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OPERADS FOR n-ARY ALGEBRAS – CALCULATIONS AND CONJECTURES

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Abstract. In [8] we studied Koszulity of a family $t\mathcal{A}ss^n_d$ of operads depending on a natural number $n \in \mathbb{N}$ and on the degree $d \in \mathbb{Z}$ of the generating operation. While we proved that, for $n \leq 7$, the operad $t\mathcal{A}ss^n_d$ is Koszul if and only if d is even, and while it follows from [4] that $t\mathcal{A}ss^n_d$ is Koszul for d even and arbitrary n, the (non)Koszulity of $t\mathcal{A}ss^n_d$ for d odd and $n \geq 8$ remains an open problem. In this note we describe some related numerical experiments, and formulate a conjecture suggested by the results of these computations.

1. Introduction

All algebraic objects will be considered over a ground field $\mathbf{k}$ of characteristic zero. In particular, the symbol $\otimes$ will denote the tensor product over $\mathbf{k}$. We assume some familiarity with operad theory, namely with Koszul duality for quadratic operads and their Koszulity, see for instance [9, Chapter II.3] or the original sources [1, 2]. In Section 3 we also refer to minimal models for operads. The necessary notions can again be found in [9, Chapter II.3] or in the original source [6]. We however recall the most basic notions at the beginning of Section 2.

The operad $t\mathcal{A}ss^n_d$ mentioned in the abstract describes algebras introduced in the following:

1.1. **Definition.** Let V be a graded vector space, $n \geq 2$, and $\mu\colon V^{\otimes n} \to V$ a degree d linear map. The couple $A = (V, \mu)$ is a *degree d totally associative n-ary algebra* if, for each $1 \leq i, j \leq n$,

$$\mu\left(1\!\!1^{\otimes i-1} \otimes \mu \otimes 1\!\!1^{\otimes n-i}\right) = \mu\left(1\!\!1^{\otimes j-1} \otimes \mu \otimes 1\!\!1^{\otimes n-j}\right)\,, \tag{1}$$

where $1\!\!1\colon V \to V$ denotes the identity map.

If we symbolize μ by an oriented corolla with one output and n inputs, then the axiom (1) can be depicted as

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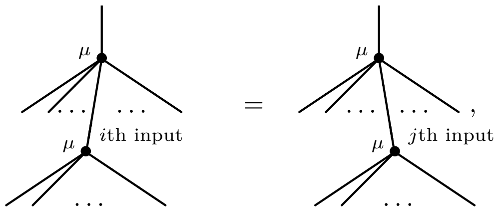


with the compositions of the indicated operations taken from the bottom up.

Therefore, in totally associative algebras, all associations of the iterated n-ary multiplication are the same. Degree 0 totally associative 2-algebras are ordinary associative algebras. Degree 0 totally associative n-algebras are usually called simply n-ary totally associative algebras.

Let $t\mathcal{A}ss^n_d$ be the *operad* for degree d totally associative n-algebras. It is not difficult to prove that the Koszulity of $t\mathcal{A}ss^n_d$ depends only on the *parity* of d. In this brief note we focus on

Conjecture A. The operad $t\mathcal{A}ss^n_d$ is Koszul if and only if d is even.

It follows from the work of Hoffbeck [4] on the Poincaré -Birkhoff-Witt criterion for operads that $t\mathcal{A}ss^n_d$ is Koszul for d even. In [8] we proved that $t\mathcal{A}ss^n_d$ is not Koszul if d is odd and $n \leq 7$. The non-Koszulity for d odd and $n \geq 8$ is therefore still conjectural.

