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INDUCED DIFFERENTIAL FORMS ON MANIFOLDS OF FUNCTIONS

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Dedicated to Peter W. Michor at the occasion of his 60th birthday

Abstract. Differential forms on the Fréchet manifold $\mathcal{F}(S,M)$ of smooth functions on a compact k-dimensional manifold S can be obtained in a natural way from pairs of differential forms on M and S by the hat pairing. Special cases are the transgression map $\Omega^p(M)\to\Omega^{p-k}(\mathcal{F}(S,M))$ (hat pairing with a constant function) and the bar map $\Omega^p(M)\to\Omega^p(\mathcal{F}(S,M))$ (hat pairing with a volume form). We develop a hat calculus similar to the tilda calculus for non-linear Grassmannians [6].

1. Introduction

Pairs of differential forms on the finite dimensional manifolds M and S induce differential forms on the Fréchet manifold $\mathcal{F}(S,M)$ of smooth functions. More precisely, if S is a compact oriented k-dimensional manifold, the hat pairing is:

$$\Omega^p(M)\times\Omega^q(S)\to\Omega^{p+q-k}(\mathcal{F}(S,M))$$
$$\widehat{\omega\cdot\alpha}=\fint_S \operatorname{ev}^*\omega\wedge\operatorname{pr}^*\alpha\,,$$

where $\operatorname{ev}\colon S\times\mathcal{F}(S,M)\to M$ denotes the evaluation map, $\operatorname{pr}\colon S\times\mathcal{F}(S,M)\to S$ the projection and $\fint_S$ fiber integration. We show that the hat pairing is compatible with the canonical Diff(M) and $\operatorname{Diff}(S)$ actions on $\mathcal{F}(S,M)$, and with the exterior derivative. As a consequence we obtain a hat pairing in cohomology.

The hat (transgression) map is the hat pairing with the constant function 1, so it associates to any form $\omega\in\Omega^p(M)$ the form $\widehat{\omega\cdot 1}=\widehat{\omega}=\fint_S\operatorname{ev}^*\omega\in\Omega^{p-k}(\mathcal{F}(S,M))$. Since $\mathfrak{X}(M)$ acts infinitesimally transitive on the open subset $\operatorname{Emb}(S,M)\subset\mathcal{F}(S,M)$ of embeddings of the k-dimensional oriented manifold S into M [7], the expression of $\widehat{\omega}$ at $f\in\operatorname{Emb}(S,M)$ is

$$\widehat{\omega}(X_1\circ f,\dots,X_{p-k}\circ f)=\int_S f^*(i_{X_{p-k}}\dots i_{X_1}\omega),\quad X_1,\dots,X_{p-k}\in\mathfrak{X}(M)\,.$$

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When S is the circle, then one obtains the usual transgression map with values in the space of $(p-1)$-forms on the free loop space of M.

Let $\mathrm{Gr}_k(M)$ be the non-linear Grassmannian of k-dimensional oriented submanifolds of M. The tilda map associates to every $\omega \in \Omega^p(M)$ a differential $(p-k)$-form on $\mathrm{Gr}_k(M)$ given by [6]

$$\tilde{\omega}(\tilde{Y}_N^1, \dots, \tilde{Y}_N^{p-k}) = \int_N i_{Y_N^{p-k}} \cdots i_{Y_N^1} \omega\,, \quad \forall \tilde{Y}_N^1, \dots, \tilde{Y}_N^{p-k} \in \Gamma(TN^\perp) = T_N \mathrm{Gr}_k(M)\,,$$

for $\tilde{Y}_N$ section of the orthogonal bundle $TN^\perp$ represented by the section Y_N of $TM|_N$. The natural map

$$\pi\colon \mathrm{Emb}(S,M) \to \mathrm{Gr}_k(M)\,, \quad \pi(f) = f(S)$$

provides a principal bundle with the group $\mathrm{Diff}_+(S)$ of orientation preserving diffeomorphisms of S as structure group.

The hat map on $\mathrm{Emb}(S,M)$ and the tilda map on $\mathrm{Gr}_k(M)$ are related by $\widehat{\omega} = \pi^* \tilde{\omega}$. This is the reason why for the hat calculus one has similar properties to those for the tilda calculus. The tilda calculus was used to study the non-linear Grassmannian of co-dimension two submanifolds as symplectic manifold [6]. We apply the hat calculus to the hamiltonian formalism for p-branes and open p-branes [1] [2].

The bar map $\bar{\omega} = \widehat{\omega \cdot \mu}$ is the hat pairing with a fixed volume form μ on S, so

$$\bar{\omega}(Y_f^1, \dots, Y_f^p) = \int_S \omega(Y_f^1, \dots, Y_f^p)\mu\,, \quad \forall Y_f^1, \dots, Y_f^p \in \Gamma(f^*TM) = T_f \mathcal{F}(S,M)\,.$$

We use the bar calculus to study $\mathcal{F}(S,M)$ with symplectic form $\bar{\omega}$ induced by a symplectic form ω on M. The natural actions of $\mathrm{Diff}_{\mathrm{ham}}(M,\omega)$ and $\mathrm{Diff}_{\mathrm{ex}}(S,\mu)$, the group of hamiltonian diffeomorphisms of M and the group of exact volume preserving diffeomorphisms of S, are two commuting hamiltonian actions on $\mathcal{F}(S,M)$. Their momentum maps form the dual pair for ideal incompressible fluid flow [12] [4].

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2. Hat pairing

We denote by $\mathcal{F}(S,M)$ the set of smooth functions from a compact oriented k-dimensional manifold S to a manifold M. It is a Fréchet manifold in a natural way [10]. Tangent vectors at $f \in \mathcal{F}(S,M)$ are identified with vector fields on M along f, i.e. sections of the pull-back vector bundle f^*TM.

Let $\mathrm{ev}\colon S \times \mathcal{F}(S,M) \to M$ be the evaluation map $\mathrm{ev}(x,f) = f(x)$ and $\mathrm{pr}\colon S \times \mathcal{F}(S,M) \to S$ the projection $\mathrm{pr}(x,f) = x$. A pair of differential forms $\omega \in \Omega^p(M)$ and $\alpha \in \Omega^q(S)$ determines a differential form $\widehat{\omega \cdot \alpha}$ on $\mathcal{F}(S,M)$ by the fiber integral over S (whose definition and properties are listed in the appendix) of the $(p+q)$-form $\mathrm{ev}^* \omega \wedge \mathrm{pr}^* \alpha$ on $S \times \mathcal{F}(S,M)$:

$$\widehat{\omega \cdot \alpha} = ⨍_S \mathrm{ev}^* \omega \wedge \mathrm{pr}^* \alpha\,. \tag{1}$$

In this way we obtain a bilinear map called the *hat pairing*:

$$\Omega^p(M) \times \Omega^q(S) \to \Omega^{p+q-k}\big(\mathcal{F}(S,M)\big)\,.$$

An explicit expression of the hat pairing avoiding fiber integration is:

$$(\widehat{\omega\cdot\alpha})_f(Y_f^1,\dots,Y_f^{p+q-k}) = \int_S f^*\big(i_{Y_f^{p+q-k}}\dots i_{Y_f^1}(\omega\circ f)\big)\wedge\alpha\,, \tag{2}$$

for $Y_f^1,\dots Y_f^{p+q-k}$ vector fields on M along $f\in\mathcal{F}(S,M)$. Here we denote by $f^*\beta_f$ the "restricted pull-back" by f of a section β_f of $f^*(\Lambda^m T^*M)$, which is a differential m-form on S given by $f^*\beta_f\colon x\in S\mapsto (\Lambda^m T_x^*f)(\beta_f(x))\in\Lambda^m T_x^*S$, where $T_x^*f\colon T_{f(x)}^*M\to T_x^*S$ denotes the dual of T_xf.

