ARCHIVUM MATHEMATICUM (BRNO)
Tomus 45 (2009), 79–82

LOWER BOUNDS FOR EXPRESSIONS OF LARGE SIEVE TYPE

Jan-Christoph Schlage-Puchta

Abstract. We show that the large sieve is optimal for almost all exponential sums.

Let a_n, $1 \leq n \leq N$ be complex numbers, and set $S(\alpha) = \sum_{n\leq N} a_n e(n\alpha)$, where $e(\alpha) = \exp(2\pi i\alpha)$. Large Sieve inequalities aim at bounding the number of places where this sum can be extraordinarily large, the basic one being the bound

$$\sum_{q\leq Q} \sum_{\substack{1\leq a\leq q\\(a,q)=1}} \left|S\Big(\frac{a}{q}\Big)\right|^2 \leq (N+Q^2) \sum_{n\leq N} |a_n|^2$$

(see e.g. [3] for variations and applications). P. Erdős and A. Rényi [1] considered lower bounds of the same type, in particular they showed that the bound

$$\sum_{q\leq Q} \sum_{(a,q)=1} \left|S\Big(\frac{a}{q}\Big)\right|^2 \ll N \sum_{n\leq N} |a_n|^2\,, \tag{1}$$

valid for $Q \ll \sqrt{N}$, is wrong for almost all choices of coefficients $a_n \in \{1,-1\}$, provided that $Q > C\sqrt{N}\log N$, and that the standard probabilistic argument fails to decide whether (1) is true in the range $\sqrt{N} < Q < \sqrt{N}\log N$. In this note, we show that (1) indeed fails throughout this range.

Theorem 1. *Let $S(\alpha)$ be as above. Then*

$$\sum_{q\leq Q} \sum_{(a,q)=1} \left|S\Big(\frac{a}{q}\Big)\right|^2 \geq \varepsilon Q^2 \sum_{n\leq N} |a_n|^2 \tag{2}$$

holds true with probability tending to 1 provided ε tends to 0, and Q^2/N tends to infinity.

Our approach differs from [1] in so far as we first prove an unconditional lower bound, which involves an awkward expression, and show then that almost always this expression is small. We show the following.

2000 *Mathematics Subject Classification*: primary 11N35; secondary 30B20.
Key words and phrases: Large sieve, random series.
Received January 8, 2008, revised January 2009. Editor R. Kučera.

Lemma 1. *Let $S(\alpha)$ be as above, and define*

$$M(x) = \sup_{\mathfrak{m}} \frac{\int\limits_{\mathfrak{m}} |S(u)|^2\, du}{\int\limits_0^1 |S(u)|^2\, du},$$

where $\mathfrak{m}$ ranges over all measurable subsets of $[0,1]$ of measure x. Then for any real parameter $A > 1$ we have the estimate

$$\sum_{q\leq Q}\sum_{(a,q)=1}\left|S\Big(\frac{a}{q}\Big)\right|^2 \geq \Big(\frac{Q^2}{A}\Big(1 - M\Big(\frac{1}{A}\Big)\Big) - 6\pi NA\Big)\sum_{n\leq N}|a_n|^2\,. \tag{3}$$

Proof. Our proof adapts Gallagher's proof of an upper bound large sieve [2]. For every $f \in C^1([0,1])$, we have

$$f(1/2) = \int\limits_0^1 f(u)\, du + \int\limits_0^{1/2} uf'(u)\, du - \int\limits_{1/2}^1 (1-u)f'(u)\, du\,.$$

Putting $f(u) = |S(u)|^2$, and using the linear substitution $u \mapsto (\alpha - \delta/2) + \delta u$, we obtain for every $\delta > 0$ and any $\alpha \in [0,1]$

$$\begin{aligned}|S(\alpha)|^2 = \frac{1}{\delta}\int\limits_{\alpha-\delta/2}^{\alpha+\delta/2} |S(u)|^2\, du &+ \frac{1}{\delta}\int\limits_{\alpha-\delta/2}^{\alpha} \big(\delta/2 - |u-\alpha|\big)\big(S'(u)S(-u) - S(u)S'(-u)\big)\, du\\ &- \frac{1}{\delta}\int\limits_{\alpha}^{\alpha+\delta/2} \big(\delta/2 - |u-\alpha|\big)\big(S'(u)S(-u) - S(u)S'(-u)\big)\, du\,.\end{aligned}$$