2. Ginzburg-Kapranov's criterion for n-ary operads

For convenience of the reader we recall, following [8], some features of the Koszul duality of *non-binary* operads. Assume $E = \{E(a)\}_{a\geq 2}$ is a Σ-module of finite type concentrated in arity n. Operads $\mathcal{P} = \Gamma(E)/(R)$, where $\Gamma(E)$ is the free operad on E and (R) the ideal generated by a subspace $R \subset \Gamma(E)(2n-1)$ are called *n-ary quadratic*. Let $E^\vee = \{E^\vee(a)\}_{a\geq 2}$ be the Σ-module with

$$E^\vee(a) := \begin{cases} \mathrm{sgn}_a \otimes \uparrow^{a-2} E(a)^{\#}, & \text{if } a = n \text{ and} \\ 0, & \text{otherwise} \end{cases}$$

where $\uparrow^{a-2}$ is the iterated suspension, sgn_a the signum representation, and $\#$ the linear dual of a graded vector space with the induced representation. There is a non-degenerate pairing

$$\langle - | - \rangle \colon \Gamma(E^\vee)(2n-1) \otimes \Gamma(E)(2n-1) \to \mathbf{k}\,.$$

Its concrete form is not relevant for this note, the details can be found in [9, page 142].

2.1. **Definition.** The *Koszul dual* of the n-ary operad $\mathcal{P} = \Gamma(E)/(R)$ is the quotient

$$\mathcal{P}^! := \Gamma(E^\vee)/(R^\perp)\,,$$

where $R^\perp \subset \Gamma(E^\vee)(2n-1)$ is the annihilator of $R \subset \Gamma(E)(2n-1)$ in the above pairing, and $(R^\perp)$ the ideal generated by $R^\perp$.

If $\mathcal{P}$ is n-ary, generated by an operation of degree d, then the generator of $\mathcal{P}^!$ has the same arity but degree $-d+n-2$, i.e. for $n \neq 2$ (the non-binary case) the Koszul duality may not preserve the degree of the generating operation. In the following standard definition, $\mathsf{D}(-)$ denotes the dual operad construction [2, (3.2.12)]. Recall that it is essentially the bar construction (which takes operads to cooperads) followed by the componentwise vector space dual (which takes cooperads to operads). In Section II.3.3 of the monograph [9], $\mathsf{D}(-)$ was called the dual bar construction.

2.2. **Definition.** A quadratic operad $\mathcal{P}$ is *Koszul* if the natural map $\mathsf{D}(\mathcal{P}^!) \to \mathcal{P}$ is a homology equivalence.

The definition below describes algebras over the Koszul dual of $t\mathcal{A}ss^n_d$.

2.3. **Definition.** Let V be a graded vector space and $\mu\colon V^{\otimes n} \to V$ a degree d linear map. The couple $A = (V, \mu)$ is a *degree d partially associative n-ary algebra* if the following single axiom is satisfied:

$$\sum_{i=1}^{n} (-1)^{(i+1)(n-1)} \mu \left(1\!\!1^{\otimes i-1} \otimes \mu \otimes 1\!\!1^{\otimes n-i} \right) = 0\,.$$

In partially associative n-ary algebras, all associations of the multiplication (with alternating signs if n is even) sum to zero. So, for $n = 2$ one has

$$((ab)c) - (a(bc)) = 0\,,$$

thus degree d partially associative 2-ary algebras are associative algebras with multiplication of degree d. For $n = 3$ one has

$$((abc)de) + (a(bcd)e) + (ab(cde)) = 0\,.$$

Degree $(n-2)$ partially associative n-ary algebras are precisely A_∞-algebras $A = (V, \mu_1, \mu_2, \ldots)$ [5, §1.4] which are *meager* in that they satisfy $\mu_k = 0$ for $k \neq n$. Their symmetrizations are *Lie n-algebras* [3].

Let $p\mathcal{A}ss^n_d$ denote the operad for degree d partially associative n-ary algebras. The following statement follows from a simple calculation.

