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ON OSCILLATION CRITERIA FOR THIRD ORDER NONLINEAR DELAY DIFFERENTIAL EQUATIONS

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Abstract. In this paper we are concerned with the oscillation of third order nonlinear delay differential equations of the form

$$\left(r_2(t)\left(r_1(t)x'\right)'\right)' + p(t)x' + q(t)f(x(g(t))) = 0.$$

We establish some new sufficient conditions which insure that every solution of this equation either oscillates or converges to zero.

1. Introduction

In this paper we consider nonlinear third order functional differential equations of the form

$$\left(r_2(t)(r_1(t)x')'\right)' + p(t)x' + q(t)f(x(g(t))) = 0\,, \tag{1.1}$$

where r_1, r_2, p, $q \in C(I,\mathbb{R})$, $I = [t_0,\infty) \subset \mathbb{R}$, $t_0 \geq 0$ is a constant such that $r_1 > 0$, $r_2 > 0$, $p(t) \geq 0$, $q(t) \geq 0$, $q(t) \not\equiv 0$ in the neighborhood of ∞, $g \in C^1(I,\mathbb{R})$ satisfies $g(t) < t$, $g'(t) \geq 0$, and $g(t) \to \infty$ as $t \to \infty$ and $f \in C(R,R)$ such that f is nondecreasing, $xf(x) > 0$ for $x \neq 0$.

We consider only those solutions of Eq. (1.1) which are defined and nontrivial for all sufficiently large t. Such a solution is called oscillatory if it has arbitrarily large zeros, otherwise it is called nonoscillatory.

Note that if x is a solution of Eq. (1.1), then $-x$ is a solution of

$$\left(r_2(t)(r_1(t)x')'\right)' + p(t)x' + q(t)f^*(x(g(t))) = 0\,,$$

where $f^*(x) = -f(-x)$ and $xf^*(x) > 0$ for all $x \neq 0$. Since f^* and f are of the same class, we may restrict our attention only to a positive solution of Eq. (1.1) whenever a nonoscillatory solution of Eq. (1.1) is concerned.

In recent years, the oscillatory and asymptotic behavior of differential equations and their applications have been and still are receiving intensive attention. In fact, there are several monographs and hundreds of research papers for ordinary and functional differential equations, see for example the monographs Agarwal et al. [1]–[2], Erbe et al. [8], Gyori and Ladas [10], and Swanson [16].

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Determining oscillation criteria in particularly for second order differential equations has received a great deal of attention in the last few years. Compared to second order differential equations, the study of oscillation and asymptotic behavior of third order differential equations has received considerably less attention in the literature. We obtain some new results in this paper are motivated by recent of [3, 4, 5, 9, 15, 17] and insure that every solution of Eq. (1.1) is oscillatory or converges to zero. For general interest on oscillation results we refer, for example, to Erbe [7], Grace et al. [9], Parhi and Das [11], Philos and Sficas [13], Seman [14], Tiryaki and Yaman [18], and the references cited therein.

In this section we state and prove some lemmas which we will use in the proof of our main results.

For the sake of brevity, we define

$$L_0x(t) = x(t)\,,\quad L_ix(t) = r_i(t)\left(L_{i-1}x(t)\right)'\,,\quad i = 1,2\,,$$

$$L_3x(t) = \left(L_2x(t)\right)' \quad \text{for} \quad t \in I\,.$$

So Eq. (1.1) can be written as

$$L_3x(t) + \frac{p(t)}{r_1(t)}L_1x(t) + q(t)f(x(g(t))) = 0\,.$$

Define the functions

$$R_1(t,s) = \int_s^t \frac{du}{r_1(u)}, \qquad R_2(t,s) = \int_s^t \frac{du}{r_2(u)}, \qquad \text{and}$$
$$R_{12}(t,s) = \int_s^t \frac{1}{r_1(\tau)}\int_s^\tau \frac{du}{r_2(u)}d\tau\,, \quad t_0 \le s \le t < \infty\,.$$