The fact that (1) and (2) provide the same differential form on $\mathcal{F}(S,M)$ can be deduced from the identity

$$(\mathrm{ev}^*\,\omega)_{(x,f)}(Y_f^1,\dots,Y_f^{p-k},X_x^1,\dots,X_x^k) = f^*(i_{Y_f^{p-k}}\dots i_{Y_f^1}(\omega\circ f))(X_x^1,\dots,X_x^k)$$

for $Y_f^1,\dots,Y_f^{p-k}\in T_f\mathcal{F}(S,M)$ and $X_x^1,\dots,X_x^k\in T_xS$.

Since $\mathfrak{X}(M)$ acts infinitesimally transitive on the open subset $\mathrm{Emb}(S,M)\subset\mathcal{F}(S,M)$ of embeddings of the k-dimensional oriented manifold S into M, we express $\widehat{\omega}$ at $f\in\mathrm{Emb}(S,M)$ as:

$$(\widehat{\omega\cdot\alpha})_f(X_1\circ f,\dots,X_{p+q-k}\circ f) = \int_S f^*(i_{X_{p+q-k}}\dots i_{X_1}\omega)\wedge\alpha\,. \tag{3}$$

One uses the fact that the "restricted pull-back" by f of $i_{X_{p+q-k}\circ f}\dots i_{X_1\circ f}(\omega\circ f)$ is $f^*(i_{X_{p+q-k}}\dots i_{X_1}\omega)$.

Next we show that the hat pairing is compatible with the exterior derivative of differential forms.

Theorem 1. *The exterior derivative* $\mathbf{d}$ *is a derivation for the hat pairing, i.e.*

$$\mathbf{d}\,(\widehat{\omega\cdot\alpha}) = (\widehat{\mathbf{d}\,\omega)\cdot\alpha} + (-1)^p\widehat{\omega\cdot\mathbf{d}\,\alpha}\,, \tag{4}$$

where $\omega\in\Omega^p(M)$ *and* $\alpha\in\Omega^q(S)$.

Proof. Differentiation and fiber integration along the boundary free manifold S commute, so

$$\begin{aligned}\mathbf{d}\,(\widehat{\omega\cdot\alpha}) &= \mathbf{d}\,\fint_S \mathrm{ev}^*\,\omega\wedge\mathrm{pr}^*\,\alpha = \fint_S \mathbf{d}\,(\mathrm{ev}^*\,\omega\wedge\mathrm{pr}^*\,\alpha)\\ &= \fint_S \mathrm{ev}^*\,\mathbf{d}\,\omega\wedge\mathrm{pr}^*\,\alpha + (-1)^p\fint_S \mathrm{ev}^*\,\omega\wedge\mathrm{pr}^*\,\mathbf{d}\,\alpha = (\widehat{\mathbf{d}\,\omega)\cdot\alpha} + (-1)^p\widehat{\omega\cdot\mathbf{d}\,\alpha}\end{aligned}$$

for all $\omega\in\Omega^p(M)$ and $\alpha\in\Omega^q(S)$. □

The differential form $\widehat{\omega\cdot\alpha}$ is exact if ω is closed and α exact (or if α is closed and ω exact). In the special case $p+q=k$ these conditions imply that the function $\widehat{\omega\cdot\alpha}$ on $\mathcal{F}(S,M)$ vanishes.

Corollary 2. *The hat pairing induces a bilinear map on de Rham cohomology spaces*

$$H^p(M)\times H^q(S)\to H^{p+q-k}\big(\mathcal{F}(S,M)\big)\,. \tag{5}$$

In particular there is a bilinear map

$$H^p(M) \times H^q(M) \to H^{p+q-k}\big(\operatorname{Diff}(M)\big)\,.$$

Remark 3. The cohomology group $H^q(S)$ is isomorphic to the homology group $H_{k-q}(S)$ by Poincaré duality. With the notation $n = k - q$, the hat pairing (5) becomes

$$H^p(M) \times H_n(S) \to H^{p-n}\big(\mathcal{F}(S,M)\big)\,,$$

and it is induced by the map $(\omega,\sigma) \mapsto \fint_\sigma \operatorname{ev}^*\omega$, for differential p-forms ω on M and n-chains σ on S.

If S is a manifold with boundary, then formula (4) receives an extra term coming from integration over the boundary. Let $i_\partial\colon \partial S \to S$ be the inclusion and $r_\partial\colon \mathcal{F}(S,M) \to \mathcal{F}(\partial S,M)$ the restriction map.

Proposition 4. *The identity*

$$\mathbf{d}\,(\widehat{\omega\cdot\alpha}) = (\widehat{\mathbf{d}\,\omega)\cdot}\alpha + (-1)^p\widehat{\omega\cdot\mathbf{d}\,\alpha} + (-1)^{p+q-k}r_\partial^*(\widehat{\omega\cdot i_\partial^*\alpha}^{\,\partial}) \tag{6}$$

holds for $\omega \in \Omega^p(M)$ and $\alpha \in \Omega^q(S)$, where the upper index ∂ assigned to the hat means the pairing

$$\Omega^p(M) \times \Omega^q(\partial S) \to \Omega^{p+q-k+1}\big(\mathcal{F}(\partial S,M)\big)\,.$$

Proof. For any differential n-form β on $S \times \mathcal{F}(S,M)$, the identity

$$\mathbf{d}\fint_S \beta - \fint_S \mathbf{d}\,\beta = (-1)^{n-k}\fint_{\partial S}(i_\partial \times 1_{\mathcal{F}(S,M)})^*\beta$$

holds because of the identity (19) from the appendix. The obvious formulas

$$\operatorname{pr}\circ(i_\partial \times 1_{\mathcal{F}(S,M)}) = i_\partial \circ \operatorname{pr}_\partial, \quad \operatorname{ev}\circ(i_\partial \times 1_{\mathcal{F}(S,M)}) = \operatorname{ev}_\partial\,,$$

for $\operatorname{ev}_\partial\colon \partial S \times \mathcal{F}(S,M) \to M$ and $\operatorname{pr}_\partial\colon \partial S \times \mathcal{F}(S,M) \to \partial S$, are used to compute