2.4. **Proposition.** *One has isomorphisms of operads*

$$(t\mathcal{A}ss^n_d)^! \cong p\mathcal{A}ss^n_{-d+n-2}\,,$$
$$(p\mathcal{A}ss^n_d)^! \cong t\mathcal{A}ss^n_{-d+n-2}\,.$$

Observe the shift of the degree of the generating operation. Since $\mathcal{P}$ is Koszul if and only if $\mathcal{P}^!$ is, one may reformulate the conjecture as

Conjecture A'. The operad $p\mathcal{A}ss^n_d$ is Koszul if and only if $n \equiv d \bmod 2$.

Recall that the *generating* or *Poincaré series* of an operad $\mathcal{P} = \{\mathcal{P}(a)\}_{a\geq 1}$ in the category of graded vector spaces of finite type is defined by

$$g_{\mathcal{P}}(t) := \sum_{a\geq 1} \frac{1}{a!} \chi(\mathcal{P}(a)) t^a\,,$$

where $\chi(-)$ denotes the Euler characteristic.

2.5. **Example.** It is not difficult to verify that the generating series for the operad $t\mathcal{A}ss_d^n$ is

$$g_{t\mathcal{A}ss_d^n}(t) := \begin{cases} t + t^n + t^{2n-1} + t^{3n-2} + t^{4n-3} + \cdots\,, & \text{if } d \text{ is even,}\\ t - t^n + t^{2n-1}, & \text{if } d \text{ is odd.}\end{cases}$$

We see that, for d odd, $t\mathcal{A}ss_d^n$ is nontrivial only in arities 1, n and $2n-1$. This is best explained by taking the simplest case $n = 2$ and analyzing the operadic desuspension $\widetilde{\mathcal{A}ss} := \mathbf{s}^{-1}t\mathcal{A}ss_1^2$.

Recall that the *operadic desuspension* $\mathbf{s}^{-1}\mathcal{P}$ of an operad $\mathcal{P} = \{\mathcal{P}(a)\}_{a\geq 1}$ is the operad $\mathbf{s}^{-1}\mathcal{P} = \{\mathbf{s}^{-1}\mathcal{P}(a)\}_{a\geq 1}$, where $\mathbf{s}^{-1}\mathcal{P}(a) := \mathrm{sgn}_a \otimes {\downarrow}^{a-1}\, \mathcal{P}(a)$, the signum representation tensored with the (ordinary) desuspension of the graded vector space $\mathcal{P}(a)$ iterated $(a-1)$ times. The structure operations of $\mathbf{s}^{-1}\mathcal{P}$ are induced by those of $\mathcal{P}$ in the obvious way. The Poincaré series of the operad $\mathcal{P}$ and its suspension $\mathbf{s}^{-1}\mathcal{P}$ are clearly related by

$$g_{\mathbf{s}^{-1}\mathcal{P}}(t) = -g_{\mathcal{P}}(-t)\,. \tag{2}$$

Algebras for the operad $\widetilde{\mathcal{A}ss}$ turn out to be *anti-associative* algebras with a degree 0 multiplication satisfying

$$a(bc) = -(ab)c\,, \quad \text{for} \quad a, b, c \in V\,.$$

While $\widetilde{\mathcal{A}ss}(1) = \mathbf{k}$, $\widetilde{\mathcal{A}ss}(2) = \mathbf{k}[\Sigma_2]$ and $\widetilde{\mathcal{A}ss}(3) = \mathbf{k}[\Sigma_3]$, the vanishing $\widetilde{\mathcal{A}ss}(4) = 0$ follows from the 'fake pentagon'

$$\begin{tikzcd}
 & -(ab)(cd) \arrow[dl, equal] \arrow[dr, equal] & \\
(((ab)c)d) \arrow[dd, equal] & & (a(b(cd))) \\
 & & -(a(b(cd))) \arrow[d, equal] \\
-((a(bc))d) \arrow[rr, equal] & & (a((bc)d))
\end{tikzcd}$$

by which all 4-fold products are trivial, as well as all a-fold products for $a \geq 4$. In other words, $\widetilde{\mathcal{A}ss}(a) = 0$ for $a \geq 4$, so the generating series for $\widetilde{\mathcal{A}ss}$ is therefore $t + t^2 + t^3$. By (2), the generating series of $t\mathcal{A}ss_1^2$ equals

$$t - t^2 + t^3$$

as claimed.