We have $|S(u)| = |S(-u)|$ and $|S'(-u)| = |S'(u)|$, thus $|S'(u)S(-u) - S(u)S'(-u)| \leq 2|S(u)S'(u)|$, and we obtain

$$\begin{aligned}|S(\alpha)|^2 &\geq \frac{1}{\delta}\int\limits_{\alpha-\delta/2}^{\alpha+\delta/2} |S(u)|^2\, du - \frac{1}{\delta}\int\limits_{\alpha-\delta/2}^{\alpha+\delta/2} 2\Big(\frac{1}{2} - \frac{|u-\alpha|}{\delta}\Big)|S(u)S'(u)|\, du\\ &\geq \frac{1}{\delta}\int\limits_{\alpha-\delta/2}^{\alpha+\delta/2} |S(u)|^2\, du - \int\limits_{\alpha-\delta/2}^{\alpha+\delta/2} |S(u)S'(u)|\, du\,.\end{aligned}$$

We now set $\delta = A/Q^2$. We can safely assume that $\delta < \frac{1}{2}$, since our claim would be trivial otherwise. Summing over all fractions $\alpha = \frac{a}{q}$ with $q \leq Q$, $(a,q)=1$, we get

$$\sum_{q\leq Q}\sum_{(a,q)=1}\Big|S\Big(\frac{a}{q}\Big)\Big|^2 \geq \frac{Q^2}{A}\int_0^1 |S(u)|^2 du - \frac{Q^2}{A}\int_{m(Q,A)} |S(u)|^2\,du \;-\; \int_0^1 R(u)|S(u)S'(u)|\,du\,, \tag{4}$$

where

$$R(u) = \#\Big\{a, q : (a,q)=1, q\leq Q, \Big|u-\frac{a}{q}\Big| \leq \frac{A}{Q^2}\Big\}\,,$$

and

$$m(Q,A) = \big\{u \in [0,1] : R(u) = 0\big\}\,.$$

To bound $R(u)$, let $\frac{a_1}{q_1} < \frac{a_2}{q_2} < \cdots < \frac{a_k}{q_k}$ be the list of all fractions with $q_i \leq Q$, $\big|u - \frac{a_i}{q_i}\big| \leq \frac{A}{Q^2}$. We have for $i \neq j$ the bound

$$\Big|\frac{a_i}{q_i} - \frac{a_j}{q_j}\Big| \geq \frac{1}{q_i q_j} \geq \frac{1}{Q^2}\,,$$

that is, the fractions $\frac{a_1}{q_1}, \dots, \frac{a_k}{q_k}$ form a set of points with distance $> \frac{1}{Q^2}$ in an interval of length $\frac{2A}{Q^2}$. There can be at most $2A+1$ such points, hence, $R(u) \leq 3A$.

Next, we bound $|m(Q,A)|$. By Dirichlet's theorem, we have that for each real number $\alpha \in [0,1]$ there exists some $q \leq Q$ and some a, such that $|\alpha - \frac{a}{q}| \leq \frac{1}{qQ}$. If $\alpha \in m(Q,A)$, we must have $\frac{1}{qQ} > \frac{A}{Q^2}$, that is, $q < Q/A$. Hence, we obtain

$$\begin{aligned}|m(Q,A)| &\leq \Big|\bigcup_{q<Q/A}\bigcup_{(a,q)=1}\Big[\frac{a}{q}-\frac{1}{qQ},\frac{a}{q}+\frac{1}{qQ}\Big]\setminus\Big[\frac{a}{q}-\frac{A}{Q^2},\frac{a}{q}+\frac{A}{Q^2}\Big]\Big|\\ &\leq \sum_{q<Q/A}\frac{\varphi(q)(2Q-2Aq)}{qQ^2} \leq \frac{1}{Q^2}\int_0^{Q/A}\big(2Q-2At\big)\,dt = \frac{1}{A}\,.\end{aligned}$$