$$\begin{aligned}
\mathbf{d}\,(\widehat{\omega\cdot\alpha}) &= \mathbf{d}\fint_S \operatorname{ev}^*\omega \wedge \operatorname{pr}^*\alpha \\
&= \fint_S \mathbf{d}\,(\operatorname{ev}^*\omega \wedge \operatorname{pr}^*\alpha) + (-1)^{p+q-k}\fint_{\partial S}(i_\partial \times 1_{\mathcal{F}(S,M)})^*(\operatorname{ev}^*\omega \wedge \operatorname{pr}^*\alpha) \\
&= \fint_S \operatorname{ev}^*\mathbf{d}\,\omega \wedge \operatorname{pr}^*\alpha + (-1)^p\fint_S \operatorname{ev}^*\omega \wedge \operatorname{pr}^*\mathbf{d}\,\alpha + (-1)^{p+q-k}\fint_{\partial S}\operatorname{ev}_\partial^*\omega \wedge \operatorname{pr}_\partial^*\, i_\partial^*\alpha \\
&= (\widehat{\mathbf{d}\,\omega)\cdot}\alpha + (-1)^p\widehat{\omega\cdot\mathbf{d}\,\alpha} + (-1)^{p+q-k}r_\partial{}^*(\widehat{\omega\cdot i_\partial^*\alpha}^{\,\partial})\,,
\end{aligned}$$

thus obtaining the requested identity. □

Left $\operatorname{Diff}(M)$ action. The natural left action of the group of diffeomorphisms $\operatorname{Diff}(M)$ on $\mathcal{F}(S,M)$ is $\varphi\cdot f = \varphi\circ f$. The infinitesimal action of $X \in \mathfrak{X}(M)$ is the vector field $\bar{X}$ on $\mathcal{F}(S,M)$:

$$\bar{X}(f) = X\circ f\,, \quad \forall f \in \mathcal{F}(S,M)\,.$$

We denote by $\bar{\varphi}$ the diffeomorphism of $\mathcal{F}(S,M)$ induced by the action of $\varphi \in \operatorname{Diff}(M)$, so $\bar{\varphi}(f) = \varphi\circ f$ is the push-forward by φ.

Proposition 5. *Given $\omega \in \Omega^p(M)$ and $\alpha \in \Omega^q(S)$, the identity*

$$\bar{\varphi}^* \widehat{\omega \cdot \alpha} = \widehat{(\varphi^* \omega) \cdot \alpha} \tag{7}$$

and its infinitesimal version

$$L_{\bar{X}} \widehat{\omega \cdot \alpha} = \widehat{(L_X \omega) \cdot \alpha} \tag{8}$$

hold for all $\varphi \in \operatorname{Diff}(M)$ and $X \in \mathfrak{X}(M)$.

Proof. Using the expression (1) of the hat pairing and identity (15) from the appendix, we have:

$$\begin{aligned} \bar{\varphi}^* \widehat{\omega \cdot \alpha} &= \bar{\varphi}^* \fint_S \operatorname{ev}^* \omega \wedge \operatorname{pr}^* \alpha = \fint_S (1_S \times \bar{\varphi})^* (\operatorname{ev}^* \omega \wedge \operatorname{pr}^* \alpha) \\ &= \fint_S \operatorname{ev}^* \varphi^* \omega \wedge \operatorname{pr}^* \alpha = \widehat{(\varphi^* \omega) \cdot \alpha}\,, \end{aligned}$$

since $\operatorname{pr} \circ (1_S \times \bar{\varphi}) = \operatorname{pr}$ and $\operatorname{ev} \circ (1_S \times \bar{\varphi}) = \varphi \circ \operatorname{ev}$. □

A similar result is obtained for any smooth map $\eta \in \mathcal{F}(M_1, M_2)$ and its push-forward $\bar{\eta} \colon \mathcal{F}(S, M_1) \to \mathcal{F}(S, M_2)$, $\bar{\eta}(f) = \eta \circ f$:

$$\bar{\eta}^* \widehat{\omega \cdot \alpha} = \widehat{\eta^* \omega \cdot \alpha}\,,$$

for all $\omega \in \Omega^p(M_2)$ and $\alpha \in \Omega^q(S)$.

Lemma 6. *For all vector fields $X \in \mathfrak{X}(M)$, the identity $i_{\bar{X}} \widehat{\omega \cdot \alpha} = \widehat{(i_X \omega) \cdot \alpha}$ holds.*

Proof. The vector field $0_S \times \bar{X}$ on $S \times \mathcal{F}(S, M)$ is ev-related to the vector field X on M, so

$$\begin{aligned} i_{\bar{X}} \widehat{\omega \cdot \alpha} &= i_{\bar{X}} \fint_S \operatorname{ev}^* \omega \wedge \operatorname{pr}^* \alpha = \fint_S i_{0_S \times \bar{X}} (\operatorname{ev}^* \omega \wedge \operatorname{pr}^* \alpha) \\ &= \fint_S \operatorname{ev}^* (i_X \omega) \wedge \operatorname{pr}^* \alpha = \widehat{(i_X \omega) \cdot \alpha}\,. \end{aligned}$$

At step two we use formula (18) from the appendix. □

Right $\operatorname{Diff}(S)$ action. The natural right action of the diffeomorphism group $\operatorname{Diff}(S)$ on $\mathcal{F}(S, M)$ can be transformed into a left action by $\psi \cdot f = f \circ \psi^{-1}$. The infinitesimal action of $Z \in \mathfrak{X}(S)$ is the vector field $\hat{Z}$ on $\mathcal{F}(S, M)$:

$$\widehat{Z}(f) = -Tf \circ Z\,, \quad \forall f \in \mathcal{F}(S, M)\,.$$

We denote by $\widehat{\psi}$ the diffeomorphism of $\mathcal{F}(S, M)$ induced by the action of ψ, so $\widehat{\psi}(f) = f \circ \psi^{-1}$ is the pull-back by ψ^{-1}.

Proposition 7. *Given $\omega \in \Omega^p(M)$ and $\alpha \in \Omega^q(S)$, the identity*

$$\widehat{\psi}^* \widehat{\omega \cdot \alpha} = \widehat{\omega \cdot \psi^* \alpha}$$

and its infinitesimal version

$$L_{\widehat{Z}} \widehat{\omega \cdot \alpha} = \widehat{\omega \cdot L_Z \alpha}$$

hold for all orientation preserving $\psi \in \operatorname{Diff}(S)$ and $Z \in \mathfrak{X}(S)$.

Proof. The obvious identities $\mathrm{ev}\circ(1_S\times\widehat{\psi})=\mathrm{ev}\circ(\psi^{-1}\times 1_{\mathcal F})$, $\mathrm{pr}\circ(1_S\times\widehat{\psi})=\mathrm{pr}$ and $\mathrm{pr}\circ(\psi\times 1_{\mathcal F})=\psi\circ\mathrm{pr}$ are used in the computation

$$\begin{aligned}\widehat{\psi}^*\widehat{\omega\cdot\alpha}&=\widehat{\psi}^*\fint_S \mathrm{ev}^*\omega\wedge\mathrm{pr}^*\alpha=\fint_S(1_S\times\widehat{\psi})^*(\mathrm{ev}^*\omega\wedge\mathrm{pr}^*\alpha)\\&=\fint_S\big((\psi^{-1}\times 1_{\mathcal F})^*\,\mathrm{ev}^*\omega\big)\wedge\mathrm{pr}^*\alpha=\fint_S\mathrm{ev}^*\omega\wedge(\psi\times 1_{\mathcal F})^*\,\mathrm{pr}^*\alpha\\&=\fint_S\mathrm{ev}^*\omega\wedge\mathrm{pr}^*\,\psi^*\alpha=\widehat{\omega\cdot\psi^*\alpha}\,,\end{aligned}$$

together with formula (17) from the appendix at step four. □

Lemma 8. *The identity* $i_{\widehat{Z}}\widehat{\omega\cdot\alpha}=(-1)^p\widehat{\omega\cdot i_Z\alpha}$ *holds for all vector fields* $Z\in\mathfrak{X}(S)$, *if* $\omega\in\Omega^p(M)$.