We assume that

$$R_1(t,t_0) \to \infty \quad \text{as} \quad t \to \infty\,, \tag{1.2}$$

$$R_2(t,t_0) \to \infty \quad \text{as} \quad t \to \infty\,, \tag{1.3}$$

and

$$R_2(t,t_0) < \infty \quad \text{as} \quad t \to \infty\,. \tag{1.4}$$

Moreover we shall assume that the function f satisfies conditions:

$$-f(-uv) \ge f(uv) \ge f(u)f(v) \quad \text{for} \quad uv > 0\,, \tag{1.5}$$

$$\frac{f(u)}{u} \ge K > 0\,, \quad K \quad \text{is a real constant,} \quad u \ne 0\,, \tag{1.6}$$

and

$$\frac{u}{f(u)} \to 0 \quad \text{as} \quad u \to 0\,. \tag{1.7}$$

Definition 1. The Eq. (1.1) is called superlinear if the function f for every $\epsilon > 0$ satisfies

$$\int_{\pm\epsilon}^{\pm\infty} \frac{du}{f(u)} < \infty\,, \tag{1.8}$$

and Eq. (1.1) is called sublinear if f satisfies

$$\int_{0}^{\pm\epsilon} \frac{du}{f(u)} < \infty \quad \text{for every} \quad \epsilon > 0\,. \tag{1.9}$$

Let us give examples of the functions which satisfy the conditions (1.5) and (1.8) or (1.9).

Example 1. The functions f_1 and $f_2\colon R \to R$, where $f_1(u) = |u|^{\alpha} \operatorname{sgn} u$, $\alpha > 0$ and $f_2(u) = \dfrac{|u|^{2\alpha} \operatorname{sgn} u}{1 + |u|^{\alpha}}$, $\alpha > 0$ are continuous on R, satisfy $uf(u) > 0$ for $u \neq 0$ and conditions nondecreasing of f and (1.5). Further, function f_1 satisfies (1.8) for $\alpha > 1$ and (1.9) for $0 < \alpha < 1$. The function f_2 satisfies (1.8) for $\alpha > 1$.

Lemma 1. *Suppose that*

$$\left(r_2(t) z'\right)' + \frac{p(t)}{r_1(t)} z = 0$$

is nonoscillatory. If x is a nonoscillatory solution of (1.1) *on* $[T, \infty)$, $T \geq t_0$, *then there exists a* $t_1 \in [T, \infty)$ *such that either* $x(t) L_1 x(t) > 0$ *or* $x(t) L_1 x(t) < 0$ *for all* $t \geq t_1$.

The reader can refer to [17, Lemma 1] for the proof of Lemma 1.

Lemma 2. *Let ρ_2 be a sufficiently smooth positive function defined on* $[t_{0,}\infty)$, *set*

$$\phi(t) = r_1(t) \left(r_2(t) \rho_2'(t)\right)' + \rho_2(t) p(t)\ ,$$

and (1.6) *hold. Suppose that there exists a* $t_1 \geq T \geq t_0$ *such that*

$$\rho_2'(t) \geq 0 =, \quad \phi(t) \geq 0\,,$$

$$\int_{t_1}^{\infty} \left(K\rho_2(s) q(s) - \phi'(s)\right) ds = \infty\,, \tag{1.10}$$

where $K\rho_2(t) q(t) - \phi'(t) \geq 0$ *for all* $t \in [t_1, \infty)$ *and not identically zero in any subinterval of* $[t_1, \infty)$. *If* (1.2) *holds and x be a nonoscillatory solution of Eq.* (1.1) *which satisfies* $x(t) L_1 x(t) \leq 0$ *for all* $t \geq t_1$, *then* $\lim\limits_{t\to\infty} x(t) = 0$.

The reader can refer to [4, Lemma 2.4] for the proof of Lemma 2.

Remark 1. When

$$\phi'(t) \leq 0 \tag{1.11}$$

in Lemma 2, we can take

$$\int^{\infty} \rho_2(s) q(s) ds = \infty \tag{1.12}$$

to replace (1.10). Hence the condition (1.6) fails.

Lemma 3. *Let the assumption* (1.3) *hold. If x is a nonoscillatory solution of Eq.* (1.1) *which satisfies $x(t)\, L_1 x(t) \geq 0$ for all large t, then there exists a $t_1 \geq t_0$ such that*

$$L_0 x(t)\, L_k x(t) > 0\,, \quad k = 0, 1, 2\,; \; L_0 x(t)\, L_3 x(t) \leq 0 \tag{1.13}$$

for all $t \geq t_1$.

A nonoscillatory solution x of Eq. (1.1) is said to have property V_2 if it satisfies the inequalities (1.13).

