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k-DIRAC OPERATOR AND THE CARTAN-KÄHLER THEOREM

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Abstract. We apply the Cartan-Kähler theorem for the k-Dirac operator studied in Clifford analysis and to the parabolic version of this operator. We show that for $k = 2$ the tableaux of the first prolongations of these two operators are involutive. This gives us a new characterization of the set of initial conditions for the 2-Dirac operator.

1. Introduction

1.1. **k-Dirac operator.** Let g be the Euclidean product on $\mathbb{R}^n$ with an orthonormal basis $\{\varepsilon_1, \dots, \varepsilon_n\}$. We denote by $\mathbb{R}_n$ the Clifford algebra for $(\mathbb{R}^n, g)$ with the defining relation $\varepsilon_\alpha\varepsilon_\beta + \varepsilon_\beta\varepsilon_\alpha = -2g_{\alpha\beta}$ and by $M(n,k,\mathbb{R})$ the affine space of matrices of size $n \times k$. We assume throughout this paper that $k \geq 2$ and $n \geq 3$. For $i = 1, \dots, k$ set

$$\partial_i\psi = \sum_{\alpha=1}^{n} \varepsilon_\alpha \cdot \partial_{\alpha i}\psi \tag{1}$$

where ψ is a smooth $\mathbb{R}_n$-valued function on $M(n,k,\mathbb{R})$. Here $\partial_{\alpha i}$ are the coordinate vector fields on $M(n,k,\mathbb{R})$. We call the operator $\partial = (\partial_1, \dots, \partial_k)$ the *k-Dirac operator in the Euclidean setting* or just the *k-Dirac operator*. The k-Dirac operator is an overdetermined, constant coefficient system of PDEs. A solution of $\partial\psi = 0$ is called a *monogenic function* or a *monogenic spinor in the Euclidean setting*. More to this operator can be found for example in [3] and [7].

In this paper we will show that the tableau associated to the first prolongation of the 2-Dirac operator is involutive. This is Theorem 1. This gives new characterization of the set of initial conditions. See Theorem 2.

We will use representation theory of symmetry group $\mathrm{SL}(k,\mathbb{R}) \times \mathrm{Spin}(n)$ of the operator ∂, see [10]. We denote this group by $\tilde{\mathrm{G}}_0^{ss}$. The notation for the symmetry group will be explained later. The group $\tilde{\mathrm{G}}_0^{ss}$ is a semi-simple Lie group with Lie algebra $\mathfrak{sl}(k,\mathbb{R}) \oplus \mathfrak{so}(n)$. We will work with complex representations of $\tilde{\mathrm{G}}_0^{ss}$ and its Lie algebra. In particular, we consider the complex spinor representations of $\mathfrak{so}(n)$ rather then the real Clifford module $\mathbb{R}_n$. For n odd, there is only one spinor module $\mathbb{S}$. If n is even there are two non-isomorphic spinor modules $\mathbb{S}_+$ and $\mathbb{S}_-$. In

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this case we set $\mathbb{S} := \mathbb{S}_+ \oplus \mathbb{S}_-$. We denote by s the dimension of the corresponding spinor module $\mathbb{S}$. Thus $s = 2^m$ if $n = 2m + 1$ or $n = 2m$.

1.2. **Parabolic k-Dirac operator.** The *parabolic k-Dirac operator D* is an invariant first order operator which lives in the world of parabolic geometries. For the purpose of this paper we will give a coordinate definition of this operator. Put $\mathcal{U} = M(n, k, \mathbb{R}) \times A(k, \mathbb{R})$ where $M(, n, k, \mathbb{R})$ is the affine space of matrices of size $n \times k$ while $A(k, \mathbb{R})$ is the affine space of skew-symmetric matrices of size k. We write coordinates as $(x_{\alpha i}, y_{rs})$ where $x_{\alpha i}$, resp. y_{rs} are coordinates on $M(n, k, \mathbb{R})$, resp. on $A(k, \mathbb{R})$. We write $\partial_{\alpha i} = \partial_{x_{\alpha i}}, \partial_{rs} = \partial_{y_{rs}}$. We use the convention that $\partial_{rs} = -\partial_{sr}$. The set $\mathcal{U}$ is isomorphic to an open affine subset of the Grassmannian of isotropic k-planes in $\mathbb{R}^{k,n+k}$. The Grassmannian is a flat model for a particular type of parabolic geometries. For $k = 2$ this geometry is refered to as Lie contact structures, see [2].

For $\alpha = 1, \dots, n$ and $i = 1, \dots, k$ put $L_{\alpha i} := \partial_{\alpha i} - \frac{1}{2} x_{\alpha j} \partial_{ji}$ where we sum over the repeated index. We will call these vector fields *left invariant vector fields*. Lie bracket is

$$[L_{\alpha i}, L_{\beta j}] = g_{\alpha\beta} \partial_{ij} \,. \tag{2}$$

These vector fields span a non-integrable distribution on $\mathcal{U}$ which is the essence of the parabolic geometry. The (graded) tangent bundle of the Grassmannian variety has a natural reduction of the structure group to $\mathrm{G}_0 := \mathrm{GL}(k, \mathbb{R}) \times \mathrm{SO}(n)$. We get a principal G_0-bundle over $\mathcal{U}$ which we denote by $\mathcal{G}_0$. This is the notation commonly used in [2]. We may (uniquely) lift the trivial bundle $\mathcal{G}_0$ to a principal $\tilde{\mathrm{G}}_0 := \mathrm{GL}(k, \mathbb{R}) \times \mathrm{Spin}(n)$-bundle $\tilde{\mathcal{G}}_0$. As $\tilde{\mathrm{G}}_0$ is a $2:1$-covering of G_0 the natural projection $\tilde{\mathcal{G}}_0 \to \mathcal{G}_0$ is a $2:1$-covering. The groups $\mathrm{G}_0, \tilde{\mathrm{G}}_0$ are reductive groups whose semi-simple parts are isomorphic to $\mathrm{G}_0^{ss}, \tilde{\mathrm{G}}_0^{ss}$ respectively. We extend the action of $\tilde{\mathrm{G}}_0^{ss}$ on $\mathbb{S}$ to the action of $\tilde{\mathrm{G}}_0$ by the choice of a generalised conformal weight, i.e. we specify the action of the center of $\mathrm{GL}(k, \mathbb{R})$, as in [9]. By associating $\mathbb{S}$ to the bundle $\tilde{\mathcal{G}}_0$ we get the spinor bundle $\mathcal{S} := \tilde{\mathcal{G}}_0 \times_{\tilde{\mathrm{G}}_0} \mathbb{S}$ over $\mathcal{U}$.

The set of vector fields $\{L_{\alpha i}, \partial_{rs} | \alpha = 1, \dots, n; i = 1, \dots, k; 1 \le r < s \le k\}$ defines a section of $\mathcal{G}_0$ and we lift it to a section of $\tilde{\mathcal{G}}_0$. This trivializes the spinor bundle $\mathcal{S}$ over $\mathcal{U}$ and thus a section $\psi \in \Gamma(\mathcal{S})$ becomes a spinor valued function over $\mathcal{U}$. With the choice of the very flat Weyl connection we can write in this trivialization $D\psi = (D_1\psi, \dots, D_k\psi)$ where

$$D_i \psi = \sum_{\alpha=1}^{n} \varepsilon_\alpha \cdot L_{\alpha i} \psi \,. \tag{3}$$

Comparing this to (1) we see that we have just replaced each $\partial_{\alpha i}$ by the corresponding left invariant vector field $L_{\alpha i}$. A solution of $D\psi = 0$ is called a (*parabolic*) *monogenic spinor*.

There is a strong and very interesting link which leads from the operator (3) to (1). First of all, a parabolic monogenic spinor ψ which does not depend on y-coordinates can be naturally viewed as a solution of $\partial\psi = 0$. A bit of work shows that there is a (unique) locally exact sequence of invariant operators starting with the operator D. We can do the same move for the whole sequence as we did with the monogenic spinors, i.e. we work only with the real analytic sections which in the preferred trivialization do not depend on y-coordinates on which we act by the invariant operators. Then we get a new sequence of operators which is still locally exact and thus descends to a resolution of ∂. This can be found in [8].