We finally formulate the (generalized) Ginzburg-Kapranov test [2]:

2.6. **Theorem.** *If a quadratic, not necessary binary, operad $\mathcal{P}$ is Koszul, then its Poincaré series and the Poincaré series of its dual $\mathcal{P}^!$ are tied by the functional equation*

$$g_{\mathcal{P}}(-g_{\mathcal{P}^!}(-t)) = t\,.$$

In other words, $-g_{\mathcal{P}^!}(-t)$ is a formal inverse of $g_{\mathcal{P}}(t)$.

The following particular form of the GK-test is a simple consequence of the above facts.

2.7. **Proposition.** *If the operad $t\mathcal{A}ss^n_d$ is Koszul, then all coefficients in the formal inverse of $t - t^n + t^{2n-1}$ are non-negative.*

The following theorem proved in [8] follows from the theory of analytic functions.

2.8. **Theorem.** *Suppose $g(z)$ is an analytic function in $\mathbb{C}$ such that $g(0) = 0$ and $g'(0) = 1$. If the equation*

$$g'(z) = 0$$

has no real solutions, then the formal inverse $g^{-1}(z)$ has at least one negative coefficient.

For the generating function $g(z) := z - z^n + z^{2n-1}$ of $t\mathcal{A}ss^n_d$, the equation $g'(z) = 0$ reads

$$g'(z) = 1 - nz^{n-1} + (2n-1)z^{2n-2} = 0$$

which, after the substitution $w := z^{n-1}$, leads to

$$1 - nw + (2n-1)w^2 = 0\,. \tag{3}$$

Fact. The discriminant $n^2 - 8n + 4$ of (3) is negative for $n \leq 7$ and positive for $n \geq 8$.

The Fact explains the distinguished rôle of $n = 7$ resp. 8. By Theorem 2.8, the inverse of $t - t^n + t^{2n-1}$ has, for $n \leq 7$, a negative coefficient so $t\mathcal{A}ss^n_d$ is for d odd and $n \leq 7$ not Koszul.

Equation (3) has, for $n = 8$, two real solutions, $\mathfrak{z}_1 = \sqrt[7]{1/3}$ and $\mathfrak{z}_2 = \sqrt[7]{1/5}$. Therefore, for $n = 8$ as well as for all higher n's, Theorem 2.8 does not apply and we are unable to prove the existence of negative coefficients in the inverse of $z - z^n + z^{2n-1}$. On the contrary, the calculations given in Section 3 indicate that all coefficients of the inverse are positive, so the Ginzburg-Kapranov criterion is not determinative.

3. Calculations, gaps and another conjecture

We computed, using `Mathematica`, the initial parts of the formal inverse of $t - t^n + t^{2n-1}$ for $n \leq 8$. We found:

$$t + t^2 + t^3 - 4t^5 - 14t^6 - 30t^7 - 33t^8 + 55t^9 + \cdots$$

for $n = 2$,

$$t + t^3 + 2t^5 + 4t^7 + 5t^9 - 13t^{11} - 147t^{13} + \cdots$$

for $n = 3$, and

$$t + t^4 + 3t^7 + 11t^{10} + 42t^{13} + 153t^{16} + 469t^{19} + 690t^{22} - 5967t^{25} + \cdots$$