Lemma 4. *Let x be a solution of* (1.1)*. If x has property V_2 for every large t, then there exists $t_1 \geq T \geq t_0$ such that either*

$$x(t) \geq R_{12}(t, t_1)\, L_2 x(t)\,, \quad t \geq t_1 \tag{1.14}$$

or

$$L_1 x(t) \geq R_2(t, t_1)\, L_2 x(t)\,, \quad t \geq t_1 \tag{1.15}$$

or

$$x(t) \geq \frac{R_{12}(t, t_1)}{R_2(t, t_1)} L_1 x(t)\,, \quad t \geq t_1\,. \tag{1.16}$$

The reader can refer to [6] for the condition (1.16) and [17, Lemma 2] for the condition (1.15).

2. Main Results

Theorem 1. *Let the hypotheses of Lemmas 1–3 and* (1.5), (1.11) *hold. If the first order delay equation*

$$y'(t) + \frac{p(t)}{r_1(t)} R_2(g(t), T)\, y(g(t)) + q(t)\, f(R_{12}(g(t), T))\, f(y(g(t))) = 0 \tag{2.1}$$

for every $T \geq t_0$ is oscillatory, then every solution x of Eq. (1.1) *is either oscillatory or satisfies $\lim_{t\to\infty} x(t) = 0$.*

Proof. Let x be a nonoscillatory solution of Eq. (1.1) on $[T, \infty)$, $T \geq t_0$. Without loss of generality, we may assume that $x(t) > 0$ and $x(g(t)) > 0$ for $t \geq T_1 \geq T$. From Lemma 1 it follows that $L_1 x(t) > 0$ or $L_1 x(t) < 0$ for $t \geq t_1 \geq T_1$. If $L_1 x(t) > 0$ for $t \geq t_1$, then x has property V_2 for large t from Lemma 3. From Lemma 4, we obtain (1.14) and (1.15). Now there exists a $t_2 \geq t_1$ such that

$$x(g(t)) \geq R_{12}(g(t), t_1)\, L_2 x(g(t)) \quad \text{and}$$
$$L_1 x(g(t)) \geq R_2(g(t), t_1)\, L_2 x(g(t)) \quad \text{for} \quad t \geq t_2\,.$$

From Eq. (1.1), we have

$$\begin{aligned}-L_3x(t) &= \frac{p(t)}{r_1(t)}L_1x(t)+q(t)f(x(g(t)))\\ &\geq \frac{p(t)}{r_1(t)}R_2(g(t),t_1)L_2x(g(t))+q(t)f(R_{12}(g(t),t_1)L_2x(g(t)))\\ &\geq \frac{p(t)}{r_1(t)}R_2(g(t),t_1)L_2x(g(t))+q(t)f(R_{12}(g(t),t_1))f(L_2x(g(t)))\,,\end{aligned}$$

for $t\geq t_2$. Setting $y(t)=L_2x(t)>0$ for $t\geq t_2$, we obtain

$$y'(t)+\frac{p(t)}{r_1(t)}R_2(g(t),t_1)y(g(t))+q(t)f(R_{12}(g(t),t_1))f(y(g(t)))\leq 0$$

for $t\geq t_2$. Integrating the above inequality from t to u and letting $u\to\infty$, we have

$$\begin{aligned}y(t)\geq \int_t^\infty \Big(&\frac{p(s)}{r_1(s)}R_2(g(s),t_1)y(g(s))\\ &+q(s)f(R_{12}(g(s),t_1))f(y(g(s)))\Big)\,ds\,.\end{aligned}$$

As in [12], it is easy to conclude that there exists a positive solution $y(t)$ of Eq. (2.1) with $\lim_{t\to\infty}y(t)=0$, which contradictions the fact that Eq. (2.1) is oscillatory.

Let $x(t)>0$, $L_1x(t)<0$, $t\geq t_1$. By Remark 1 we have $\lim_{t\to\infty}x(t)=0$. The proof is complete. □

Corollary 1. *Let the hypotheses of Lemmas 1–3 hold. If the first order delay equation*

$$y'(t)+\left(Kq(t)R_{12}(g(t),T)+\frac{p(t)}{r_1(t)}R_2(g(t),T)\right)y(g(t))=0 \tag{2.2}$$

for some $K>0$ and every $T\geq t_0$ is oscillatory, then every solution x of Eq. (1.1) *is either oscillatory or satisfies* $\lim_{t\to\infty}x(t)=0$.