1.3. **Motivation and summary of results.** Motivation for this paper is hidden in the set of initial conditions for these two systems of PDEs. It is not hard to see that any monogenic spinor (in the Euclidean setting) ψ is uniquely determined by its restriction to the set $M(n-1,k,\mathbb{R}) \cong \{x_{11} = x_{12} = \ldots = x_{1k} = 0\}$. Moreover on this set the restriction $\psi|_{M(n-1,k,\mathbb{R})}$ has to satisfy for each $i,j = 1,\ldots,k$:

$$[\tilde{\partial}_i, \tilde{\partial}_j]\psi|_{M(n-1,k,\mathbb{R})} = 0 \tag{4}$$

where $\tilde{\partial}_i = \sum_{\alpha=2}^{n} \varepsilon_\alpha \partial_{\alpha i}$. This is a consequence of the fact that the coordinate vector fields commute. On the other hand given a real analytic spinor valued function φ on $M(n-1,k,\mathbb{R})$ converging on open neighbouhood of $x \in M(n-1,k,\mathbb{R})$ and which satisfies (4) then there is a unique monogenic spinor on $M(n,k,\mathbb{R})$ converging on some open neighbouhood of x whose restriction to $M(n-1,k,\mathbb{R})$ coincides with φ.

Conjecture 1. Given arbitrary real analytic spinor ψ in $x_{\alpha i}$-variables with $\alpha \geq 2$ converging on some open subset $\mathcal{V}$ of $M(n-1,k,\mathbb{R})$ there is a unique (parabolic) monogenic spinor Ψ, i.e. $D\Psi = 0$, converging on some open neighbouhood of $\mathcal{V}$ in $\mathcal{U}$ whose restriction to the set $M(n-1,k,\mathbb{R}) \cong \{x_{11} = \cdots = x_{1k} = y_{12} \cdots = y_{k-1,k} = 0\}$ coincides ψ.

If Conjecture 1 is true then the system (3) will have a nicer set of initial conditions then (1). Starting with the k-Dirac operator in the Euclidean setting and looking for a new system of PDEs such that any quadratic real analytic spinor on $M(n-1,k,\mathbb{R})$ extends to a unique solution of the new system then one can derive the Lie bracket (2) and the right dimension of the set $\mathcal{U}$. This is already a link from the operator (1) to the operator (3). The only question is how good this link is?

I hoped this result would follow from the Cartan-Kähler theorem. Unfortunatelly it does not. Nevertheless the Cartan-Kähler theorem gives us some other interesting results. In this paper we show that both systems are involutive after the first prolongation if $k = 2$. These are Theorems 1 and 3. For $k \geq 3$ this is no longer true and one has to continue on prolongating. I do not know when the involutivity is attained. A closer look on the proof of involutivity for the parabolic 2-Dirac operator also explains why the Cartan-Kähler theorem does not give Conjecture 1. We will comment more on this in Remark 1 at the very end of the paper.

In the next section we cover basic machinery and terminology needed for the Cartan-Kähler theorem. This short summary is taken mostly from [5]. For more on the Cartan-Kähler theorem and exterior differential systems see [1].

We will simply refer to the k-Dirac operator if it is clear from the context whether we mean D or ∂. Similarly we say just a monogenic spinor ψ whether it is clear if we mean $\partial\psi = 0$ or $D\psi = 0$.

2. Exterior differential systems

We assume throughout the paper that all structures are real analytic. Then we can apply the machinery of the Cartan-Kähler theorem.

Let M be a manifold. An *exterior differential system* (EDS) on M is a graded differential ideal $\mathcal{I} \subset \Omega^*(M)$. Recall that $\Omega^*(M)$ is naturally graded by the degree of differential forms. A graded differential ideal is a graded ideal closed under the de Rham differential. We denote the k-th homogeneous piece by $\mathcal{I}^k$. We are interested in *integral manifolds* of $\mathcal{I}$. These are submanifolds $i : N \hookrightarrow M$ such that $i^*\alpha = 0$ for any $\alpha \in \mathcal{I}$. Many interesting problems can be formulated in the language of EDSs and integral manifolds. For a fixed $x \in M$, the set of *integral elements* for $\mathcal{I}$ of rank k at x is the set $\{E \subset T_xM : \dim(E) = k, \forall\, \alpha \in \mathcal{I}^k : \alpha|_E = 0\}$.

We will be interested in EDSs with independence condition. An independence condition for $\mathcal{I}$ is given by a set of 1-forms $\{\omega^1, \dots, \omega^n\}$. We consider only those integral manifolds of $\mathcal{I}$ for which $i^*(\omega^1 \wedge \ldots \wedge \omega^n)$ is a non-vanishing form on N.

We will be interested only in the differential ideals with an independence condition which are (locally) generated as differential ideals by a set of linearly independent 1-forms. We will call such a differential ideal a *Pfaffian system.* Let $\mathcal{I}$ be a Pfaffian system with a generating set of linearly independent 1-forms $\{\theta^1, \dots, \theta^s\}$. We denote by I the subbundle of T^*M spanned by these forms. We may assume that $\{\omega^1, \dots, \omega^n, \theta^1, \dots, \theta^s\}$ is a set of everywhere linearly independent 1-forms on M. This set spans a subbundle of T^*M which we denote by J. So I is a subbundle of J. Let $\{\pi^1, \dots, \pi^t\}$ be a set of 1-forms such that the set $\{\omega^i, \theta^j, \pi^\varepsilon\}$ with $i = 1, \dots, n, j = 1, \dots, s, \varepsilon = 1 \dots, r$ gives a basis of T^*_xM for each $x \in M$. Let us now fix $x \in M$. We put $\mathrm{V}^* := (J/I)_x, \mathrm{W}^* := I_x$. We write $v^i = \omega^i_x, w^j = \theta^j_x$ and denote the dual elements by v_i and w_j. We call the Pfaffian system $\mathcal{I}$ a *linear Pfaffian system* if

$$d\theta^a = A^a_{\varepsilon i}\pi^\varepsilon \wedge \omega^i + T^a_{ij}\omega^i \wedge \omega^j \tag{5}$$

holds for some functions $A^a_{\varepsilon b}, T^a_{ij}$ modulo the algebraic ideal in $\Omega^*(M)$ generated by I. The *tableau* at the point x is equal to $A_x := \langle A^a_{\varepsilon i} v^i \otimes w_a \subset \mathrm{V}^* \otimes \mathrm{W} | 1 \leq \varepsilon \leq r\rangle$. Here $\langle\ \rangle$ denotes the linear span. We drop the subscript x and write A instead of A_x.

Let $\delta\colon \mathrm{V}^* \otimes \mathrm{V}^* \otimes \mathrm{W} \to \Lambda^2\mathrm{V}^* \otimes \mathrm{W}$ be the natural projection. Set $H^{0,2}(A) := \Lambda^2\mathrm{V}^* \otimes \mathrm{W}/\delta(\mathrm{V}^* \otimes A)$. The *torsion* of (I, J) at x is defined as the class $[T]_x := [T^a_{ij}(x)v^i \wedge v^j \otimes w_a] \in H^{0,2}(A)$. If $[T_x] = 0$ we say that the torsion is *absorable* at x. This means that we can replace π^ε by new forms $\pi'^\epsilon, \epsilon = 1, \dots, t$ such that $\{\omega^i, \theta^j, \pi'^\varepsilon\}$ is still a basis of T^*_xM for each $x \in M$ and $T'^a_{ij} = 0$ for all a, i, j at the point x with respect to the new basis. Later on we will need the following lemma.

Lemma 1. *For a linear Pfaffian system on M with an independence condition, the necessary and sufficient condition for vanishing of the torsion $[T_x]$ at a point $x \in M$ is that there is an integral element over $x \in M$ satisfying the independence condition.*

Proof. See Proposition 5.14. from [1]. $\square$

If follows from Lemma 1 that the torsion is a primary obstruction for existence of integral manifolds. Suppose that $[T] = 0$ on an open neighbouhood of x. Put $A_k := A \cap \langle v^{k+1}, \ldots, v^n \rangle \otimes \mathrm{W}$ for $k = 1, \ldots, n$. Put $A^{(1)} := S^2\mathrm{V}^* \otimes \mathrm{W} \cap \mathrm{V}^* \otimes A$. We call $A^{(1)}$ the (*first*) *prolongation of the tableau* A. Then we have the inequality

$$\dim(A^{(1)}) \leq \dim(A) + \dim(A_1) + \cdots + \dim(A_{n-1})\,. \tag{6}$$

We say that the tableau is *involutive* if the equality holds for some choice of a basis of V^*. It is convenient to introduce the *Cartan characters* $s_1, \ldots, s_n$ of the tableau by requiring that $\dim(A) - \dim(A_k) = s_1 + \cdots + s_k$ holds for each $k = 1, \ldots, n$. Then the inequality (6) becomes

$$\dim(A^{(1)}) \leq s_1 + 2s_2 + \cdots + ns_n\,. \tag{7}$$

If the tableau is involutive then the Cartan-Kähler theorem applies. The Cartan-Kähler theorem guarantees existence of n-dimensional integral manifolds passing through x satisfying the independence condition. Moreover we can read from the Cartan characters "how many" such local manifolds there are.