Proof. The infinitesimal version of the first identity in the proof of Proposition 7 is $T\,\mathrm{ev}\,.(0_S\times\widehat{Z})=T\,\mathrm{ev}\,.(-Z\times 0_{\mathcal{F}(S,M)})$, so we compute:

$$\begin{aligned}i_{\widehat{Z}}\widehat{\omega\cdot\alpha}&=i_{\widehat{Z}}\fint_S\mathrm{ev}^*\omega\wedge\mathrm{pr}^*\alpha=\fint_S i_{0_S\times\widehat{Z}}(\mathrm{ev}^*\omega\wedge\mathrm{pr}^*\alpha)\\&=\fint_S(i_{0_S\times\widehat{Z}}\,\mathrm{ev}^*\omega)\wedge\mathrm{pr}^*\alpha=\fint_S(i_{-Z\times 0_{\mathcal{F}(S,M)}}\,\mathrm{ev}^*\omega)\wedge\mathrm{pr}^*\alpha\\&=\fint_S i_{-Z\times 0_{\mathcal{F}(S,M)}}(\mathrm{ev}^*\omega\wedge\mathrm{pr}^*\alpha)-\fint_S(-1)^p\,\mathrm{ev}^*\omega\wedge i_{-Z\times 0_{\mathcal{F}(S,M)}}\,\mathrm{pr}^*\alpha\\&=(-1)^p\fint_S\mathrm{ev}^*\omega\wedge\mathrm{pr}^*(i_Z\alpha)=(-1)^p\widehat{\omega\cdot i_Z\alpha}\,.\end{aligned}$$

At step two we use formula (18) from the appendix. □

3. Tilda map and hat map

Let $\mathrm{Gr}_k(M)$ be the non-linear Grassmannian (or differentiable Chow variety) of compact oriented k-dimensional submanifolds of M. It is a Fréchet manifold [10] and the tangent space at $N\in\mathrm{Gr}_k(M)$ can be identified with the space of smooth sections of the normal bundle $TN^\perp=(TM|_N)/TN$. The tangent vector at N determined by the section $Y_N\in\Gamma(TM|_N)$ is denoted by $\tilde{Y}_N\in T_N\,\mathrm{Gr}_k(M)$.

The *tilda map* [6] associates to any p-form ω on M a $(p-k)$-form $\tilde{\omega}$ on $\mathrm{Gr}_k(M)$ by:

$$\tilde{\omega}_N(\tilde{Y}_N^1,\dots,\tilde{Y}_N^{p-k})=\int_N i_{Y_N^{p-k}}\cdots i_{Y_N^1}\omega\,. \tag{9}$$

Here all $\tilde{Y}_N^j$ are tangent vectors at $N\in\mathrm{Gr}_k(M)$, i.e. sections of $TN^\perp$ represented by sections Y_N^j of $TM|_N$. Then $i_{Y_N^{p-k}}\cdots i_{Y_N^1}\omega\in\Omega^k(N)$ does not depend on representatives Y_N^j of $\tilde{Y}_N^j$, and integration is well defined since $N\in\mathrm{Gr}_k(M)$ comes with an orientation.

Let S be a compact oriented k-dimensional manifold. The *hat map* is the hat pairing with the constant function $1\in\Omega^0(S)$. It associates to any form $\omega\in\Omega^p(M)$

the form $\widehat{\omega} \in \Omega^{p-k}(\mathcal{F}(S,M))$:

$$\widehat{\omega} = \widehat{\omega \cdot 1} = \fint_S \mathrm{ev}^* \omega\,. \tag{10}$$

On the open subset $\mathrm{Emb}(S,M) \subset \mathcal{F}(S,M)$ of embeddings, formula (2) gives

$$\widehat{\omega}(X_1 \circ f, \ldots, X_{p-k} \circ f) = \int_S f^*(i_{X_{p-k}} \ldots i_{X_1}\omega)\,. \tag{11}$$

Remark 9. The hat map induces a transgression on cohomology spaces

$$H^p(M) \to H^{p-k} = \big(\mathcal{F}(S,M)\big)\,.$$

When S is the circle, then one obtains the usual transgression map with values in the $(p-1)$-th cohomology space of the free loop space of M.

Let π denote the natural map

$$\pi\colon \mathrm{Emb}(S,M) \to \mathrm{Gr}_k(M), \quad \pi(f) = f(S)\,.$$

where the orientation on $f(S)$ is chosen such that the diffeomorphism $f\colon S \to f(S)$ is orientation preserving. The image $\pi(\mathrm{Emb}(S,M))$ is the manifold $\mathrm{Gr}_k^S(M)$ of k-dimensional submanifolds of M of type S. Then $\pi\colon \mathrm{Emb}(S,M) \to \mathrm{Gr}_k^S(M)$ is a principal bundle over $\mathrm{Gr}_k^S(M)$ with structure group $\mathrm{Diff}_+(S)$, the group of orientation preserving diffeomorphisms of S.

Note that there is a natural action of the group $\mathrm{Diff}(M)$ on the non-linear Grassmannian $\mathrm{Gr}_k(M)$ given by $\varphi \cdot N = \varphi(N)$. Let $\tilde{\varphi}$ be the diffeomorphism of $\mathrm{Gr}_k(M)$ induced by the action of $\varphi \in \mathrm{Diff}(M)$. Then $\tilde{\varphi} \circ \pi = \pi \circ \bar{\varphi}$ for the restriction of $\bar{\varphi}(f) = \varphi \circ f$ to a diffeomorphism of $\mathrm{Emb}(S,M) \subset \mathcal{F}(S,M)$. As a consequence, the infinitesimal generators for the $\mathrm{Diff}(M)$ actions on $\mathrm{Gr}_k(M)$ and on $\mathrm{Emb}(S,M)$ are π-related. This means that for all $X \in \mathfrak{X}(M)$, the vector fields $\tilde{X}$ on $\mathrm{Gr}_k(M)$ given by $\tilde{X}(N) = X|_N$ and $\bar{X}$ on $\mathrm{Emb}(S,M)$ given by $\bar{X}(f) = X \circ f$ are π-related.

Proposition 10. *The hat map on* $\mathrm{Emb}(S,M)$ *and the tilda map on* $\mathrm{Gr}_k(M)$ *are related by* $\widehat{\omega} = \pi^*\tilde{\omega}$*, for any* k*-dimensional oriented manifold* S.