Theorem 2. *Let the hypotheses of Lemmas 1–3 hold. If*

$$\limsup_{t\to\infty}\int_{g(t)}^t\left(Kq(s)R_{12}(g(s),T)+\frac{p(s)}{r_1(s)}R_2(g(s),T)\right)ds>1 \tag{2.3}$$

for some $K>0$ and every $T\geq t_0$, then every solution x of Eq. (1.1) *is either oscillatory or satisfies* $\lim_{t\to\infty}x(t)=0$.

Proof. Proceeding as in the proof of Theorem 1, we obtain x has property V_2 for large t. From Lemma 4, we obtain (1.14) and (1.15). Now there exists a $t_2\geq t_1$ such that

$$x(g(t))\geq R_{12}(g(t),t_1)L_2x(g(t))\quad\text{and}$$
$$L_1x(g(t))\geq R_2(g(t),t_1)L_2x(g(t))\quad\text{for}\quad t\geq t_2\,.$$

Integrating Eq. (1.1) from $g(t)$ to t, we have

$$-L_2x(t) + L_2x(g(t)) = \int_{g(t)}^{t} \left(\frac{p(s)}{r_1(s)} L_1x(s) + q(s) f(x(g(s))) \right) ds$$

$$\begin{aligned} L_2x(g(t)) &\geq \int_{g(t)}^{t} \left(\frac{p(s)}{r_1(s)} L_1x(g(s)) + Kq(s)\, x(g(s)) \right) ds \\ &\geq \int_{g(t)}^{t} \left(\frac{p(s)}{r_1(s)} R_2(g(s), t_1)\, L_2x(g(s)) + Kq(s)\, R_{12}(g(s), t_1)\, L_2x(g(s)) \right) ds \\ &\geq L_2x(g(t)) \int_{g(t)}^{t} \left(Kq(s)\, R_{12}(g(s), t_1) + \frac{p(s)}{r_1(s)} R_2(g(s), t_1) \right) ds\,. \end{aligned}$$

Hence,

$$1 \geq \int_{g(t)}^{t} \left(Kq(s)\, R_{12}(g(s), t_1) + \frac{p(s)}{r_1(s)} R_2(g(s), t_1) \right) ds \quad \text{for} \quad t \geq t_2\,.$$

Taking limsup of both sides of the above inequality as $t \to \infty$, we arrive at a contraction to condition (2.3).

Let $x(t) > 0$, $L_1x(t) < 0$, $t \geq t_1$. By Lemma 2 we have $\lim_{t\to\infty} x(t) = 0$. The proof is complete. □

Example 2. Consider the third order delay equation

$$x'''(t) + \frac{1}{4t^2} x'(t) + \left(1 - \frac{1}{4t^2}\right) x\left(t - \frac{3\pi}{2}\right) = 0\,, \quad t \geq \frac{3\pi}{2}\,. \tag{2.4}$$

It is easy to check that all conditions of Theorem 2 are satisfied and hence every solution $x(t)$ of Eq. (2.4) is either oscillatory or satisfies $\lim_{t\to\infty} x(t) = 0$. An example of such a solution is $x(t) = \sin t$.

Theorem 3. *Let the hypotheses of Lemmas 1–3 hold. If*

$$\liminf_{t\to\infty} \int_{g(t)}^{t} \left(Kq(s)\, R_{12}(g(s), T) + \frac{p(s)}{r_1(s)} R_2(g(s), T) \right) ds > \frac{1}{e} \tag{2.5}$$

for some $K > 0$ *and any* $T \geq t_0$, *then every solution* x *of Eq.* (1.1) *is either oscillatory or satisfies* $\lim_{t\to\infty} x(t) = 0$.