If the tableau is not involutive one has to start over on the pullback of the canonical system on the Grassmann bundle to the space of integral elements. In calculation this means that we add elements from $A^{(1)}$ as new variables and add new forms $\theta^a_i := A^a_{\varepsilon i}\pi^\varepsilon - p^a_{ij}\omega^j$ where $p^a_{ij} v^i \otimes v^j \otimes w_a \in A^{(1)}$ to the ideal I.

EDSs naturally arise with PDEs. Suppose that we are given a system of PDEs of order k. Then we take M to be the space of k-jets of solutions of the PDE and we pull back the canonical system which lives on the space of jets of vector valued functions.

Computation simplifies in the case of a constant coefficient system. The torsion vanishes and in the case of a linear, constant coefficient, homogeneous system of PDEs the tableau A is at any point isomorphic to the space of linear solutions of this system. The first prolongation of the tableau is naturally isomorphic to the space of quadratic solutions. Set $A^{(0)} := A$ and inductively for $j = 1, 2, \ldots$ put $A^{(j)} := S^j\mathrm{V}^* \otimes \mathrm{W} \cap \mathrm{V}^* \otimes A^{(j-1)}$. Then $A^{(j)}$ is naturally isomorphic the to space of homogeneous solutions of the system of homogeneity $j+1$ and $A^{(j+1)} \cong (A^{(j)})^{(1)}$. For more see [5].

3. k-Dirac operator (in the Euclidean setting) and the Cartan-Kähler theorem

For this paper we will need to understand the space of linear, quadratic and cubic monogenic spinors. Recall that $\mathbb{S}$ is the complex spinor representation of $\mathfrak{so}(n)$ defined in Section 1. There is an isomorphism $M(k, n, \mathbb{R}) \otimes_{\mathbb{R}} \mathbb{S} \cong M(k, n, \mathbb{C}) \otimes_{\mathbb{C}} \mathbb{S}$. We will work with complex representations of Lie algebra of $\tilde{\mathrm{G}}_0^{ss}$ and take tensor product over complex numbers. We will denote the Cartan product by $\boxtimes$. This is the irreducible subspace with the highest weight in the tensor product of irreducible representations.

Let E, resp. F be the defining representation of $\mathfrak{sl}(k, \mathbb{C})$, resp. of $\mathfrak{so}(n, \mathbb{C})$. We choose a basis $\{e_1, \ldots, e_k\}$ of E. We denote by $\{\varepsilon_1, \ldots, \varepsilon_n\}$ an orhonormal basis of F. Let g be the $\mathrm{SO}(n, \mathbb{C})$-invariant scalar product on F. If $n = 2m$ is even we denote by $\{v_1, \dots, v_m, w_1, \dots, w_m\}$ a null basis of F such that $g(v_i, w_j) = \delta_{ij}$. If $n = 2m+1$ is odd we denote by $\{v_1, \dots, v_m, w_1, \dots, w_m, u\}$ a basis such that the relations on v_i, w_j are same as for $n = 2m$ and $g(u,u) = 1, g(u, w_i) = g(u, v_j) = 0$.

In the case of the k-Dirac operator we have that $(\mathrm{V}_x)_{\mathbb{C}} := (\mathrm{V}_x^*) \otimes_{\mathbb{R}} \mathbb{C} \cong \mathrm{E} \otimes \mathrm{F}$ and $\mathrm{W}_x \cong \mathbb{S}$ for each $x \in M(n,k,\mathbb{R})$. The tableau is isomorphic to $A_x \cong \mathrm{E} \otimes \mathbb{T}$ where $\mathbb{T} \cong \mathrm{F} \boxtimes \mathbb{S}$ is the Cartan component. We will drop the subscript x. We call $\mathbb{T}$ the *twistor representation* of $\mathfrak{so}(n)$. The subspace $\mathbb{T}$ is invariantly defined as the kernel of the canonical projection $\pi : \mathrm{F} \otimes \mathbb{S} \to \mathbb{S}$ which is on decomposable elements given by the Clifford multiplication $\pi(\varepsilon \otimes s) = \varepsilon \cdot s$. By induction on i we get that $A^{(i)}$ is the intersection of $S^{i+1}(\mathrm{E} \otimes \mathrm{F}) \otimes \mathbb{S}$ with the kernel of the projection

$$\mathrm{E}^{\otimes^{i+1}} \otimes \mathrm{F}^{\otimes^{i+1}} \otimes \mathbb{S} \xrightarrow{Id_{\mathrm{M}} \otimes \pi} \mathrm{E}^{\otimes^{i+1}} \otimes \mathrm{F}^{\otimes^{i}} \otimes \mathbb{S} \tag{8}$$

where $\mathrm{M} = \mathrm{E}^{\otimes^{i+1}} \otimes \mathrm{F}^{\otimes^{i}}$.

3.1. The space of polynomials on $M(n,k,\mathbb{C})$ as a $\mathbf{GL}(k,\mathbb{C}) \times \mathbf{GL}(n,\mathbb{C})$-module.

Let us consider the action of $\mathrm{GL}(k,\mathbb{C}) \times \mathrm{GL}(n,\mathbb{C})$ on the space of polynomials on $M(n,k,\mathbb{C})$ given by $((g,h) \cdot f)(x) = f(hxg^T)$ where $g \in \mathrm{GL}(k,\mathbb{C}), h \in \mathrm{GL}(n,\mathbb{C}), x \in M(n,k,\mathbb{C})$ and f is a polynomial on $M(n,k,\mathbb{C})$. The space of linear polynomials is isomorphic to $\mathrm{E}' \otimes \mathrm{F}'$ where E', F' is the defining representation of $\mathrm{GL}(k,\mathbb{C})$, resp. of $\mathrm{GL}(n,\mathbb{C})$. The space of quadratic polynomials is then isomorphic to $S^2(\mathrm{E}' \otimes \mathrm{F}') \cong S^2\mathrm{E}' \otimes S^2\mathrm{F}' \oplus \Lambda^2\mathrm{E}' \otimes \Lambda^2\mathrm{F}'$. The set $\{e_i \odot e_j \otimes \varepsilon_\alpha \odot \varepsilon_\beta, e_i \wedge e_j \otimes \varepsilon_\alpha \wedge \varepsilon_\beta | i,j = 1,\dots,k; \alpha,\beta = 1,\dots,n\}$ is a basis of $S^2(\mathrm{E}' \otimes \mathrm{F}')$. Here we are using the bases introduced in the previous paragraph. The corresponding polynomials are $x_{\alpha i}x_{\beta j} + x_{\beta i}x_{\alpha j} \in S^2\mathrm{E}' \otimes S^2\mathrm{F}'$, resp. $x_{\alpha i}x_{\beta j} - x_{\beta i}x_{\alpha j} \in \Lambda^2\mathrm{E}' \otimes \Lambda^2\mathrm{F}'$.

With respect to the usual choices on Lie algebra of the semi-simple part of $\mathrm{GL}(k,\mathbb{C}) \times \mathrm{GL}(n,\mathbb{C})$, i.e. the Cartan subalgebra consists of diagonal matrices and positive roots span the strictly upper triangular matrices, the polynomials $x_{11}, x_{11}x_{22} - x_{12}x_{21}$ are highest weight vectors of $\mathrm{E}' \otimes \mathrm{F}'$, resp. of $\Lambda^2\mathrm{E}' \otimes \Lambda^2\mathrm{F}'$. If $k = 2$ then Theorem 5.2.7 on $\mathrm{GL}(k,\mathbb{C}) \times \mathrm{GL}(n,\mathbb{C})$-duality from [4] shows that any highest weight polynomial of an $\mathrm{GL}(k,\mathbb{C}) \times \mathrm{GL}(n,\mathbb{C})$-irreducible subspace is up to a scalar multiple a product of x_{11} and $x_{11}x_{22} - x_{12}x_{21}$.

3.2. Non-involutivity of the tableau of the k-Dirac operator.

Recall that we have denoted by $\{\varepsilon_1, \dots, \varepsilon_n\}$ an orthonormal basis of $\mathbb{C}^n$. We consider it also as an orthonormal basis of $\mathbb{R}^n$ and $\{e_1, \dots, e_k\}$ as a basis of $\mathbb{R}^k$. Then $\{e_i \otimes \varepsilon_\alpha\}$ is a basis of V^*. Let us order it by $\{e_1 \otimes \varepsilon_1, \dots, e_1 \otimes \varepsilon_k, e_2 \otimes \varepsilon_1, \dots, e_n \otimes \varepsilon_k\}$. Let $\{s^\mu | \mu = 1, \dots, s\}$ be a basis of $\mathbb{S}$. Then $\{\varepsilon_\alpha \otimes s^\mu + \varepsilon_n \otimes \varepsilon_n.\varepsilon_\alpha.s^\mu | \alpha < n\}$ is a basis of $\mathbb{T}$. In particular $\dim(\mathbb{T}) = s(n-1)$. The Cartan characters with respect to the ordered basis of V^* are equal to $s_1 = \ldots = s_{k(n-1)} = s, s_{k(n-1)+1} = \ldots = s_{nk} = 0$. It is clear that these characters minimaze the right hand side of (7). So the right hand side in (7) is equal to $s\binom{k(n-1)+1}{2}$.