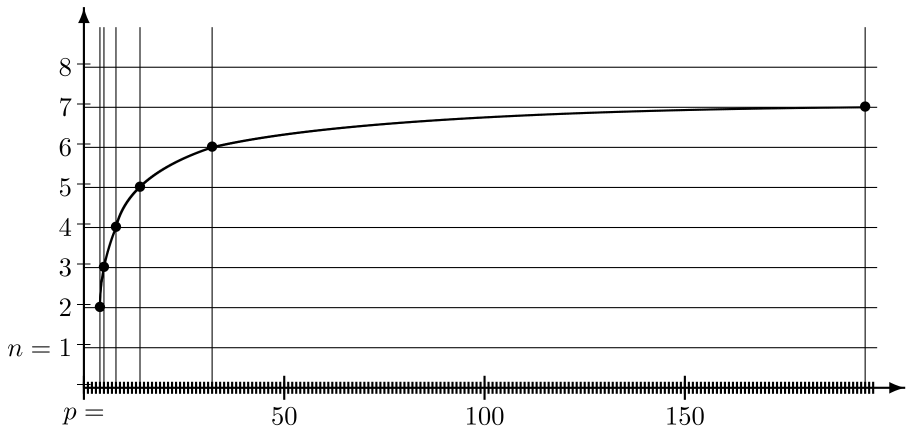
for $n = 4$.

The first negative coefficient in the inverse of $t - t^n + t^{2n-1}$ was at t^{57} for $n = 5$, at t^{161} for $n = 6$, and at t^{1171} for $n = 7$. For $n = 8$ we did not find any negative term of degree less than 10 000.

To appreciate the growth of the first negative coefficient, we introduce $\langle p \rangle := p(n-1)+1$, $p \geq 0$, the arity of an operation composed of p instances of an n-ary multiplication. The following table shows n and the corresponding p such that the first negative coefficient occurs at $t^{\langle p \rangle}$:

$n =$	2	3	4	5	6	7	8
$p =$	4	5	8	14	32	195	∞?

The dependence of p on n is plotted in the following table that clearly indicates that $p = \infty$ for $n \geq 8$, i.e. that there are no negative coefficients in the inverse of $t - t^n + t^{2n-1}$:



Although the GK-test does not apply for $n \geq 8$, there are some other indications that the operad $t\mathcal{A}ss^n_d$, d even, may not be Koszul.

3.1. **Example.** In [8] we explicitly established the initial part of the minimal model of $p\mathcal{A}ss^2_1 = (t\mathcal{A}ss^2_{-1})^!$,

$$(p\mathcal{A}ss^2_1, 0) \leftarrow (\Gamma(E_2, E_3, \sqcup, E_5, \ldots), \partial)\,. \tag{4}$$

Here E_2 is an one-dimensional space placed in arity 2, E_3 is one-dimensional placed in arity 3, and E_5 is 4-dimensional in arity 5.

It was the first non-trivial calculation of part of the minimal model of a non-Koszul operad. As shown in [8], the restriction $\partial|_{E_5}$ is not quadratic but ternary. It then follows from the construction of [7] that the L_∞-deformation complex for $p\mathcal{A}ss^2_1$-algebras has a non-trivial l_3-term.

The *gap* $\sqcup$ in arity 4 generators is caused by the 'wrong' signs in the pentagon, see Example 2.5. The fact that it is followed by a nontrivial space E_5 shows that

$p\mathcal{A}ss^2_1$ is not Koszul, as follows from a proposition below which we formulate for the n-ary case, for arbitrary $n \geq 2$.

Recall $\langle p \rangle := p(n-1)+1$, $p \geq 0$. If $\mathcal{P}$ is n-ary, then $\mathcal{P}(n) \neq 0$ only for $n = \langle p \rangle$ for some $p \geq 0$, and for the generators E of the minimal model $(\mathcal{P}, 0) \leftarrow (\Gamma(E), \partial)$ clearly the same holds:

$$E(n) \neq 0 \text{ only for } n \text{ of the form } n = \langle p \rangle \text{ for some } p \geq 0.$$

3.2. **Definition.** The minimal model of an n-ary operad has a *gap of length* $d \geq 1$ if there is $q \geq 2$ such that

$$E(\langle p \rangle) = 0 \text{ for } q \leq p \leq q+d-1$$

while

$$E(\langle q-1 \rangle) \neq 0 \neq E(\langle q+d \rangle)\,.$$

The model of $p\mathcal{A}ss^2_1$ is of the form $(\Gamma(E_{\langle 1 \rangle}, E_{\langle 2 \rangle}, \sqcup, E_{\langle 4 \rangle}, \ldots), \partial)$ with non-trivial $E_{\langle 2 \rangle}$ and $E_{\langle 4 \rangle}$. So it has a gap of length 1 – with $d = 1$, $q = 3$ in the above definition.