Proof. Proceeding as in the proof of Theorem 2, we obtain

$$-L_3x(t) = \frac{p(t)}{r_1(t)} L_1x(t) + q(t) f(x(g(t)))$$

$$\begin{aligned} -L_3x(t) &\geq \frac{p(t)}{r_1(t)} L_1x(t) + Kq(t)\, x(g(t)) \\ &\geq \frac{p(t)}{r_1(t)} R_2(g(t), t_1)\, L_2x(g(t)) + Kq(t)\, R_{12}(g(t), t_1)\, L_2x(g(t))\,, \end{aligned}$$

for $t \geq t_2$. Setting $y(t) = L_2 x(t) > 0$ for $t \geq t_2$, we obtain

$$y'(t) + \frac{p(t)}{r_1(t)} R_2(g(t), t_1)\, y(g(t)) + K q(t) R_{12}(g(t), t_1)\, y(g(t)) \leq 0$$

$$y'(t) + \left(K q(t) R_{12}(g(t), t_1) + \frac{p(t)}{r_1(t)} R_2(g(t), t_1) \right) y(g(t)) \leq 0$$

for $t \geq t_2$. By known results, see [2, 10, 12], we arrive at the desired contradiction.

Let $x(t) > 0$, $L_1 x(t) < 0$, $t \geq t_1$. By Lemma 2 we have $\lim_{t\to\infty} x(t) = 0$. The proof is complete. □

Example 3. Consider the third order equation

$$x'''(t) + e^{-2t+2} x'(t) + \frac{1}{e} x(t-1)\left(1 + x^2(t-1)\right) = 0\,, \quad t \geq 1\,. \tag{2.6}$$

It is easy to check that all conditions of Theorem 3 are satisfied and hence every solution $x(t)$ of Eq. (2.6) is either oscillatory or satisfies $\lim_{t\to\infty} x(t) = 0$. One such solution of Eq. (2.6) is $x(t) = e^{-t}$.

Theorem 4. *Let the hypotheses of Lemmas 1–3 and* (1.5), (1.7), (1.11) *hold. If*

$$\limsup_{t\to\infty} P(t) \int_{g(t)}^{t} q(s) f\left(R_{12}(g(s), T)\right) ds > 0\,, \tag{2.7}$$

where $P(t) = 1 \Big/ \left(1 - \int_{g(t)}^{t} \frac{p(s)}{r_1(s)} R_2(g(s), T)\, ds\right) \geq 0$ *for every* $t \geq T \geq t_0$ *and not identically zero in any subinterval of* $[T, \infty)$, *then every solution* x *of Eq.* (1.1) *is either oscillatory or satisfies* $\lim_{t\to\infty} x(t) = 0$.

Proof. Proceeding as in the proof of Theorem 1, we obtain

$$\begin{aligned} -L_3 x(t) &= \frac{p(t)}{r_1(t)} L_1 x(t) + q(t) f(x(g(t))) \\ &\geq \frac{p(t)}{r_1(t)} R_2(g(t), t_1) L_2 x(g(t)) + q(t) f(R_{12}(g(t), t_1)) f(L_2 x(g(t)))\,, \end{aligned}$$

for $t \geq t_2 \geq t_1$. Integrating the above inequality from $g(t)$ to t, we have

$$\begin{aligned} -L_2x(t) + L_2x(g(t)) \geq \int_{g(t)}^{t} \Big(&\frac{p(s)}{r_1(s)} R_2(g(s), t_1) L_2 x(g(s)) \\ &+ q(s) f(R_{12}((s), t_1)) f(L_2 x(g(s))) \Big)\, ds \end{aligned}$$

$$\begin{aligned} L_2x(g(t)) \geq L_2 x(g(t)) \int_{g(t)}^{t} \frac{p(s)}{r_1(s)} R_2(g(s), t_1)\, ds + f(L_2 x(g(t))) \\ \times \int_{g(t)}^{t} q(s) f(R_{12}(g(s), t_1))\, ds \end{aligned}$$

$$\frac{L_2 x(g(t))}{f(L_2 x(g(t)))} \geq P(t) \int_{g(t)}^{t} q(s) f(R_{12}(g(s), t_1))\, ds\,, \quad t \geq t_2 \geq t_1\,.$$

Taking limsup of both sides of the above inequality as $t \to \infty$, we arrive at a contraction to condition (2.7).