We now have to compute the dimension $\dim(A^{(1)})$. We know that $A^{(1)}$ is naturally isomorphic to the space of quadratic monogenic spinors.

Lemma 2. *The dimension of $A^{(1)}$ is equal to $s\binom{k(n-1)+1}{2} - s\binom{k}{2}$. In particular $\dim A^{(1)} < s\binom{k(n-1)+1}{2}$ and the tableau associated to the k-Dirac operator is not involutive.*

Proof. First let us consider the piece $A^{(1)} \cap S^2\mathrm{E} \otimes S^2\mathrm{F} \otimes \mathbb{S}$. Let $a^{\mu}_{ij\alpha\beta} \in \mathbb{C}$ with $\alpha, \beta > 1$ be artibrary such that $a^{\mu}_{ij\alpha\beta} = a^{\mu}_{ij\beta\alpha} = a^{\mu}_{ji\alpha\beta}$. Consider the element $a^{\mu}_{ij\alpha\beta} e_i \odot e_j \otimes (\varepsilon_\alpha \odot \varepsilon_\beta \otimes s^\mu + \varepsilon_\alpha \odot \varepsilon_1 \otimes \varepsilon_1 \cdot \varepsilon_\beta \cdot s^\mu + \varepsilon_\beta \odot \varepsilon_1 \otimes \varepsilon_1 \cdot \varepsilon_\alpha \cdot s^\mu - 2\delta^{\alpha\beta} \varepsilon_1 \otimes \varepsilon_1 \otimes s^\mu)$.

Then from (8) it follows that this is an element of $S^2\mathrm{E} \otimes S^2\mathrm{F} \otimes \mathbb{S} \cap A^{(1)}$ with prescribed components $a^{\mu}_{ij\alpha\beta} e_i \odot e_j \otimes \varepsilon_\alpha \odot \varepsilon_\beta \otimes s^\mu$ where α, $\beta > 1$. On the other hand any element of $S^2\mathrm{E} \otimes S^2\mathrm{F} \otimes \mathbb{S} \cap A^{(1)}$ is uniquely determined by the coefficients $a^{\mu}_{ij\alpha\beta} \in \mathbb{C}$ with α, $\beta > 1$. This shows that $\dim(S^2\mathrm{E} \otimes S^2\mathrm{F} \otimes \mathbb{S} \cap A^{(1)}) = s\binom{n}{2}\binom{k+1}{2}$. As a $\mathfrak{sl}(k, \mathbb{C}) \oplus \mathfrak{so}(n, \mathbb{C})$-module this piece is isomorphic to $S^2\mathrm{E} \otimes S^2_0\mathrm{F} \boxtimes \mathbb{S}$ where $S^2_0\mathrm{F}$ denotes the trace-free part of $S^2\mathrm{F}$.

Let $a^{\mu}_{ij\alpha\beta} \in \mathbb{C}$ with $\alpha, \beta > 1$ be such that $a^{\mu}_{ij\alpha\beta} = -a^{\mu}_{ij\beta\alpha} = -a^{\mu}_{ji\alpha\beta}$. Then the element $a^{\mu}_{ij\alpha\beta} e_i \wedge e_j \otimes (\varepsilon_\alpha \wedge \varepsilon_\beta \otimes s^\mu + \varepsilon_\alpha \wedge \varepsilon_1 \otimes \varepsilon_1 \cdot \varepsilon_\beta \cdot s^\mu - \varepsilon_\beta \wedge \varepsilon_1 \otimes \varepsilon_1 \cdot \varepsilon_\alpha \cdot s^\mu)$ belongs to $A^{(1)}$ iff for all i, $j = 1, \dots, k : \sum_{\alpha\beta} a^{\mu}_{ij\alpha\beta} \varepsilon_\alpha \cdot \varepsilon_\beta \cdot s^\mu = 0$. These are $s\binom{k}{2}$ linearly independent equations. This shows that $\dim(\Lambda^2\mathrm{E} \otimes \Lambda^2\mathrm{F} \otimes \mathbb{S} \cap A^{(1)}) = s(\binom{k}{2}\binom{n-1}{2} - \binom{k}{2})$. As a $\mathfrak{sl}(k, \mathbb{C}) \oplus \mathfrak{so}(n, \mathbb{C})$-module this piece is isomorphic to $\Lambda^2\mathrm{E} \otimes \Lambda^2\mathrm{F} \boxtimes \mathbb{S}$. This space is irreducible if $n > 4$. If $n = 4$ it is the direct sum of two irreducible pieces. Summing up we get that $\dim(A^{(1)}) = s\binom{k(n-1)+1}{2} - s\binom{k}{2}$. $\square$

3.3. **Involutivity of the tableau of the first prolongation for $k = 2$.** Now we show that the tableau associated to the first prolongation is involutive when $k = 2$. We replace A by $A^{(1)}$ and $A^{(1)}$ by $A^{(2)}$ and repeat the algorithm. This means that we have to compare the sum of the Cartan characters with respect to a suitable filtration on the space of quadratic monogenic spinors (right hand side of (7)) to the dimension of the space of cubic monogenic spinors.

According to Section 3.1 the space of homogeneous polynomials of degree 3 on $M(n, 2, \mathbb{C})$ decomposes into $S^3\mathrm{E}' \otimes S^3\mathrm{F}' \oplus (\mathrm{E}' \boxtimes \Lambda^2\mathrm{E}') \otimes (\mathrm{F}' \boxtimes \Lambda^2\mathrm{F}')$ as a $\mathrm{GL}(2, \mathbb{C}) \times \mathrm{GL}(n, \mathbb{C})$-module. If we restrict $S^3\mathrm{F}'$ to $\mathfrak{so}(n, \mathbb{C})$ then it decomposes into $S^3_0\mathrm{F} \oplus \mathrm{F}$ where $S^3_0\mathrm{F}$ is the trace-free part of $S^3\mathrm{F}$. The trace-free part is the kernel of the canonical contraction. If $n > 4$ then $\Lambda^2\mathrm{F}'$ is irreducible also under the action of $\mathfrak{so}(n, \mathbb{C})$. If $n = 4$ then $\mathfrak{so}(4, \mathbb{C}) \cong \mathfrak{so}(3, \mathbb{C}) \oplus \mathfrak{so}(3, \mathbb{C})$ and the defining representation $\mathbb{C}^4$ is isomorphic $\mathbb{C}^4 \cong \mathbb{C}^2 \otimes \mathbb{C}^2$ where we use that $\mathfrak{so}(3, \mathbb{C}) \cong \mathfrak{sl}(2, \mathbb{C})$. The space $\Lambda^2\mathbb{C}^4$ decomposes into $\Lambda^2\mathbb{C}^2 \otimes S^2\mathbb{C}^2 \oplus S^2\mathbb{C}^2 \otimes \Lambda^2\mathbb{C}^2$. This is a splitting of 2-forms into self-dual and anti-self-dual part. The spinor representations are $\mathbb{S}_+ \cong \mathbb{C}^2 \otimes \mathbb{C}$, $\mathbb{S}_- \cong \mathbb{C} \otimes \mathbb{C}^2$, i.e. it is the defining representation of one summand times the trivial representation of the second summand. The projection $\pi\colon (\mathbb{C}^2 \otimes \mathbb{C}^2) \otimes \mathbb{S}_+ \to \mathbb{S}_-$ is then the obvious skew-symmetrization in the first factor times the identity on the latter factor. Similarly for $\mathbb{S}_-$.

Lemma 3. *Let us write $n = 2m$ if n is even and $n = 2m + 1$ if n is odd. The space of cubic monogenic spinors contains the following list of irreducible $\mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{so}(n, \mathbb{C})$-modules*

$$n \geq 3: \qquad S^3\mathrm{E} \otimes (S^3_0\mathrm{F} \boxtimes \mathbb{S}) \qquad \frac{2^{m+1}(n+1)n(n-1)}{3}$$
$$n \geq 5: \quad (\mathrm{E} \boxtimes \Lambda^2\mathrm{E}) \otimes ((\Lambda^2\mathrm{F} \boxtimes \mathrm{F}) \boxtimes \mathbb{S}) \quad \frac{2^{m-1}(n+1)(n-1)(n-3)}{3}$$

where on the right is the dimension of the module. For $n = 4$ there is also the module $(\mathrm{E} \boxtimes \Lambda^2\mathrm{E}) \otimes (\mathbb{C}^2 \otimes S^4\mathbb{C}^2 \oplus S^4\mathbb{C}^2 \otimes \mathbb{C}^2)$.