3.3. **Proposition.** *Suppose that the minimal model of a quadratic n-ary operad $\mathcal{P}$ has a gap of finite length. Then $\mathcal{P}$ is not Koszul.*

Proof. Suppose that $\mathcal{P}$ is Koszul and let $(\mathcal{P}, 0) \leftarrow (\Gamma(E), \partial)$ be its minimal model. It follows from Definition 2.2 and the uniqueness of the minimal model for operads [9, Theorem II.3.126] that the collection E is the (suitably suspended) Koszul dual $\mathcal{P}^!$. The operad $\mathcal{P}^!$ is n-ary, too, so $\mathcal{P}^!(\langle q \rangle) = 0$ for some $q \geq 2$ implies $\mathcal{P}^!(\langle p \rangle) = 0$ for all $p \geq q$. Thus $\mathcal{P}^!$ and therefore also E cannot have a gap of a *finite* length. $\square$

The strategy we suggest is to study the gaps in the minimal model of $p\mathcal{A}ss^n_d$ with $n \not\equiv d \bmod 2$. Their existence would imply non-Koszulity of $p\mathcal{A}ss^n_d$, as well as the non-Koszulity of their Koszul duals $t\mathcal{A}ss^n_d$, d odd, thus establishing Conjecture A. It is not difficult to prove the following:

3.4. **Proposition.** *Let $\mathcal{P}$ be an arbitrary, not necessarily Koszul, operad with $\mathcal{P}(1) = \mathbf{k}$, and $(\mathcal{P}, 0) \leftarrow (\Gamma(E), \partial)$ its minimal model. The Poincaré series $g_{\mathcal{P}}(t)$ of $\mathcal{P}$ is related with the generating function*

$$g_E(t) := -t + \sum_{a \geq 2} \frac{1}{a!} \chi(E(a)) t^a$$

of the Σ-module $\{E(a)\}_{a \geq 2}$ by the functional equation

$$g_{\mathcal{P}}(-g_E(t)) = t\,.$$

The above theorem enables one to calculate the Poincaré series of the collection of generators of the minimal model of $\mathcal{P}$ from the generating series of $\mathcal{P}$. It clearly implies the GK-criterion.

3.5. **Example.** It happens that $p\mathcal{A}ss^2_1 = t\mathcal{A}ss^2_1$, so the generating series of $p\mathcal{A}ss^2_1$ is

$$g_{p\mathcal{A}ss^2_1}(t) = t - t^2 + t^3\,.$$

One can compute the formal inverse of this function as

$$t + t^2 + t^3 - 4t^5 - 14t^6 - 30t^7 - 33t^8 + 55t^9 + \cdots .$$

The absence of the t^4-term together with the presence of the t^5-term "shows" the gap of length 1 in the minimal model of $p\mathcal{A}ss_1^2$.