Let $x(t) > 0$, $L_1 x(t) < 0$, $t \geq t_1$. By Remark 1 we have $\lim_{t\to\infty} x(t) = 0$. The proof is complete. □

Corollary 2. *When Theorem 4 doesn't have the condition* (1.11), *we can take either*

$$\limsup_{t\to\infty} \int_{g(t)}^{t} \left(Kq(s) f\left(R_{12}(g(s), T)\right) + \frac{p(s)}{r_1(s)} R_2(g(s), T) \right) ds > 1 \tag{2.8}$$

or

$$\limsup_{t\to\infty} \int_{g(t)}^{t} \left(K^2 q(s) R_{12}(g(s), T) + \frac{p(s)}{r_1(s)} R_2(g(s), T) \right) ds > 1$$

or

$$\limsup_{t\to\infty} K^2 P(t) \int_{g(t)}^{t} q(s) f\left(R_{12}(g(s), T)\right) ds > 1$$

to replace (2.7).

Example 4. Consider

$$x'''(t) + \frac{1}{4t^2} x'(t) + t^{1-2\gamma} x^{\gamma}(t-1) = 0, \quad t \geq 1, \tag{2.9}$$

where γ is the ratio of two positive odd integers, $0 < \gamma < 1$. By choosing $\rho_2(t) = t^{2\gamma}$, we see that all conditions of Theorem 4 are satisfied. Then, every solution $x(t)$ of Eq. (2.9) is either oscillatory or satisfies $\lim_{t\to\infty} x(t) = 0$.

Now, we consider $g(t) \leq t$.

Theorem 5. *Let the hypotheses of Lemmas 1–3 and $g(t) \leq t$,* (1.5), (1.11) *hold. If the second order equation*

$$(r_2(t) y'(t))' + \frac{p(t)}{r_1(t)} y(g(t)) + q(t) f\left(\frac{R_{12}(g(t), T)}{R_2(g(t), T)} \right) f(y(g(t))) = 0 \tag{2.10}$$

for every $T \geq t_0$ is oscillatory, then every solution x of Eq. (1.1) *is either oscillatory or satisfies* $\lim_{t\to\infty} x(t) = 0$.

Proof. Proceeding as in the proof of Theorem 1, we obtain x has property V_2 for large t. From Lemma 4, we obtain (1.16). Now there exists a $t_2 \geq t_1$ such that

$$x(g(t)) \geq \frac{R_{12}(g(t), t_1)}{R_2(g(t), t_1)} L_1 x(g(t)) \quad \text{for} \quad t \geq t_2.$$

From Eq. (1.1), we have

$$\begin{aligned}-L_3x(t) &= \frac{p(t)}{r_1(t)}L_1x(t)+q(t)f(x(g(t)))\\ &\geq \frac{p(t)}{r_1(t)}L_1x(g(t))+q(t)f\left(\frac{R_{12}(g(t),t_1)}{R_2(g(t),t_1)}L_1x(g(t))\right)\\ &\geq \frac{p(t)}{r_1(t)}L_1x(g(t))+q(t)f\left(\frac{R_{12}(g(t),t_1)}{R_2(g(t),t_1)}\right)f(L_1x(g(t)))\end{aligned}$$

and so

$$L_1x(t)\left\{L_3x(t)+\frac{p(t)}{r_1(t)}L_1x(g(t))+q(t)f\left(\frac{R_{12}(g(t),t_1)}{R_2(g(t),t_1)}\right)f(L_1x(g(t)))\right\}\leq 0$$

for every $t\geq t_2\geq t_1$. By Theorem 1 in [14] the Eq. (2.10) is oscillatory if and only if the inequality

(2.11)

$$y(t)\left\{(r_2(t)y'(t))'+\frac{p(t)}{r_1(t)}y(g(t))+q(t)f\left(\frac{R_{12}(g(t),t_1)}{R_2(g(t),t_1)}\right)f(y(g(t)))\right\}\leq 0$$

is oscillatory, too. This is a contradiction, since $y=L_1x(t)$ is a nonoscillatory solution of (2.11) for large t.

Let $x(t)>0$, $L_1x(t)<0$, $t\geq t_1$. By Remark 1 we have $\lim_{t\to\infty}x(t)=0$. The proof is complete. □

Corollary 3. *Let the hypotheses of Lemmas 1–3 and $g(t)\leq t$ hold. If the second order equation*

$$(r_2(t)y'(t))'+\left(Kq(t)\frac{R_{12}(g(t),T)}{R_2(g(t),T)}+\frac{p(t)}{r_1(t)}\right)y(g(t))=0$$

for some $K>0$ and every $T\geq t_0$ is oscillatory, then every solution x of Eq. (1.1) *is either oscillatory or satisfies* $\lim_{t\to\infty}x(t)=0$.