Proof. Let $v_1 \in \mathrm{F}$ be the first basis element introduced in Section 3. Then v_1 is null and we may assume that it is a highest weight vector in F. Let $s \in \mathbb{S}$ be a highest weight vector. Then $v_1 \cdot s = 0$. This shows that $S^3\mathrm{E} \otimes (S^3\mathrm{F} \boxtimes \mathbb{S}) \subset A^{(1)}$ if $n \geq 3$. If $n = 4$ then $\mathrm{F} \otimes \Lambda^2\mathrm{F}$ contains two unique pieces $\mathbb{C}^2 \otimes S^3\mathbb{C}^2 \oplus S^3\mathbb{C}^2 \otimes \mathbb{C}^2$. Then $S^3\mathbb{C}^2 \otimes \mathbb{C}^2 \boxtimes \mathbb{S}_+ \cong S^4\mathbb{C}^2 \otimes \mathbb{C}^2$ is in the kernel of the map (8). Similarly for $\mathbb{S}_-$. This shows that $(2, 1) \otimes (\mathbb{C}^2 \otimes S^4\mathbb{C}^2 \oplus S^4\mathbb{C}^2 \otimes \mathbb{C}^2)$ belongs to $A^{(1)}$. If $n > 4$ and $v_2 \in \mathbb{C}^n$ is the second basis from Section 3. Then with the usual convention $v_1 \wedge v_2 \in \Lambda^2\mathrm{F}$ is a highest weight vector and $v_2 \cdot s = 0$. This shows that $(\mathrm{E} \boxtimes \Lambda^2\mathrm{E}) \otimes ((\Lambda^2\mathrm{F} \boxtimes \mathrm{F}) \boxtimes \mathbb{S}) \subset A^{(1)}$. This proves the first part of the lemma. Now we use the Weyl dimension formula to compute the dimension of each module from the list.

The Weyl dimension formula works for semi-simple complex Lie algebras. We denote by Φ^+ the set of all positive roots, by ρ the lowest form. Let V_λ be an irreducible module with highest weight λ. The Weyl dimension formula is

$$\dim(\mathrm{V}_\lambda) = \frac{\prod_{\alpha \in \Phi^+} \langle \rho + \lambda, \alpha \rangle}{\prod_{\alpha \in \Phi^+} \langle \rho, \alpha \rangle}\,. \tag{9}$$

We will use the same notation as in [2]. Suppose that $n = 2m$. Then $\Phi^+ = \{e_i \pm e_j | 1 \leq i < j \leq m\}$ and $\rho = (m-1, m-2, \dots, 1, 0)$. The denominator is $\prod_{\alpha \in \Phi^+} \langle \rho, \alpha \rangle = (2m-3)!(2m-5)! \dots 3!1!(m-1)!$. Suppose that $m \geq 3$. The highest weight of $\mathrm{V}_\lambda = S^3\mathrm{F} \boxtimes \mathbb{S}_+$ is $(3 + \frac{1}{2}, \frac{1}{2}, \dots, \frac{1}{2})$. Thus the nominator is $\prod_{\alpha \in \Phi^+} \langle \lambda + \rho, \alpha \rangle = \frac{1}{6}(2m+1)!(2m-4)!(2m-6)! \dots 2!$. So we get that $\dim(S^3\mathrm{F} \boxtimes \mathbb{S}_+) = \frac{2^{m-2}}{3}(2m+1)(2m)(2m-1)$. The highest weight of $\mathrm{V}_\mu = \mathrm{F} \boxtimes \Lambda^2\mathrm{F} \boxtimes \mathbb{S}_+$ is $\mu = (2 + \frac{1}{2}, 1 + \frac{1}{2}, \frac{1}{2}, \dots, \frac{1}{2})$. Then $\prod_{\alpha \in \Phi^+} \langle \mu + \rho, \alpha \rangle = \frac{1}{6}(2m+1)(2m-1)!(2m-3)!(2m-6)!(2m-8)! \dots 2!$. This gives that $\dim(\mathrm{V}_\mu) = \frac{(2m+1)(2m-1)(2m-3)2^{m-1}}{3}$.

Let us now consider the case $n = 2m+1$. Then $\Phi^+ = \{e_i \pm e_j, e_i | 1 \leq i < j \leq m\}$ and $\rho = (m - \frac{1}{2}, m - \frac{3}{2}, \dots, \frac{3}{2}, \frac{1}{2})$. The denominator is $\prod_{\alpha \in \Phi^+} \langle \rho, \alpha \rangle = (2m-2)!(2m-4)! \dots 2!(m-\frac{1}{2})(m-\frac{3}{2}) \dots \frac{3}{2}\frac{1}{2}$. Suppose that $m \geq 2$. The highest weight of $\mathrm{V}_\sigma = S^3\mathrm{F} \boxtimes \mathbb{S}$ is $(3 + \frac{1}{2}, \frac{1}{2}, \dots, \frac{1}{2})$. Thus the nominator is $\prod_{\alpha \in \Phi^+} \langle \sigma + \rho, \alpha \rangle = \frac{1}{6}(2m+2)!(2m-3)!(2m-5)! \dots 3!1!$. So we get that $\dim(S^3\mathrm{F} \boxtimes \mathbb{S}) = \frac{2^m}{6}(2m+2)(2m+1)2m$. The highest weight of $\mathrm{V}_\nu = \mathrm{F} \boxtimes \Lambda^2\mathrm{F} \boxtimes \mathbb{S}$ is $\nu = (2 + \frac{1}{2}, 1 + \frac{1}{2}, \frac{1}{2}, \dots, \frac{1}{2})$. Then $\prod_{\alpha \in \Phi^+} \langle \nu + \rho, \alpha \rangle = \frac{1}{3}(2m+2)(2m)!(2m-2)!(2m-5)!(2m-7)! \dots 3!1!$. This gives that $\dim(\mathrm{V}_\nu) = \frac{(m+1)m(m-1)2^{m+3}}{3}$. □

Now we need to find the Cartan characters.

Lemma 4. *Let $k = 2$ and $n \geq 3$. Then the sequence of Cartan characters is equal to $(2n-2)s, (2n-3)s, \dots, 3s, 2s, 0, 0, 0$.*

Proof. Let us first consider the case $n = 3$. Let us choose the ordered basis $e_1 \otimes \varepsilon_1$, $e_2 \otimes \varepsilon_2$, $(e_1 + e_2) \otimes \varepsilon_3$, $(e_1 - e_2) \otimes \varepsilon_3$, $e_2 \otimes \varepsilon_1$, $e_1 \otimes \varepsilon_2$ of V^*. The corresponding affine coordinates are

$$(t_1,\dots,t_6)\mapsto\begin{pmatrix} t_1 & t_5\\ t_6 & t_2\\ \frac{1}{2}(t_3+t_4) & \frac{1}{2}(t_3-t_4)\end{pmatrix}. \tag{10}$$

We have to show that the corresponding Cartan characters are 8, 6, 4, 0, 0, 0. The proof of Lemma 2 shows that the space of quadratic monogenic spinors is an irreducible $\mathfrak{sl}(2,\mathbb{C})\times\mathfrak{so}(3,\mathbb{C})$-module isomorphic to $S^2\mathrm{E}\otimes(S^2_0\mathrm{F}\boxtimes\mathbb{S})$. The complex dimension of this module is equal to 18. From the description of this module given in Lemma 2. follows that the first two Cartan characters are equal to 8, 6. It suffices to show that the last three Cartan characters are zero. That is: there is no monogenic quadratic spinor in the variables t_4,t_5,t_6.

The description of the basis of $S^2\mathrm{E}\otimes S^2\mathrm{F}$ given in Lemma 2 shows that a polynomial $f\in S^2\mathrm{E}\otimes S^2\mathrm{F}\cap\mathbb{C}[t_4,t_5,t_6]$ is necessarily of the form $f=at_4^2+bt_5^2+ct_6^2$ for $a,b,c\in\mathbb{C}$. So we have to consider a spinor of the form $(a^\mu(e_1-e_2)\odot(e_1-e_2)\otimes\varepsilon_3\odot\varepsilon_3+b^\mu e_2\odot e_2\otimes\varepsilon_1\odot\varepsilon_1+c^\mu e_1\odot e_1\otimes\varepsilon_2\odot\varepsilon_2)\otimes s^\mu$ where $\mu=1,2$. This element belongs to the kernel of the map (8) iff all coefficients a^μ, b^μ, c^μ are zero. This proves the claim for $n=3$.