We do not know any closed formula for the generating series of $p\mathcal{A}ss_d^n$ with $n \not\equiv d \bmod 2$, $n > 2$. We however wrote a script for `Mathematica` that calculates it, but its applicability is drastically limited by computers available. We established the generating series for $p\mathcal{A}ss_0^3$ as

$$t + t^3 + 2t^5 + 4t^7 + 5t^9 + 6t^{11} + 7t^{13} + 8t^{15} + \cdots ,$$

the generating series of $p\mathcal{A}ss_1^4$ as

$$t - t^4 + 3t^7 - 11t^{10} + 42t^{13} - 153t^{16} + 565t^{19} + \cdots ,$$

the generating series of $p\mathcal{A}ss_0^5$ as

$$t + t^5 + 4t^9 + 21t^{13} + 123t^{17} + 759t^{21} + \cdots ,$$

the generating series of $p\mathcal{A}ss_0^7$ as

$$t + t^7 + 6t^{13} + 50t^{19} + 481t^{25} + \cdots$$

and the generating series of $p\mathcal{A}ss_0^9$ as

$$t + t^9 + 8t^{17} + 91t^{25} + 1207t^{33} + \cdots .$$

By calculating the formal inverses of the above series, we get the following Poincaré series of the generators for the minimal models:

$$t + t^2 + t^3 + \boxed{0\, t^4} - 4t^5 - 14t^6 - 30t^7 - 33t^8 + 55t^9 + \cdots$$

for $p\mathcal{A}ss_1^2$ (we already know this),

$$t - \ t^3 + \ t^5 + \boxed{0\, t^7 + 0\, t^9} - 19\, t^{11} + 112\, t^{13} - 336\, t^{15} + \cdots$$

for $p\mathcal{A}ss_0^3$,

$$t + \ t^4 + \ t^7 + \boxed{0\, t^{10} + 0\, t^{13} + 0\, t^{16}} - 96\, t^{19} + \cdots$$

for $p\mathcal{A}ss_1^4$,

$$t - \ t^5 + \ t^9 + \boxed{0\, t^{13} + 0\, t^{17} + 0\, t^{21} + \ ?\ } + \mathrm{O}[t^{25}]$$

for $p\mathcal{A}ss_0^5$. The vanishing of the boxed terms imply, by Proposition 3.4, that the Euler characteristics of the corresponding pieces of the generating collection is zero.

It indicates that the minimal models are of the form

$$
\begin{aligned}
(p\mathcal{A}ss^2_1, 0) &\leftarrow (\Gamma(E_{\langle 1\rangle}, E_{\langle 2\rangle}, \textvisiblespace, E_{\langle 4\rangle}, \ldots), \partial),\\
(p\mathcal{A}ss^3_0, 0) &\leftarrow (\Gamma(E_{\langle 1\rangle}, E_{\langle 2\rangle}, \textvisiblespace, \textvisiblespace, E_{\langle 5\rangle}, \ldots), \partial),\\
(p\mathcal{A}ss^4_1, 0) &\leftarrow (\Gamma(E_{\langle 1\rangle}, E_{\langle 2\rangle}, \textvisiblespace, \textvisiblespace, \textvisiblespace, E_{\langle 6\rangle}, \ldots), \partial),\\
(p\mathcal{A}ss^5_0, 0) &\leftarrow (\Gamma(E_{\langle 1\rangle}, E_{\langle 2\rangle}, \textvisiblespace, \textvisiblespace, \textvisiblespace, \overset{?}{\textvisiblespace}, \boxed{?}, \ldots), \partial).
\end{aligned}
$$

Our computation did not go beyond $n \geq 6$, due to limitations of computer memory. The results for small n's however suggest that the gap grows *linearly* with n, leading to

Conjecture B. The minimal model of $p\mathcal{A}ss^n_d$, $n \not\equiv d \bmod 2$, has a gap of length $n - 1$.

The conjecture would obviously imply the non-Koszulity of $t\mathcal{A}ss^n_d$, for d odd. If it is so, then $t\mathcal{A}ss^8_1$ will be the first example of a non-Koszul operad whose non-Koszulity was not established by the Ginzburg-Kapranov criterion.

3.6. **Remark.** We followed a suggestion of the referee and compared the sequences arising in this section with The Online Encyclopedia of Integer Sequences. It recognized the generating series for $p\mathcal{A}ss^2_1$. This was not surprising as we know a closed formula. It also identified the initial part of the generating series for $p\mathcal{A}ss^3_0$ to a subsequence of the sequence $\{u_s\}_{s\geq 1}$, where u_s is the number of times 1 is used in writing out all the numbers 1 through s. We do not have any explanation for this fact. The remaining sequences were not recognized.