Proof. For the proof we use the fact that $\mathfrak{X}(M)$ acts infinitesimally transitive on $\mathrm{Emb}(S,M)$, so $T_f\,\mathrm{Emb}(S,M) = \{X \circ f\colon X \in \mathfrak{X}(M)\}$. With (9) and (11) we compute:

$$\begin{aligned}(\pi^*\tilde{\omega})_f(X_1 \circ f, \ldots, X_{p-k} \circ f) &= \tilde{\omega}_{f(S)}(X_1|_{f(S)}, \ldots, X_{p-k}|_{f(S)}) \\ &= \int_{f(S)} i_{X_{p-k}} \ldots i_{X_1}\omega = \int_S f^*(i_{X_{p-k}} \ldots i_{X_1}\omega) = \widehat{\omega}_f(X_1 \circ f, \ldots, X_{p-k} \circ f)\,,\end{aligned}$$

since $\bar{X}$ and $\tilde{X}$ are π-related. □

From the properties of the hat pairing presented in Proposition 5, Lemma 6 and Theorem 1, a hat calculus follows easily:

Proposition 11. *For any* $\omega \in \Omega^p(M)$*,* $\varphi \in \mathrm{Diff}(M)$*,* $X \in \mathfrak{X}(M)$*, and* $\eta \in \mathcal{F}(M',M)$ *with push-forward* $\bar{\eta}\colon \mathcal{F}(S,M') \to \mathcal{F}(S,M)$*, the following identities hold:*

(1) $\bar{\varphi}^*\widehat{\omega} = \widehat{\varphi^*\omega}$ *and* $\bar{\eta}^*\widehat{\omega} = \widehat{\eta^*\omega}$

(2) $L_{\bar{X}}\widehat{\omega} = \widehat{L_X\omega}$

(3) $i_{\bar{X}}\widehat{\omega} = \widehat{i_X\omega}$

(4) $\mathbf{d}\,\widehat{\omega} = \widehat{\mathbf{d}\,\omega}$.

Remark 12. If S is a manifold with boundary, then the formula 4. above receives an extra term coming from integration over the boundary ∂S as in Proposition 4:

$$\mathbf{d}\,\widehat{\omega} = \widehat{\mathbf{d}\,\omega} + (-1)^{p-k} r_\partial^* \widehat{\omega}^\partial \tag{12}$$

for $\omega \in \Omega^p(M)$. As before, $r_\partial \colon \mathcal{F}(S,M) \to \mathcal{F}(\partial S, M)$ denotes the restriction map on functions and $\omega \in \Omega^p(M) \mapsto \widehat{\omega}^\partial \in \Omega^{p-k+1}(\mathcal{F}(\partial S, M))$.

Now the properties of the tilda calculus follow imediately from Proposition 11.

Proposition 13. [6] *For any* $\omega \in \Omega^p(M)$, $\varphi \in \mathrm{Diff}(M)$ *and* $X \in \mathfrak{X}(M)$, *the following identities hold:*

(1) $\tilde{\varphi}^*\tilde{\omega} = \widetilde{\varphi^*\omega}$

(2) $L_{\tilde{X}}\tilde{\omega} = \widetilde{L_X\omega}$

(3) $i_{\tilde{X}}\tilde{\omega} = \widetilde{i_X\omega}$

(4) $\mathbf{d}\,\tilde{\omega} = \widetilde{\mathbf{d}\,\omega}$.

Proof. We verify the identities 1. and 4. From relation 1. from Proposition 11 we get that

$$\pi^*\tilde{\varphi}^*\tilde{\omega} = \bar{\varphi}^*\pi^*\tilde{\omega} = \bar{\varphi}^*\widehat{\omega} = \widehat{\varphi^*\omega} = \pi^*\widetilde{\varphi^*\omega}\,,$$

and this implies the first identity. Using identity 4. from Proposition 11 we compute

$$\pi^*\mathbf{d}\,\tilde{\omega} = \mathbf{d}\,\pi^*\tilde{\omega} = \mathbf{d}\,\widehat{\omega} = \widehat{\mathbf{d}\,\omega} = \pi^*\widetilde{\mathbf{d}\,\omega}\,,$$

which shows the last identity. □

Hamiltonian formalism for p-branes. In this section we show how the hat calculus appears in the hamiltonian formalism for p-branes and open p-branes [1] [2].

Let S be a compact oriented p-dimensional manifold. The phase space for the p-brane world volume $S \times \mathbb{R}$ is the cotangent bundle $T^*\mathcal{F}(S,M)$, where the canonical symplectic form is twisted. The twisting consists in adding a magnetic term, namely the pull-back of a closed 2-form on the base manifold, to the canonical symplectic form on a cotangent bundle [11]. These twisted symplectic forms appear also in cotangent bundle reduction.

We consider a closed differential form $H \in \Omega^{p+2}(M)$. Since $\dim S = p$, the hat map (10) provides a closed 2-form $\widehat{H}$ on $\mathcal{F}(S,M)$. If $\pi_{\mathcal{F}} : T^*\mathcal{F}(S,M) \to \mathcal{F}(S,M)$ denotes the canonical projection, the twisted symplectic form on $T^*\mathcal{F}(S,M)$ is

$$\Omega_H = -\mathbf{d}\,\Theta_{\mathcal{F}} + \frac{1}{2}\pi_{\mathcal{F}}^*\widehat{H}\,,$$

where $\Theta_{\mathcal{F}}$ is the canonical 1-form on $T^*\mathcal{F}(S,M)$.

For the description of open branes one considers a compact oriented p-dimensional manifold S with boundary ∂S and a submanifold D of M. The phase space is in this case the cotangent bundle $T^*\mathcal{F}_D(S,M)$ over the manifold [13]

$$\mathcal{F}_D(S,M) = \{f : S \to M | f(\partial S) \subset D\}.$$

The twisting of the canonical symplectic form is done with a closed differential form $H \in \Omega^{p+2}(M)$ with $i^*H = \mathbf{d}\, B$ for some $B \in \Omega^{p+1}(D)$, where $i\colon D \to M$ denotes the inclusion. The twisted symplectic form on $T^*\mathcal{F}_D(S,M)$ is

$$\Omega_{(H,B)} = -\mathbf{d}\,\Theta_{\mathcal{F}_D} + \frac{1}{2}\pi^*_{\mathcal{F}_D}(\widehat{H} - \partial^*\widehat{B}^\partial)$$

with $\partial\colon \mathcal{F}_D(S,M) \to \mathcal{F}(\partial S, D)$ the restriction map and $\pi_{\mathcal{F}_D}\colon T^*\mathcal{F}_D(S,M) \to \mathcal{F}_D(S,M)$. To distinguish between the hat calculus for $\mathcal{F}(S,M)$ and the hat calculus for $\mathcal{F}(\partial S, M)$, we denote $\widehat{\ }^\partial\colon \Omega^n(M) \to \Omega^{n-p+1}(\mathcal{F}(\partial S, M))$.

The only thing we have to verify is the closedness of $\widehat{H} - \partial^*\widehat{B}^\partial$. We first notice that (12) implies $\mathbf{d}\,\widehat{H} = \widehat{\mathbf{d}\,H} + r_\partial^*\widehat{H}^\partial$, where $r_\partial\colon \mathcal{F}(S,M) \to \mathcal{F}(\partial S, M)$ denotes the restriction map, and identity 4 from Proposition 11 implies $\widehat{\mathbf{d}\,B}^\partial = \mathbf{d}\,\widehat{B}^\partial$. On the other hand identity 1 from Proposition 11 ensures that $\widehat{i^*H}^\partial = \bar{i}^*\widehat{H}^\partial$, with $\bar{i}\colon \mathcal{F}(\partial S, D) \to \mathcal{F}(\partial S, M)$ denoting the push-forward by $i\colon D \to M$. Knowing that $r_\partial = \bar{i} \circ \partial$, we compute:

$$\mathbf{d}\,\widehat{H} = \widehat{\mathbf{d}\,H} + r_\partial^*\widehat{H}^\partial = \partial^*\bar{i}^*\widehat{H}^\partial = \partial^*\widehat{i^*H}^\partial = \partial^*\widehat{\mathbf{d}\,B}^\partial = \mathbf{d}\,\partial^*\widehat{B}^\partial\,,$$

so the closed 2-form $\widehat{H} - \partial^*\widehat{B}^\partial$ provides a twist for the canonical symplectic form on the cotangent bundle $T^*\mathcal{F}_D(S,M)$.