For $n>3$ we choose the ordered basis $e_1\otimes\varepsilon_1$, $e_2\otimes\varepsilon_1$, $e_1\otimes\varepsilon_2$, $\dots$, $e_2\otimes\varepsilon_{n-3}$, $e_1\otimes\varepsilon_{n-2}$, $e_2\otimes\varepsilon_{n-1}$, $(e_1+e_2)\otimes\varepsilon_n$, $(e_1-e_2)\otimes\varepsilon_n$, $e_2\otimes\varepsilon_{n-2}$, $e_1\otimes\varepsilon_{n-1}$. The affine coordinates are then

$$(t_1,t_2,t_3,\dots,t_{2n})\mapsto\begin{pmatrix} t_1 & t_2\\ t_3 & \dots\\ \dots & \dots\\ t_{2n-5} & t_{2n-1}\\ t_{2n} & t_{2n-4}\\ \frac{1}{2}(t_{2n-3}+t_{2n-2}) & \frac{1}{2}(t_{2n-3}-t_{2n-2})\end{pmatrix}. \tag{11}$$

The claim follows from the description of the space of quadratic monogenic spinors in the proof of Lemma 2 and the case $n=3$. □

Theorem 1. *The first prolongation of the tableau associated to the 2-Dirac operator is involutive.*

Proof. By the previous lemma the right hand side of the Cartan test (7) is equal to $s\sum_{i=1}^{2(n-1)}i(2n-1-i)-2s(n-1)=s\binom{2n}{3}-2s(n-1)$.

We used that $\sum_{i=1}^{n}i(n+1-i)=\binom{n+2}{3}$. Now we use the lower bound on $\dim(A^{(1)})$ from Lemma 3. Recall that $s=\dim(\mathbb{S}_+\oplus\mathbb{S}_-)=2^m$ where $n=2m$ if n is even while $n=2m+1$ if n is odd. For $n\geq5$ we have that $\dim(A^{(1)})\geq\frac{2^{m+1}(n+1)n(n-1)+2^{m-1}(n+1)(n-1)(n-3)}{3}=2^m\binom{2n}{3}-2\cdot2^m(n-1)$. Thus we have the equality in the Cartan test and the tableau is involutive. Let us consider $n=4$. The module $S^3_0\mathrm{F}\boxtimes\mathbb{S}_+$ is isomorphic to $S^3\mathbb{C}^2\otimes S^3\mathbb{C}^2\boxtimes\mathbb{S}_+\cong S^4\mathbb{C}^2\otimes S^3\mathbb{C}^2$. The dimension is equal to 20. The dimension of the latter piece from Lemma 3 is clearly 40. Since $\dim S^3\mathrm{E}=4$ we get that $\dim(A^{(1)})\geq4\cdot40+40=200$. On the other hand the sum of Cartan characters is equal to $4\binom{8}{3}-2\cdot4\cdot3=200$ and this is again an involutive tableau. This completes the proof for n even.

The last remaining case is $n = 3$. Recall that $\mathfrak{sl}(2,\mathbb{C}) \cong \mathfrak{so}(3,\mathbb{C})$ and $\mathbb{S} \cong \mathbb{C}^2, \mathrm{F} \cong \mathfrak{sl}(3,\mathbb{C})$. Then $\dim(S^3\mathrm{E} \otimes S^3_0\mathrm{F} \boxtimes \mathbb{S}) = 32$. The sum of the Cartan characters is $2(4 + 2 \cdot 3 + 3 \cdot 2) = 32$. $\square$

3.4. **Initial conditions for the 2-Dirac operator.** We now use the Cartan-Kähler theorem to characterize the set of initial conditions for the 2-Dirac operator. Let us first consider the case $n = 3$. Already on the lowest dimensional case we can illustrate the power of the Cartan-Kähler theorem. The general case will be given below.

Let us recall that we chose in the proof of Lemma 4 an ordered basis of $M(3,2,\mathbb{R})$ with affine coordinates in (10). Let us now denote the natural coordinates on the space $J^2\mathbb{S}$ of 2-jets of spinors by $\{t_i, s^\mu, u^\mu_j, p^\mu_{ij}\}$ so that the canonical contact system is $\theta^\mu = ds^\mu - u^\mu_j\omega_j$, $\theta^\mu_i = du^\mu_i - p^\mu_{ij}\omega_j$ where $\omega_i = dt_i$, $p^\mu_{ij} = p^\mu_{ji}$, $\mu = 1,2$, $i,j = 1,\dots,6$. Let Σ be the subset of $J^2\mathbb{S}$ of 2-jets of monogenic spinors. Then local solutions of $\partial f = 0$ are in $1-1$ correspondence with integral manifols in Σ of the pullback of the canonical system satisfying the independence condition given by $\omega_1,\dots,\omega_6$. On such manifold we can consider s^μ, u^μ_j, p^μ_{ij} as functions of $t_1,\dots,t_6$.

For $\mu = 1,2$, $i = 1,2,3$ and $j = i, i+1,\dots,4$ let f^μ_{ij} be arbitrary real analytic functions of variables $t_1,\dots,t_i$. Now the proof of the Cartan-Kähler theorem gives that there is a unique integral manifold of the canonical system satisfying the independence condition passing through the point $t_1 = \dots = t_6 = s^\mu = u^\mu_1 = \dots = u^\mu_6 = 0$ such that the following set of equations

$$\begin{aligned} p^\mu_{1j}(t_1,0,0,0,0,0) &= f^\mu_{1j}(t_1); j = 1,2,3,4 \\ p^\mu_{2j}(t_1,t_2,0,0,0,0) &= f^\mu_{2j}(t_1,t_2); j = 2,3,4 \\ p^\mu_{3j}(t_1,t_2,t_3,0,0,0) &= f^\mu_{3j}(t_1,t_2,t_3); j = 3,4 \end{aligned} \tag{12}$$

holds on the integral manifold. Recall that this gives existence of a monogenic spinor which satisfies the system of initial conditions (12) on an open neighbouhood of the given point.

Since the k-Dirac operator is a constant coefficient system it suffices to understand homogeneous parts of monogenic spinors. The system of equations (12) is equivalent to the following: given arbitrary homogeneous spinors f_1, f_2 of homogeneity r, resp. $r-1$ where $r \geq 2$ in variables t_1, t_2, t_3 then there is a unique monogenic spinor $f_1 + t_4 f_2 + g$ where g is a homogeneous spinor of degree r such that the sum of degrees of the variables t_4, t_5, t_6 of each monomial appearing in a component of g is at least equal to 2.

For example consider quadratic monogenic spinors. The space of quadratic monogenic spinors is naturally isomorphic to the vector space $\{a^\mu_{ij}t_it_j + b^\mu_l t_4 t_l\}$ where $a^\mu_{ij} \in \mathbb{C}, b^\mu_l \in \mathbb{C}$ are arbitrary constants symmetric in $i,j,l = 1,2,3$. Note that the dimension of this space is $2(\binom{4}{2} + 3) = 18$ which agrees with the previous computations. Cubic monogenic spinors are naturally isomorphic to the space $\{a^\mu_{ijl}t_it_jt_l + b^\mu_{uv}t_4t_ut_v\}$ where $a^\mu_{ijl} \in \mathbb{C}, b^\mu_{uv} \in \mathbb{C}$ are arbitrary constants symmetric in $i,j,l,u,v = 1,2,3$. The dimension of the space of these coefficients is $2(\binom{5}{3} + \binom{4}{2}) = 32$.

For general n we use the same coordinates as in (11). Let g_1, g_2 be homogeneous spinors on $M(n,2,\mathbb{R})$ of homogeneity r, resp. $r-1$ with $r \geq 2$ in variables $t_1,\dots,t_{2n-3}$. Then there is a unique monogenic spinor $g_1 + t_{2n-2}g_2 + g$ where g is a homogeneous spinor of degree r such that the sum of degrees of the variables t_{2n-2}, t_{2n-1}, t_{2n} of each monomial appearing in a component of g is at least equal to 2. We have proved the following theorem.

Theorem 2. *The vector space of homogeneous monogenic spinors of degree $r \geq 2$ for the 2-Dirac operator in dimension $n \geq 3$ is naturally isomorphic to the direct sum of vector spaces of homogeneous spinors of degree r and $r-1$ in the variables $t_1, t_2, \dots, t_{2n-3}$ from* (11).