3.7. **Remark.** Degree 0 totally associative n-algebras, i.e. algebras over the operad $t\mathcal{A}ss^n_0$, generalize, for $n \geq 3$, associative algebras in a straightforward manner. The referee formulated an intriguing question whether there exists an analog of the associahedra for these algebras. In this remark we argue that this might indeed be possible.

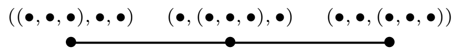
Recall that Stasheff's operad of associahedra $\mathcal{K} = \{K_a\}_{a\geq 1}$ is an operad in the category of polyhedra. Its most important property is that its operad of cellular chains is isomorphic to the minimal model of the associative operad [6, Example 4.8].

Let us try to start constructing the 'ternary' associahedron $\mathcal{K}^3 = \{K^3_a\}_{a\geq 1}$ for totally associative 3-algebras, mimicking the construction of the classical Stasheff operad. It is clear that the first nontrivial piece of $\mathcal{K}^3$ is the point K^3_3 in arity 3 that represents the ternary multiplication.

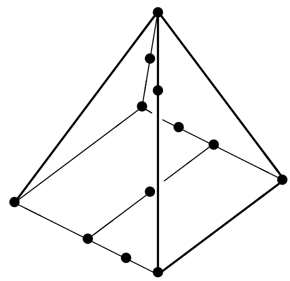
The next piece K^3_5 of the 3-associahedron must have three vertices corresponding to the three possible bracketing of five variables, namely

$$v_1 := ((\bullet, \bullet, \bullet), \bullet, \bullet),\ v_2 := (\bullet, (\bullet, \bullet, \bullet), \bullet),\ \text{ and } v_3 := (\bullet, \bullet, (\bullet, \bullet, \bullet)).$$

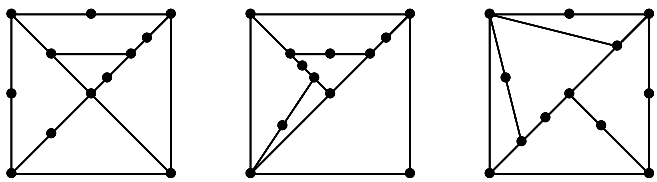
For the edges of K_5^3 we need to choose two of the three relations killing the differences $v_1 - v_2$, $v_2 - v_3$ and $v_1 - v_3$, because the resulting K_5^3 must be acyclic. If we chose e.g. the first two ones, we get the following picture of K_5^3:



So K_5^3 is the interval divided into two subintervals. Having K_3^3 and K_5^3 as above, the 1 skeleton of K_7^3 is the graph:



Now we have to choose five cycles, out of six, of this 1-skeleton and fill them by 2-dimensional faces. Since the figure above has an obvious left-right mirror symmetry, there are precisely three essentially independent choices. Depending on the choice, we get the following three combinatorially distinct K_7^3's:



They are convex polyhedra with twelve vertices, sixteen edges and five 2-dimensional faces.

Each choice of K_3^3, K_5^3 and K_7^3 determines the 2-skeleton of K_9^3. To perform the next step, we need to kill the generators of the second homotopy group of this 2-skeleton by choosing fourteen 3-dimensional faces, etc.

The fundamental difference from the construction of the classical associahedron is that at each step we need to make a choice, and that the combinatorial type of the resulting polyhedra depends on these choices. We formulate the last conjecture of this note:

Conjecture C. For each $n \geq 3$, there exists an operad $\mathcal{K}^n$ in the category of contractible polyhedra such that the minimal model of the operad for degree 0 n-ary totally associative algebras is isomorphic to the cellular chain operad of $\mathcal{K}^n$.

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