We can now estimate the right hand side of (4). The first summand is $\frac{Q^2}{A}\sum_{n\leq N}|a_n|^2$, while the second is by definition at most $\frac{Q^2}{A}M(1/A)$. For the third we apply the Cauchy-Schwarz-inequality to obtain

$$\begin{aligned}\Big(\int_0^1 |S(u)S'(u)|\,du\Big)^2 &\leq \Big(\int_0^1 |S(u)|^2\,du\Big)\Big(\int_0^1 |S'(u)|^2\,du\Big)\\ &= \Big(\sum_{n\leq N}|a_n^2|\Big)\Big(\sum_{n\leq N}(2\pi n)^2|a_n^2|\Big)\\ &\leq (2\pi N)^2\Big(\sum_{n\leq N}|a_n^2|\Big)^2.\end{aligned}$$

Hence, the last term in (4) is bounded above by $3A(2\pi N)\sum_{n\leq N}|a_n|^2$, and inserting our bounds into (4) yields the claim of our lemma. □

Now we deduce Theorem 1. Let $S(\alpha)$ be a random sum in the sense that the coefficients $a_n \in \{1,-1\}$ are chosen at random. We compute the expectation of the fourth moment of $S(\alpha)$.

$$\begin{aligned}\mathrm{E}\int_0^1 |S(u)|^4\,du &= \mathrm{E}\sum_{\substack{\mu_1+\mu_2=\nu_1+\nu_2\\ \mu_1,\mu_2,\nu_1,\nu_2\leq N}} a_{\nu_1}a_{\nu_2}a_{\mu_1}a_{\mu_2}\\ &= \#\big\{\mu_1,\mu_2,\nu_1,\nu_2\leq N : \{\mu_1,\mu_2\}=\{\nu_1,\nu_2\}\big\}\\ &= 2N^2-N.\end{aligned}$$

If $m\subseteq[0,1]$ is of measure x, then $\int_m |S(u)|^2\;du \leq \sqrt{x}\big(\int_m |S(u)|^4\;du\big)^{1/2}$, thus $\mathrm{E}M(x)\leq\sqrt{2x}$. In particular, we have $M(x)\leq 1/2$ with probability $\geq 1-\sqrt{8x}$. Let $\delta>0$ be given, and set $A=8\delta^{-2}$. Then with probability $\geq 1-\delta$ we have $M(1/A)\leq 1/2$, and (3) becomes

$$\begin{aligned}\sum_{q\leq Q}\sum_{(a,q)=1}\left|S\left(\frac{a}{q}\right)\right|^2 &\geq \left(\frac{Q^2\delta^2}{16}-48\delta^{-2}\pi N\right)\sum_{n\leq N}|a_n|^2\\ &\geq \frac{Q^2\delta^2}{32}\sum_{n\leq N}|a_n|^2,\end{aligned}$$

provided that $Q^2 > 1536\delta^4 N$. Hence, for fixed ϵ, the relation (2) becomes true with probability $1-\sqrt{1024\epsilon}$, provided that Q^2/N is sufficiently large. Hence, our claim follows.

I would like to thank the referee for improving the quality of this paper.

References

[1] Erdős, P., Rényi, A., *Some remarks on the large sieve of Yu. V. Linnik*, Ann. Univ. Sci. Budapest. Eötvös Sect. Math. **11** (1968), 3–13.

[2] Gallagher, P. X., *The large sieve*, Mathematika **14** (1967), 14–20.

[3] Montgomery, H. L., *Multiplicative Number Theory*, Lecture Notes in Math. **227** (1971).

Department of Pure Mathematics and Computer Algebra
Universiteit Gent
Krijgslaan 281, 9000 Gent, Belgium
E-mail: `jcsp@cage.ugent.be`