4. The Cartan-Kähler theorem for the parabolic k-Dirac operator

4.1. **The canonical linear Pfaffian system on $J^2\mathbb{S}$.** Let us recall that we are working on the affine set $\mathcal{U}$ from Section 1.2. We write coordinates on $M(n,k,\mathbb{R})$ as $x_{\alpha i}$ and on $A(k,\mathbb{R})$ as y_{rs}. Then $\partial_{\alpha i}$ and $\partial_{rs} = -\partial_{sr}$ stand for the coordinate vector fields. We use convention $dy_{rs}(\partial_{ij}) = \delta_{ir}\delta_{js} - \delta_{jr}\delta_{is}$. Then we can write $dy_{rs} = -dy_{sr}$. We set $L_{\alpha i} = \partial_{\alpha i} - \frac{1}{2}x_{\alpha j}\partial_{ij}$ and call them left invariant vector fields.

As in the case of the Euclidean 2-Dirac operator we will need the first prolongation of the canonical linear Pfaffian system living on the space of 1-jets of monogenis spinors, i.e. we will need to work on the space of 2-jets of monogenic spinors on $\mathcal{U}$. We write coordinates on the space $J^2\mathbb{S}$ of 2-jets of spinors over $\mathcal{U}$ as $\{x_{\alpha i}, y_{rs}, s^\mu, u^\mu_{\alpha i}, v^\mu_{rs}, a^\mu_{\alpha i\beta j}, b^\mu_{\alpha irs}, c^\mu_{rsuv}\}$ with the relations $v^\mu_{rs} = v^\mu_{sr}, a^\mu_{\alpha i\beta j} = a^\mu_{\beta j\alpha i}, b^\mu_{\alpha irs} = -b^\mu_{\alpha isr}, c^\mu_{rsuv} = c^\mu_{uvrs} = -c^\mu_{sruv} = -c^\mu_{rsvu}$. The canonical Pfaffian system $\mathcal{I}$ is generated by 1-forms $\theta^\mu = ds^\mu - u^\mu_{\alpha i}dx_{\alpha i} - \frac{1}{2}v^\mu_{rs}dy_{rs}, \theta^\mu_{\alpha i} = du^\mu_{\alpha i} - a^\mu_{\alpha i\beta j}dx_{\beta j} - \frac{1}{2}b^\mu_{\alpha irs}dy_{rs}, \theta^\mu_{uv} = dv^\mu_{uv} - b^\mu_{\alpha iuv}dx_{\alpha i} - \frac{1}{2}c^\mu_{uvrs}dy_{rs}$. Here we are summing over all r, $s = 1,\dots,k$ and so the factor $\frac{1}{2}$ appears there.

We will need to introduce new coordinates which are more adapted for the operator D. In the first place we have to find the dual 1-forms to the vector fields $L_{\alpha i}$, ∂_{rs}, i.e. we are looking for 1-forms such that $\omega_{\alpha i}(L_{\beta j}) = \delta_{\alpha\beta}\delta_{ij}$, $\omega_{\alpha i}(\partial_{rs}) = \omega_{rs}$ $(L_{\alpha i}) = 0$, $\omega_{rs}(\partial_{ij}) = \delta_{ir}\delta_{js} - \delta_{is}\delta_{jr}$. These forms will give for each $x \in \mathcal{U}$ an isomorpism $T^*_x\mathcal{U} \cong \mathrm{E}\otimes\mathrm{F}\oplus\Lambda^2\mathrm{E}$ where $\Lambda^2\mathrm{E}$ is isomorphic to the span of all $(\omega_{rs})_x$ and $\mathrm{E}\otimes\mathrm{F}$ is the span of all $(\omega_{\alpha i})_x$. We will not distinguish between complex and real representations as carefully as we did in the previous sections. The meaning should be clear from the context. We find that $\omega_{\alpha i} = dx_{\alpha i}$ and $\omega_{rs} = dy_{rs} - \frac{1}{2}(x_{\beta r}dx_{\beta s} - x_{\beta s}dx_{\beta r})$. We have $d\omega_{\alpha i} = 0, d\omega_{rs} = \sum_\alpha \omega_{\alpha s}\wedge\omega_{\alpha r}$.

Substituting $\omega_{\alpha i}$, ω_{rs} into the formula for θ^μ we obtain $\theta^\mu = ds^\mu - \sigma^\mu_{\alpha i}\omega_{\alpha i} - \frac{1}{2}v^\mu_{rs}\omega_{rs}$ where $\sigma^\mu_{\alpha i} = u^\mu_{\alpha i} - \frac{1}{2}x_{\alpha j}v^\mu_{ij}$. We set $A^\mu_{\alpha i\beta j} = a^\mu_{\alpha i\beta j} - \frac{1}{2}(x_{\alpha s}b^\mu_{\beta jis} + x_{\beta t}b^\mu_{\alpha ijt}) + \frac{1}{4}x_{\alpha s}x_{\beta t}c^\mu_{risj} - \frac{1}{2}\delta_{\alpha\beta}v^\mu_{ij}$, $B^\mu_{\alpha ijs} = b^\mu_{\alpha ijs} - \frac{1}{2}x_{\alpha t}c^\mu_{itjs}$, $C^\mu_{rskl} = c^\mu_{rskl}$. Then $A^\mu_{\alpha i\beta j} - A^\mu_{\beta j\alpha i} = \delta_{\alpha\beta}v^\mu_{ij}$. This is compatible with (2). The forms $\theta^\mu_{\alpha i}$, θ^μ_{rs} are then $\theta^\mu_{\alpha i} = d\sigma^\mu_{\alpha i} + \frac{1}{2}x_{\alpha j}\theta_{ij} - A^\mu_{\alpha i\beta j}\omega_{\beta j} - B^\mu_{\alpha irs}\omega_{rs}$, $\theta^\mu_{rs} = dv^\mu_{rs} - B^\mu_{\beta jrs}\omega_{\beta j} - \frac{1}{2}C^\mu_{rsuv}\omega_{uv}$.

4.2. **Vanishing of torsion.** In this section we argue that the torsion of the linear Pfaffian system associated to the k-Dirac operator and to its prolongation vanishes. We state a necessary lemma from [8] and set the notation. We define a grading on the space of polynomials $\mathbb{C}[x_{\alpha i}, y_{rs}]$. The *weighted degree* of linear polynomials is $\deg_w(x_{\alpha i}) = 1$, $\deg_w(y_{rs}) = 2$. We extend this to the space of monomials such that $\deg_w$ is a morphism of $(\mathbb{C}[x_{\alpha i}, y_{rs}], \cdot) \to (\mathbb{Z}, +)$. Then $\deg_w(f) = r$ iff f is a sum of monomials of the weighted degree r. We say that a spinor ψ on $\mathcal{U}$ is of the weighted degree r if each component of ψ is a weighted polynomial of degree r in the preferred trivialization.

Lemma 5. *Let ψ be a homogeneous monogenic spinor (in the Euclidean setting) of degree r on $M(n, k, \mathbb{R})$, i.e. $\partial\psi = 0$. Let $g \in \mathbb{C}[y_{rs}]$ be an arbitrary homogeneous polynomial of degree l. Then there is a parabolic monogenic spinor Ψ homogeneous of weighted degree $r + 2l$ on $\mathcal{U}$, i.e. $D\Psi = 0$, which is of the form $\Psi = g\psi + l.o.t.$ where l.o.t. stands for a spinor on $\mathcal{U}$ whose components are polynomials which are of degree strictly smaller than l in y-variables.*

Proof. This is Lemma 8.6.2 from [8]. □

Let us choose $k \in \{1, 2\}$. We denote the space of k-jets of monogenic spinors by Σ. The lemma implies that there is an integral manifold passing through any point in the fibre of the canonical projection $\Sigma \to \mathcal{U}$ over the origin $0 \in \mathcal{U}$. The tangent space of the integral manifold is an integral element and so by Lemma 1 the torsion $[T] = 0$ vanishes identically in the fibre over $0 \in \mathcal{U}$.

The flow of (the projection of) a right invariant vector field X on the homogeneous space is symmetry of the operator D. By flows of such vector fields we can move any point $x \in \mathcal{U}$ to any given point $x' \in \mathcal{U}$. The induced action on Σ is compatible with the canonical projection to $\mathcal{U}$. The flow of the field X preserve the ideal I and thus also the tableau and the torsion is invariant along the flow lines. Since the torsion vanishes in the fibre over $0 \in \mathcal{U}$ it has to vanish everywhere on Σ.

