\gamma+1)\sqrt{\log t}\big)\,, \quad q_2(t) = g(t)\,.$$

We assume that $\alpha_2\beta_1=\beta\gamma<1$ and that $f\in\mathrm{RV}(\lambda)$ and $g\in\mathrm{RV}(\mu)$. As is easily seen,

$$\mu_1+1+\beta_1(\lambda_2+1) = -2(1-\beta\gamma)<0\,, \quad \alpha_2(\mu_1+1)+\lambda_2+1 = -(1-\beta\gamma)<0\,,$$

which means that (4.19) holds true and the constants ρ and σ defined by (4.20) reduce to $\rho=-2$ and $\sigma=-1$. Moreover, the functions $X(t)$ and $Y(t)$ defined by (4.22) are shown to satisfy the relations

$$X(t)\sim t^{-2}\exp\big(\sqrt{\log t}\big)\,, \qquad Y(t)\sim t^{-1}\exp\big(-\sqrt{\log t}\big), \quad t\to\infty\,.$$

Finally note that condition (4.24) amounts to requiring that $\lambda_1=\lambda$ and $\mu_2=\mu$ satisfy

$$\lambda<2\alpha-3, \qquad \mu<\beta-2\,.$$

Taking above remarks into account and applying Theorem 4.7, we conclude that if $\beta\gamma<1$, then for any regularly varying functions $f\in\mathrm{RV}(\lambda)$ with $\lambda<2\alpha-3$ and $g\in\mathrm{RV}(\mu)$ with $\mu<\beta-2$ the above system possesses type-(II,II) regularly varying solutions $(x(t),y(t))$ of index $(-2,-1)$, all of which enjoy the unique asymptotic behavior

$$x(t)\sim t^{-2}\exp\big(\sqrt{\log t}\big)\,, \qquad y(t)\sim t^{-1}\exp\big(-\sqrt{\log t}\big)\,, \quad t\to\infty\,.$$

We conclude with the remark that qualitative theory of diagonal systems of the form (A) with emphasis on oscillation properties has been developed in depth by Mirzov [15].

Appendix. Regularly varying functions

For the reader's convenience we summarize here the definition and some basic properties of regularly varying functions (in the sense of Karamata) which are used in establishing the precise asymptotic behavior of type-(II,II) solutions for system (A) in Sections 4.

Definition A.1. A measurable function $f\colon[0,\infty)\to(0,\infty)$ is called *regularly varying of index* $\rho\in\mathbf{R}$ if

$$\lim_{t\to\infty}\frac{f(\lambda t)}{f(t)} = \lambda^{\rho} \quad \text{for all} \quad \lambda>0\,.$$

The totality of regularly varying functions of index ρ is denoted by $\mathrm{RV}(\rho)$. We often use the symbol SV to denote $\mathrm{RV}(0)$ and call members of SV *slowly varying functions.* Any function $f(t) \in \mathrm{RV}(\rho)$ is expressed as $f(t) = t^\rho g(t)$ with $g(t) \in \mathrm{SV}$, and so the class SV of slowly varying functions is of fundamental importance in the theory of regular variation. One of the most important properties of regularly varying functions is the following *representation theorem.*

Proposition A.1. *$f(t) \in \mathrm{RV}(\rho)$ if and only if $f(t)$ is represented in the form*

$$f(t) = c(t) \exp\Big\{ \int_{t_0}^{t} \frac{\delta(s)}{s} ds \Big\}, \quad t \geq t_0 ,$$

for some $t_0 > 0$ and for some measurable functions $c(t)$ and $\delta(t)$ such that

$$\lim_{t\to\infty} c(t) = c_0 \in (0, \infty) \quad \text{and} \quad \lim_{t\to\infty} \delta(t) = \rho .$$

If in particular $c(t) \equiv c_0$ for $t \geq t_0$, then $f(t)$ is referred to as a *normalized* regularly varying function of index ρ.