Non-linear Grassmannians as symplectic manifolds. In this subsection we recall properties of the co-dimension two non-linear Grassmannian as a symplectic manifold.

Proposition 14 ([8])**.** *Let M be a closed m-dimensional manifold with volume form ν. The tilda map provides a symplectic form $\tilde{\nu}$ on* $\mathrm{Gr}_{m-2}(M)$

$$\tilde{\nu}_N(\tilde{X}_N, \tilde{Y}_N) = \int_N i_{Y_N} i_{X_N}\nu\,,$$

for $\tilde{X}_N$ and $\tilde{Y}_N$ sections of $TN^\perp$ determined by sections X_N and Y_N of $TM|_N$.

Proof. The 2-form $\tilde{\nu}$ is closed since $\mathbf{d}\,\tilde{\nu} = \widetilde{\mathbf{d}\,\nu}$ by the tilda calculus. To verify that it is also (weakly) non-degenerate, let X_N be an arbitrary vector field along N such that $\int_N i_{Y_N} i_{X_N}\nu = 0$ for all vector fields Y_N along N. Then X_N must be tangent to N, so $\tilde{X}_N = 0$. □

In dimension $m = 3$ the symplectic form $\tilde{\nu}$ is known as the Marsden–Weinstein symplectic from on the space of unparameterized oriented links, see [12], [3].

Hamiltonian $\mathrm{Diff}_{\mathrm{ex}}(M,\nu)$ action. The action of the group $\mathrm{Diff}(M,\nu)$ of volume preserving diffeomorphisms of M on $\mathrm{Gr}_{m-2}(M)$ preserves the symplectic form $\tilde\nu$:

$$\tilde\varphi^*\tilde\nu = \widetilde{\varphi^*\nu} = \tilde\nu\,, \quad \forall\varphi\in\mathrm{Diff}(M,\nu)\,.$$

The subgroup $\mathrm{Diff}_{\mathrm{ex}}(M,\nu)$ of exact volume preserving diffeomorphisms acts in a hamiltonian way on the symplectic manifold $(\mathrm{Gr}_{m-2}(M),\tilde\nu)$. Its Lie algebra is $\mathfrak{X}_{\mathrm{ex}}(M,\nu)$, the Lie algebra of exact divergence free vector fields, i.e. vector fields X_α such that $i_{X_\alpha}\nu = \mathbf{d}\,\alpha$ for a potential form $\alpha\in\Omega^{m-2}(M)$. The infinitesimal action of X_α is the vector field $\tilde X_\alpha$. By the tilda calculus $\tilde\alpha\in\mathcal{F}(\mathrm{Gr}_{m-2}(M))$ is a hamiltonian function for the hamiltonian vector field $\tilde X_\alpha$:

$$i_{\tilde X_\alpha}\tilde\nu = \widetilde{i_{X_\alpha}\nu} = \widetilde{\mathbf{d}\,\alpha} = \mathbf{d}\,\tilde\alpha\,.$$

It depends on the particular choice of the potential α of X_α. A fixed continuous right inverse $b:\mathbf{d}\,\Omega^{m-2}(M)\to\Omega^{m-2}(M)$ to the differential $\mathbf{d}$ picks up a potential $b(\mathbf{d}\,\alpha)$ of X_α. The corresponding momentum map is:

$$\mathbf{J}:\mathcal{M}\to\mathfrak{X}_{\mathrm{ex}}(M,\nu)^*\,,\quad \langle\mathbf{J}(N),X_\alpha\rangle = \widetilde{b(\mathbf{d}\,\alpha)}(N) = \int_N b(\mathbf{d}\,\alpha)\,.$$

On the connected component $\mathcal{M}$ of $N\in\mathrm{Gr}_{m-2}(M)$, the non-equivariance of $\mathbf{J}$ is measured by the Lie algebra 2-cocycle on $\mathfrak{X}_{\mathrm{ex}}(M,\nu)$

$$\begin{aligned}\sigma_N(X,Y) &= \langle\mathbf{J}(N),[X,Y]^{\mathrm{op}}\rangle - \tilde\nu(\tilde X,\tilde Y)(N) = (b\,\widetilde{\mathbf{d}\,i_Y i_X\nu})(N) - (\widetilde{i_Y i_X\nu})(N)\\ &= (\widetilde{P i_X i_Y\nu})(N) = \int_N P i_X i_Y\nu\,.\end{aligned}$$

Here $P = 1_{\Omega^{m-2}(M)} - b\circ\mathbf{d}$ is a continuous linear projection on the subspace of closed $(m-2)$-forms and $(X,Y)\mapsto[P i_Y i_X\nu]\in H^{m-2}(M)$ is the universal Lie algebra 2-cocycle on $\mathfrak{X}_{\mathrm{ex}}(M,\nu)$ [14]. The cocycle σ_N is cohomologous to the Lichnerowicz cocycle

$$\sigma_\eta(X,Y) = \int_M \eta(X,Y)\nu\,, \tag{13}$$

where η is a closed 2-form Poincaré dual to N [15].

If ν is an integral volume form, then σ_N is integrable [8]. The connected component $\mathcal{M}$ of $\mathrm{Gr}_{m-2}(M)$ is a coadjoint orbit of a 1-dimensional central Lie group extension of $\mathrm{Diff}_{\mathrm{ex}}(M,\nu)$ integrating σ_N, and $\tilde\nu$ is the Kostant-Kirillov-Souriau symplectic form. [6].

4. Bar map

When a volume form μ on the compact k-dimensional manifold S is given, one can associate to each differential p-form on M a differential p-form on $\mathcal{F}(S,M)$

$$\bar\omega(Y_f^1,\dots,Y_f^p) = \int_S \omega(Y_f^1,\dots,Y_f^p)\mu\,,\quad \forall Y_f^i\in T_f\mathcal{F}(S,M)\,,$$

where $\omega(Y_f^1,\dots,Y_f^p)\colon x\mapsto \omega_{f(x)}(Y_f^1(x),\dots,Y_f^p(x))$ defines a smooth function on S. In this way a *bar map* is defined. Formula (2) assures that this bar map is just the hat pairing of differential forms on M with the volume form μ

$$\bar\omega = \widehat{\omega\cdot\mu} = \fint_S \mathrm{ev}^*\,\omega\wedge \mathrm{pr}^*\,\mu\,. \tag{14}$$

From the properties of the hat pairing presented in Proposition 5, Lemma 6 and Theorem 1, one can develop a bar calculus.