Example 5. Consider

$$x'''(t)+\frac{p_0}{t^\delta}x'(t)+\frac{q_0}{t^\beta}x(\lambda t)=0,\ t\geq 1,\qquad 0<\lambda\leq 1, \tag{2.12}$$

where $0\leq p_0\leq\frac{1}{4}$, $q_0>0$, $\delta\geq 2$, and $\beta<3$ are some constants. Equation $z''+\frac{p_0}{t^\delta}z=0$ is nonoscillatory (see [16, pp. 45]) and also since $y''(t)+\frac{q_0}{t^\beta}\frac{\lambda t-1}{2}y(\lambda t)=0$ is oscillatory (see [14, Theorem 6]), equation $y''(t)+\left(\frac{p_0}{t^\delta}+\frac{q_0}{t^\beta}\frac{\lambda t-1}{2}\right)y(\lambda t)=0$ is oscillatory by the generalized Sturm comparison theorem (see [14, Theorem 2]). If we also choose $\rho_2(t)=t^2$, from Theorem 5, every solution $x(t)$ of Eq. (2.12) is either oscillatory or satisfies $x(t)\to 0$ as $t\to\infty$. If we take $\delta=2$, $\beta=3$, $\lambda=1$, $p_0=\frac{1}{4}$ and $q_0=\frac{25}{4}$, $x_1(t)=\frac{1}{t}$, $x_2(t)=t^2\cos\left(\frac{3}{2}\ln t\right)$, and $x_3(t)=t^2\sin\left(\frac{3}{2}\ln t\right)$ are solutions of Euler Eq. (2.12) and all hypotheses of Theorem 5 are satisfied.

Theorem 6. *Let the hypotheses of Lemmas 1–3 and $g(t) \le t$, (1.5), (1.8), (1.11) hold. If*

$$\int_T^\infty q(s) R_2(g(s), T) f\left(\frac{R_{12}(g(s), T)}{R_2(g(s), T)}\right) ds = \infty \quad \text{for} \quad T \ge t_0, \tag{2.13}$$

then every solution x of Eq. (1.1) is either oscillatory or satisfies $\lim_{t\to\infty} x(t) = 0$.

Proof. Proceeding as in the proof of Theorem 1, we obtain x has property V_2 for large t. Now there exists a $t_2 \ge t_1$ such that

$$x(g(t)) \ge \frac{R_{12}(g(t), t_1)}{R_2(g(t), t_1)} L_1 x(g(t)) \quad \text{for} \quad t \ge t_2.$$

From Eq. (1.1), we have

$$\begin{aligned} -\frac{d}{dt} L_2 x(t) &= \frac{p(t)}{r_1(t)} L_1 x(t) + q(t) f(x(g(t))) \\ &\ge q(t) f\left(\frac{R_{12}(g(t), t_1)}{R_2(g(t), t_1)} L_1 x(g(t))\right) \\ &\ge q(t) f\left(\frac{R_{12}(g(t), t_1)}{R_2(g(t), t_1)}\right) f(L_1 x(g(t))), \quad t \ge t_2. \end{aligned}$$

Then integrating from t to $u \ge t \ge t_2$, we get

$$L_2 x(t) \ge L_2 x(t) - L_2 x(u) \ge \int_t^u q(s) f\left(\frac{R_{12}(g(s), t_1)}{R_2(g(s), t_1)}\right) f(L_1 x(g(s))) ds$$

and from this

$$L_2 x(t) \ge \int_t^\infty q(s) f\left(\frac{R_{12}(g(s), t_1)}{R_2(g(s), t_1)}\right) f(L_1 x(g(s))) ds \quad \text{for} \quad t \ge t_2.$$

Setting $y(t) = L_1 x(t) > 0$ for $t \ge t_2$, we obtain

$$r_2(t) y'(t) \ge \int_t^\infty q(s) f\left(\frac{R_{12}(g(s), t_1)}{R_2(g(s), t_1)}\right) f(y(g(s))) ds \quad \text{for} \quad t \ge t_2. \tag{2.14}$$

Since g, y, and f are nondecreasing functions and $r_2(t) y'(t)$ is nonincreasing, we get

$$r_2(g(t)) y'(g(t)) \ge f(y(g(t))) \int_t^\infty q(s) f\left(\frac{R_{12}(g(s), t_1)}{R_2(g(s), t_1)}\right) ds \quad \text{for} \quad t \ge t_2.$$