4.3. **Non-involutivity of the tableau associated to k-Dirac operator D.** The space of 1-jets of spinors on $\mathcal{U}$ is the set $J^1\mathbb{S} = \{(x_{\alpha i}, y_{rs}, s^\mu, \sigma^\mu_{\alpha i}, v^\mu_{rs})\}$ with canonical linear Pfaffian system generated by the forms θ^μ and the indepence condition $\omega_{\alpha i}, \omega_{rs}$. The structure equations are $d\theta^\mu = -d\sigma^\mu_{\alpha i} \wedge \omega_{\alpha i} - \frac{1}{2} dv^\mu_{rs} \wedge \omega_{rs} - \frac{1}{2} v^\mu_{rs} d\omega_{rs}$. We use abstract index notation and the Einstein summation convention. We can then write $(\varepsilon_\alpha \cdot) : \mathbb{S} \to \mathbb{S}, (\varepsilon \cdot s)^\mu = (\varepsilon \cdot)^\mu_\nu s^\nu$ for any spinor $s^\nu \in \mathbb{S}$ and $\varepsilon \in \mathrm{F}$.

Then a 1-jet from $J^1\mathbb{S}$ is a 1-jet of monogenic spinor iff $\sum_\alpha (\varepsilon_\alpha)^\nu_\mu \sigma^\mu_{\alpha i} = 0$ for all $i = 1, \ldots, k$. We may take π^ε to be the forms $d\sigma^\mu_{\alpha i}$ with $\alpha > 1$ and dv^μ_{rs} with $r < s$. For each $x \in \mathcal{U} : \mathrm{V}^*_x \cong \mathrm{E} \otimes \mathrm{F} \oplus \Lambda^2\mathrm{E}$ and $\mathrm{W}_x \cong \mathbb{S}$. The tableau is at any point isomorphic to $\mathrm{E} \otimes \mathbb{T} \oplus \Lambda^2\mathrm{E} \otimes \mathbb{S}$ while the torsion is represented by $[-\frac{1}{2} v^\mu_{rs} d\omega_{rs}]$. From the discussion in the previous section follows that the torsion vanishes identically.

The Cartan characters with respect to the ordered basis $e_1 \otimes \varepsilon_1, \ldots, e_1 \otimes \varepsilon_k, \ldots, e_{n-1} \otimes \varepsilon_1, e_{n-1} \otimes \varepsilon_k, e_1 \wedge e_2, \ldots, e_{k-1} \wedge e_k$, $e_n \otimes \varepsilon_1, \ldots, e_n \otimes \varepsilon_k$ of V^* are $s_i = s$, $s_j = 0$ for $i \leq k(n-1) + \binom{k}{2} < j$ and so the right hand side of (7) is equal to $s\binom{k(n-1)+\binom{k}{2}+1}{2}$. The first prolongation is clearly isomorphic to

$$A^{(1)} \cong \mathrm{M} \oplus \Lambda^2\mathrm{E} \otimes \mathrm{E} \otimes \mathbb{T} \oplus S^2(\Lambda^2\mathrm{E}) \otimes \mathbb{S} \tag{13}$$

where M is the space of the quadratic monogenic spinors (in the Euclidean setting) described in the proof of Lemma 2. The dimension of the prolongation is $\dim(A^{(1)}) = s\binom{k(n-1)+\binom{k}{2}+1}{2} - s\binom{k}{2}$.

We see that we do not have equality in the Cartan test (7) and thus the tableau is not involutive. We have to prolong this system as we in the case of the 2-Dirac operator ∂. The interpretation of the tableau and its prolongations is the following. Let $J^i_x\mathcal{M}$ be the space of i-jets of monogenic spinors at a point $x \in \Sigma$. The tableau is isomorphic to the kernel of the canonical projection $J^1_x\mathcal{M} \to J^0_x\mathcal{M}$, the first prolongation of the tableau is isomorphic to the kernel of $J^2_x\mathcal{M} \to J^1_x\mathcal{M}$, the prolongation of the first prolongation is then isomorphic to the kernel of $J^3_x\mathcal{M} \to J^2_x\mathcal{M}$ and so on.

4.4. **Involutivity of the first prolongation of the parabolic 2-Dirac operator D.** The structure equations on $J^2\mathbb{S}$ are $d\theta^\mu = 0$, $d\theta^\mu_{\alpha i} = \frac{1}{2}x_{\alpha j}d\theta_{ij} - dA^\mu_{\alpha i\beta j} \wedge \omega_{\beta j} - \frac{1}{2}dB^\mu_{\alpha irs} \wedge \omega_{rs} - \frac{1}{2}B^\mu_{\alpha irs}d\omega_{rs}$, $d\theta^\mu_{rs} = -dB^\mu_{\beta jrs} \wedge \omega_{\beta j} - \frac{1}{2}dC^\mu_{rsuv} \wedge \omega_{uv} - \frac{1}{2}C^\mu_{rsuv}d\omega_{uv}$ all modulo I.

The space of 2-jets of monogenic spinors is the subset of $J^2\mathbb{S}$ where the following set of relations holds. In the first place: $\sum_\alpha(\varepsilon_\alpha\cdot)^\mu_\nu\sigma^\nu_{\alpha i} = 0, \sum_\alpha(\varepsilon_\alpha\cdot)^\mu_\nu B^\nu_{\alpha irs} = 0$ holds for all i, r, s. From the variables $A^\mu_{\alpha i\beta j}$ only those with $\alpha, \beta > 1$ are free. There is one more system of equations $\sum_{\alpha,\beta>1}[(\varepsilon_\alpha\cdot)^\mu_\rho, (\varepsilon_\beta\cdot)^\rho_\nu]A^\nu_{\alpha i\beta j} = (-n+2)v^\mu_{ji}$.

We find that for any $x \in \mathcal{U}$: $\mathrm{V}^* \cong \mathrm{E} \otimes \mathrm{F}$, $\mathrm{W} \cong \mathrm{E} \otimes \mathbb{T} \oplus \Lambda^2\mathrm{E} \otimes \mathbb{S}$ and that the tableau A is isomorphic to the first prolongation (13) from the previous section. The torsion vanishes identically on Σ by the same argument as in the previous section.

Lemma 6. *The Cartan characters of the tableau A are* $(2n-1)s, (2n-2)s, (2n-3)s, \ldots, 3s, 2s, 0, 0, 0$.

Proof. Let us choose the origin $x = 0 \in \Sigma$. Let us order basis of V^* by putting the vector $e_1 \wedge e_2 \in \Lambda^2\mathrm{E}$ in the first place and then we put the basis of $\mathrm{E} \otimes \mathrm{F}$ ordered in the same way as in Lemma 4. Then the first Cartan character is equal to the dimension of $S^2(\Lambda^2\mathrm{E}) \otimes \mathbb{S} \oplus \Lambda^2\mathrm{E} \otimes \mathrm{E} \otimes \mathbb{T}$. This number is equal to $s(1 + 2(n-1))$. The other Cartan characters clearly coincide with the Cartan characters from Lemma 4. □

Lemma 7. *The first prolongation $A^{(1)}$ is isomorphic to the direct sum of the corresponding irreducible $\mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{so}(n)$-modules from the table* (3) *and* $\Lambda^2E \otimes M \oplus S^2(\Lambda^2E) \otimes E \otimes \mathbb{T} \oplus S^3(\Lambda^2E) \otimes \mathbb{S}$ *where M is the module isomorphic the space of quadratic monogenic spinors* (*in the Euclidean setting*) *from Lemma 2.*

Proof. This follows from the definition of $A^{(1)} := \mathrm{V}^* \otimes A \cap S^2\mathrm{V}^* \otimes \mathrm{W}$. □

Theorem 3. *The tableau of the first prolongation of the parabolic* 2*-Dirac operator is involutive.*

Proof. The right hand side of the Cartan test (7) is $s\sum_{i=1}^{2n-1} i(2n-i) - (2n-1)s = s\frac{(2n-1)}{6}(4n^2 + 2n - 6)$. The left hand side is equal to $\dim(A^{(1)}) = s\binom{2n}{3} - 2s(n -$

$1) + s(\binom{2(n-1)+1}{2} - 1) + 2s(n-1) + s = s(2n-1)\frac{4n^2+2n-6}{6}$. Here we have used that $\dim(\mathrm{M}) = s(\binom{2(n-1)+1}{2} - 1)$ proved in Lemma 1 and that the dimension of the space of cubic monogenic spinors (in the Euclidean setting) is $s\binom{2n}{3} - 2s(n-1)$ which was shown in the proof of Theorem 1. □

Remark 1. Thus we conclude that the machinery of the Cartan-Kähler theorem reproves that the set of initial condtitions for the 2-Dirac operator D is the one stated in Lemma 5. This follows directly from Lemma 6. knowing that the tableau is involutive. So we see that the set of initial conditions from Conjecture 1 cannot be obtained from the machinery of the Cartan-Kähler theorem but a modification of the Cartan-Kähler theorem for weighted differential operators is needed. See [6].

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