Proposition 15. *For any $\omega\in\Omega^p(M)$, $\varphi\in\mathrm{Diff}(M)$ and $X\in\mathfrak{X}(M)$, the following identities hold:*

(1) $\bar\varphi^*\bar\omega = \overline{\varphi^*\omega}$
(2) $L_{\bar X}\bar\omega = \overline{L_X\omega}$
(3) $i_{\bar X}\bar\omega = \overline{i_X\omega}$
(4) $\mathbf{d}\,\bar\omega = \overline{\mathbf{d}\,\omega}$.

$\mathcal{F}(S,M)$ **as symplectic manifold.** Let (M,ω) be a connected symplectic manifold and S a compact k-dimensional manifold with a fixed volume form μ, normalized such that $\int_S\mu=1$. The following fact is well known:

Proposition 16. *The bar map provides a symplectic form $\bar\omega$ on $\mathcal{F}(S,M)$:*

$$\bar\omega_f(X_f,Y_f) = \int_S \omega(X_f,Y_f)\mu\,.$$

Proof. That $\bar\omega$ is closed follows from the bar calculus: $\mathbf{d}\,\bar\omega=\overline{\mathbf{d}\,\omega}=0$. The (weakly) non-degeneracy of $\bar\omega$ can be verified as follows. If the vector field X_f on M along S is non-zero, then $X_f(x)\neq 0$ for some $x\in S$. Because ω is non-degenerate, one can find another vector field Y_f along f such that $\omega(X_f,Y_f)$ is a bump function on S. Then $\bar\omega(X_f,Y_f)=\int_S\omega(X_f,Y_f)\mu\neq 0$, so X_f does not belong to the kernel of $\bar\omega$, thus showing that the kernel of $\bar\omega$ is trivial. □

Hamiltonian action on M. Let G be a Lie group acting in a hamiltonian way on M with momentum map $J\colon M\to\mathfrak{g}^*$. Then $\mathcal{F}(S,M)$ inherits a G-action: $(g\cdot f)(x)=g\cdot(f(x))$ for any $x\in S$. The infinitesimal generator is $\xi_{\mathcal{F}}=\bar\xi_M$ for any $\xi\in\mathfrak{g}$, where ξ_M denotes the infinitesimal generator for the G-action on M. The bar calculus shows quickly that G acts in a hamiltonian way on $\mathcal{F}(S,M)$ with momentum map

$$\mathbf{J}=\bar J\colon \mathcal{F}(S,M)\to\mathfrak{g}^*\,,\quad \bar J(f)=\int_S (J\circ f)\mu\,,\quad \forall f\in\mathcal{F}(S,M)\,.$$

Indeed, for all $\xi\in\mathfrak{g}$

$$i_{\xi_{\mathcal{F}}}\bar\omega = i_{\bar\xi_M}\bar\omega = \overline{i_{\xi_M}\omega} = \overline{\mathbf{d}\,\langle J,\xi\rangle} = \mathbf{d}\,\langle \bar J,\xi\rangle\,.$$

Let M be connected and let σ be the $\mathbb{R}$-valued Lie algebra 2-cocycle on $\mathfrak{g}$ measuring the non-equivariance of J, i.e.

$$\sigma(\xi,\eta)=\langle J(x),[\xi,\eta]\rangle-\omega(\xi_M,\eta_M)(x)\,,\quad x\in M\,,$$

(both terms are hamiltonian function for the vector field $[\xi,\eta]_M = -[\xi_M,\eta_M]$). Then the non-equivariance of $\mathbf{J} = \bar{J}$ is also measured by σ: for all $f \in \mathcal{F}(S,M)$

$$\langle \bar{J}(f), [\xi,\eta]\rangle - \bar{\omega}(\xi_{\mathcal{F}},\eta_{\mathcal{F}})(f) = \overline{\langle J,[\xi,\eta]\rangle}(f) - \overline{\omega(\xi_M,\eta_M)}(f) = \sigma(\xi,\eta)\,.$$

Hamiltonian $\mathrm{Diff}_{\mathrm{ham}}(M,\omega)$ action. The action of the group $\mathrm{Diff}(M,\omega)$ of symplectic diffeomorphisms preserves the symplectic form $\bar{\omega}$:

$$\bar{\varphi}^*\bar{\omega} = \overline{\varphi^*\omega} = \bar{\omega}\,, \quad \forall \varphi \in \mathrm{Diff}(M,\omega)\,.$$

The subgroup $\mathrm{Diff}_{\mathrm{ham}}(M,\omega)$ of hamiltonian diffeomorphisms of M acts in a hamiltonian way on the symplectic manifold $\mathcal{F}(S,M)$. The infinitesimal action of $X_h \in \mathfrak{X}_{\mathrm{ham}}(M,\omega)$, $h \in \mathcal{F}(M)$, is the hamiltonian vector field $\bar{X}_h$ on $\mathcal{F}(S,M)$ with hamiltonian function $\bar{h}$. This follows by the bar calculus:

$$\mathbf{d}\,\bar{h} = \overline{\mathbf{d}\,h} = \overline{i_{X_h}\omega} = i_{\bar{X}_h}\bar{\omega}\,.$$

The hamiltonian function $\bar{h}$ of $\bar{X}_h$ depends on the particular choice of the hamiltonian function h. To solve this problem we fix a point $x_0 \in M$ and we choose the unique hamiltonian function h with $h(x_0) = 0$, since M is connected. The corresponding momentum map is

$$\mathbf{J} : \mathcal{F}(S,M) \to \mathfrak{X}_{\mathrm{ham}}(M,\omega)^*\,, \quad \langle \mathbf{J}(f), X_h\rangle = \bar{h}(f) = \int_S (h\circ f)\mu\,.$$

The Lie algebra 2-cocycle on $\mathfrak{X}_{\mathrm{ham}}(M,\omega)$ measuring the non-equivariance of the momentum map is

$$\sigma(X,Y) = -\omega(X,Y)(x_0)\,,$$

by the bar calculus

$$\begin{aligned}\sigma(X,Y)(f) &= \langle \mathbf{J}(f), [X,Y]^{\mathrm{op}}\rangle - \bar{\omega}(X_{\mathcal{F}},Y_{\mathcal{F}})(f)\\ &= \overline{\omega(X,Y) - \omega(X,Y)(x_0)}(f) - \bar{\omega}(\bar{X},\bar{Y})(f) = -\omega(X,Y)(x_0)\,.\end{aligned}$$

This is a Lie algebra cocycle describing the central extension

$$0 \to \mathbb{R} \to \mathcal{F}(M) \to \mathfrak{X}_{\mathrm{ham}}(M,\omega) \to 0$$

where $\mathcal{F}(M)$ is enowed with the canonical Poisson bracket. A group cocycle on $\mathrm{Diff}_{\mathrm{ham}}(M,\omega)$ integrating the Lie algebra cocycle σ if ω exact is studied in [9].