Typical examples of slowly varying functions are: all functions tending to some positive constants as $t \to \infty$,

$$\prod_{n=1}^{N} (\log_n t)^{\alpha_n}, \quad \alpha_n \in \mathbf{R}, \qquad \text{and} \qquad \exp\Big\{ \prod_{n=1}^{N} (\log_n t)^{\beta_n} \Big\}, \quad \beta_n \in (0,1),$$

where $\log_n t$ denotes the n-th iteration of the logarithm. It is known that the function

$$L(t) = \exp\big\{ (\log t)^\theta \cos\, (\log t)^\theta \big\}, \qquad \theta \in \Big(0, \frac{1}{2}\Big),$$

is a slowly varying function which is oscillating in the sense that

$$\limsup_{t\to\infty} L(t) = \infty \quad \text{and} \quad \liminf_{t\to\infty} L(t) = 0 .$$

The following result illustrates operations which preserve slow variation.

Proposition A.2. *Let $L(t)$, $L_1(t)$, $L_2(t)$ be slowly varying. Then, $L(t)^\alpha$ for any $\alpha \in \mathbf{R}$, $L_1(t) + L_2(t)$, $L_1(t)L_2(t)$ and $L_1(L_2(t))$ (if $L_2(t) \to \infty$) are slowly varying.*

A slowly varying function may grow to infinity or decay to 0 as $t \to \infty$. But its order of growth or decay is severely limited as is shown in the following

Proposition A.3. *Let $f(t) \in \mathrm{SV}$. Then, for any $\varepsilon > 0$,*

$$\lim_{t\to\infty} t^\varepsilon f(t) = \infty , \qquad \lim_{t\to\infty} t^{-\varepsilon} f(t) = 0 .$$

A simple criterion for determining the regularity of differentiable positive functions follows.

Proposition A.4. *A differentiable positive function $f(t)$ is a normalized regularly varying function of index ρ if and only if*

$$\lim_{t\to\infty} t \frac{f'(t)}{f(t)} = \rho .$$

The following result called Karamata's integration theorem is of highest importance in handling slowly and regularly varying functions analytically.

Proposition A.5. *Let* $L(t) \in \mathrm{SV}$. *Then,*

(i) *if* $\alpha > -1$,

$$\int_a^t s^\alpha L(s)\, ds \sim \frac{1}{\alpha+1} t^{\alpha+1} L(t)\,, \quad t \to \infty\,;$$

(ii) *if* $\alpha < -1$,

$$\int_t^\infty s^\alpha L(s)\, ds \sim -\frac{1}{\alpha+1} t^{\alpha+1} L(t)\,, \quad t \to \infty\,;$$

(iii) *if* $\alpha = -1$,

$$l(t) = \int_a^t \frac{L(s)}{s}\, ds \in \mathrm{SV} \quad \text{and} \quad m(t) = \int_t^\infty \frac{L(s)}{s}\, ds \in \mathrm{SV}\,,$$

provided $L(t)/t$ *is integrable near the infinity in the latter case.*

A vector function $(x(t), y(t))$ is said to be *regularly varying of index* (ρ, σ) if $x(t)$ and $y(t)$ are regularly varying of indices ρ and σ. If $\rho < 0$ and $\sigma < 0$, then $(x(t), y(t))$ is called regularly varying of negative indices (ρ, σ).

For the most complete exposition of theory of regular variation and its applications we refer to the book of Bingham, Goldie and Teugels [1]. See also Seneta [16]. A comprehensive survey of results up to the year 2000 on the asymptotic analysis of second order ordinary differential equations by means of regular variation can be found in the monograph of Marić [14]. Since the publication of [14] there has been an increasing interest in the analysis of ordinary differential equations by means of regularly varying functions, and thus theory of regular variation has proved to be a powerful tool of determining the accurate asymptotic behavior of positive solutions for a variety of nonlinear differential equations of Emden-Fowler and Thomas-Fermi types. See, for example, the papers [5]–[13].

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