Hamiltonian $\mathrm{Diff}_{\mathrm{ex}}(S,\mu)$ action. The (left) action of the group $\mathrm{Diff}(S,\mu)$ of volume preserving diffeomorphisms preserves the symplectic form $\bar{\omega}$:

$$\widehat{\psi}^*\bar{\omega} = \widehat{\psi}^*\widehat{\omega\cdot\mu} = \widehat{\omega\cdot\psi^*\mu} = \widehat{\omega\cdot\mu} = \bar{\omega}\,, \quad \forall\psi \in \mathrm{Diff}(S,\mu)\,.$$

The subgroup $\mathrm{Diff}_{\mathrm{ex}}(S,\mu)$ of exact volume preserving diffeomorphisms acts in a hamiltonian way on the symplectic manifold $\mathcal{F}(S,M)$. The infinitesimal action of the exact divergence free vector field $X_\alpha \in \mathfrak{X}_{\mathrm{ex}}(S,\mu)$ with potential form $\alpha \in \Omega^{k-2}(S)$ is the hamiltonian vector field $\widehat{X}_\alpha$ on $\mathcal{F}(S,M)$ with hamiltonian function $\widehat{\omega\cdot\alpha}$. Indeed, from $i_{X_\alpha}\mu = \mathbf{d}\,\alpha$ follows by the hat calculus that

$$\mathbf{d}\,(\widehat{\omega\cdot\alpha}) = \widehat{\mathbf{d}\,\omega\cdot\alpha} + \widehat{\omega\cdot\mathbf{d}\,\alpha} = \widehat{\omega\cdot i_{X_\alpha}\mu} = i_{\widehat{X}_\alpha}\widehat{\omega\cdot\mu} = i_{\widehat{X}_\alpha}\bar{\omega}\,.$$

If the symplectic form ω is exact, then the corresponding momentum map is

$$\mathbf{J}\colon \mathcal{F}(S,M) \to \mathfrak{X}_{\mathrm{ex}}(S,\mu)^*, \quad \langle \mathbf{J}(f), X_\alpha \rangle = \widehat{(\omega \cdot \alpha)}(f) = \int_S f^*\omega \wedge \alpha.$$

It takes values in the regular part of $\mathfrak{X}_{\mathrm{ex}}(S,\mu)^*$, which can be identified with $\mathbf{d}\,\Omega^1(S)$, so we can write $\mathbf{J}(f) = f^*\omega$ under this identification.

In general the hamiltonian function $\widehat{\omega \cdot \alpha}$ of $\widehat{X}_\alpha$ depends on the particular choice of the potential form α of X_α. To fix this problem we consider as in Section 3 a continuous right inverse $b : \mathbf{d}\,\Omega^{m-2}(M) \to \Omega^{m-2}(M)$ to the differential $\mathbf{d}$, so $b(\mathbf{d}\,\alpha)$ is a potential for X_α. The corresponding momentum map is

$$\mathbf{J} : \mathcal{F}(S,M) \to \mathfrak{X}_{\mathrm{ex}}(S,\mu)^*\,, \quad \langle \mathbf{J}(f), X_\alpha \rangle = (\widehat{\omega \cdot b\mathbf{d}\,\alpha})(f) = \int_S f^*\omega \wedge b(\mathbf{d}\,\alpha)\,.$$

On a connected component $\mathcal{F}$ of $\mathcal{F}(S,M)$, the non-equivariance of $\mathbf{J}$ is measured by the Lie algebra 2-cocycle

$$\begin{aligned}\sigma_{\mathcal{F}}(X,Y) &= \langle \mathbf{J}(f), [X,Y] \rangle - \bar{\omega}(\hat{X},\hat{Y})(f) = (\omega \cdot b\mathbf{d}\, i_Y i_X \mu)\hat{}\,(f) - (\omega \cdot i_Y i_X \mu)\hat{}\,(f)\\ &= (\omega \cdot P i_X i_Y \mu)\hat{}\,(f) = \int_S f^*\omega \wedge P i_X i_Y \mu\end{aligned}$$

on the Lie algebra of exact divergence free vector fields, for $P = 1 - b\mathbf{d}$ the projection on the subspace of closed $(m-2)$-forms. It does not depend on $f \in \mathcal{F}$, because the cohomology class $[f^*\omega] \in H^2(S)$ does not depend on the choice of f. The cocycle $\sigma_{\mathcal{F}}$ is cohomologous to the Lichnerowicz cocycle $\sigma_{f^*\omega}$ defined in (13) [15]. Since $\int_S \mu = 1$, the cocycle $\sigma_{\mathcal{F}}$ is integrable if and only if the cohomology class of $f^*\omega$ is integral [8].

Remark 17. The two equivariant momentum maps on the symplectic manifold $\mathcal{F}(S,M)$, for suitable central extensions of the hamiltonian group $\mathrm{Diff}_{\mathrm{ham}}(M,\omega)$ and of the group $\mathrm{Diff}_{\mathrm{ex}}(S,\mu)$ of exact volume preserving diffeomorphisms, form the dual pair for ideal incompressible fluid flow [12] [4].

5. Appendix: Fiber integration

Chapter VII in [5] is devoted to the concept of integration over the fiber in locally trivial bundles. We particularize this fiber integration to the case of trivial bundles $S \times M \to M$, listing its main properties without proofs.

Let S be a compact k-dimensional manifold. Fiber integration over S assigns to $\omega \in \Omega^n(S \times M)$ the differential form $\fint_S \omega \in \Omega^{n-k}(M)$ defined by

$$(\fint_S \omega)(x) = \int_S \omega_x \in \Lambda^{n-k} T_x^* M\,, \quad \forall x \in M\,,$$

where $\omega_x \in \Omega^k(S, \Lambda^{n-k} T_x^* M)$ is the retrenchment of ω to the fiber over x:

$$\langle \omega_x(Z_s^1, \dots, Z_s^{n-k}), X_x^1 \wedge \dots \wedge X_x^k \rangle = \omega_{(s,x)}(X_x^1, \dots, X_x^k, Z_s^1, \dots, Z_s^{n-k})$$

for all $X_x^i \in T_x M$ and $Z_s^j \in T_s S$.

The properties of the fiber integration used in the text are special cases of the Propositions (VIII) and (X) in [5]:

- Pull-back of fiber integrals:

$$f^* \fint_S \omega = \fint_S (1_S \times f)^* \omega\,, \quad \forall f \in \mathcal{F}(M', M)\,, \tag{15}$$

with infinitesimal version

$$L_X \fint_S \omega = \fint_S L_{0_S \times X} \omega\,, \quad \forall X \in \mathfrak{X}(M)\,. \tag{16}$$

- Invariance under pull-back by orientation preserving diffeomorphisms of S:

$$\fint_S (\varphi \times 1_M)^* \omega = \fint_S \omega\,, \quad \forall \varphi \in \mathrm{Diff}_+(S)\,, \tag{17}$$

with infinitesimal version $\fint_S L_{Z \times 0_M} \omega = 0\,, \quad \forall Z \in \mathfrak{X}(S)$.

- Insertion of vector fields into fiber integrals:

$$i_X \fint_S \omega = \fint_S i_{0_S \times X} \omega, \quad \forall X \in \mathfrak{X}(M). \tag{18}$$

- Integration along boundary free manifolds commutes with differentiation. When ∂S denotes the boundary of the k-dimensional compact manifold S and $i_\partial \colon \partial S \to S$ the inclusion,

$$\mathbf{d} \fint_S \beta - \fint_S \mathbf{d}\, \beta = (-1)^{n-k} \fint_{\partial S} (i_\partial \times 1_M)^* \beta \tag{19}$$

holds for any differential n-form β on $S \times M$.

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