Multiplying this inequality by $g'(t)$ and dividing it by $r_2(g(t)) f(y(g(t)))$ and then integrating it from t_2 to $t \ge t_2$, we have

$$\int_{t_2}^t \frac{y'(g(s)) g'(s)}{f(y(g(s)))} ds \ge \int_{t_2}^t \frac{g'(s)}{r_2(g(s))} \left(\int_s^\infty q(u) f\left(\frac{R_{12}(g(u), t_1)}{R_2(g(u), t_1)}\right) du\right) ds$$

and from this

$$\begin{aligned}\int_{y(g(t_2))}^{\infty}\frac{du}{f(u)} &\geq \int_{y(g(t_2))}^{y(g(t))}\frac{du}{f(u)}\\ &\geq \int_{t_2}^{t}\frac{g'(s)}{r_2(g(s))}\left(\int_s^t q(u)\, f\left(\frac{R_{12}(g(u),t_1)}{R_2(g(u),t_1)}\right)du\right)ds\\ &= \int_{t_2}^{t}\left[R_2(g(s),t_2)-R_2(g(t_2),t_2)\right]q(s)\, f\left(\frac{R_{12}(g(s),t_1)}{R_2(g(s),t_1)}\right)ds\\ &\geq \frac{1}{2}\int_{t_3}^{t} q(s)\, R_2(g(s),t_2)\, f\left(\frac{R_{12}(g(s),t_1)}{R_2(g(s),t_1)}\right)ds\end{aligned}$$

for $t \geq t_3$, where $t_3 \geq t_2$ is such that $R_2(g(t_2),t_2) \leq \dfrac{R_2(g(t),t_2)}{2}$ for $t \geq t_3$. The last inequality contradicts the assumption (2.13) for large t.

Let $x(t) > 0$, $L_1 x(t) < 0$, $t \geq t_1$. By Remark 1 we have $\lim_{t\to\infty} x(t) = 0$. The proof is complete. □

Example 6. Consider the third order equation

$$x'''(t) + \frac{1}{t^3}x'(t) + \frac{2^\alpha + \left(\sqrt{t}-1\right)^{2\alpha}}{t\left(\sqrt{t}-1\right)^{2\alpha+1}}\,\frac{\left|x\left(\sqrt{t}\right)\right|^{2\alpha}\operatorname{sgn} x\left(\sqrt{t}\right)}{1+\left|x\left(\sqrt{t}\right)\right|^{\alpha}} = 0\,, \tag{2.15}$$

for $t \geq 1$, $\alpha > 1$. Equation $z'' + \dfrac{1}{t^3}z = 0$ is nonoscillatory (see [16, pp.45]). If we choose $\rho_2(t) = t^2$, from Theorem 6, then every solution $x(t)$ of Eq. (2.15) is either oscillatory or satisfies $\lim_{t\to\infty} x(t) = 0$.

Remark 2. Let $g(t) \leq t$, (1.3), and (1.8) hold. If

$$\int_T^{\infty} q(s)\, R_2(g(s),T)\, f\left(\frac{R_{12}(g(s),T)}{R_2(g(s),T)}\right)ds = \infty \quad \text{for} \quad T \geq t_0\,,$$

then equation

$$\left(r_2(t)\, y'(t)\right)' + q(t)\, f\left(\frac{R_{12}(g(t),T)}{R_2(g(t),T)}\right) f\left(y(g(t))\right) = 0$$

is oscillatory (see [14, Theorem 4]).

Theorem 7. *Let the hypotheses of Lemmas 1–3 and $g(t) \leq t$, (1.5), (1.8), (1.11) hold. Let there exists a nondecreasing function $G \in C(R,R)$ such that $f(x) = |x|\, G(x)$ for $x \in R$. Then, if*

$$\begin{aligned}&\int_T^{\infty} q(s)\, R_2^2(g(s),T)\, f\left(\frac{R_{12}(g(s),T)}{R_2(g(s),T)}\right)\\ &\quad\times\left(\int_{g(s)}^{\infty} q(u)\, f\left(\frac{R_{12}(g(u),T)}{R_2(g(u),T)}\right)du\right)ds = \infty\end{aligned} \tag{2.16}$$

for $T \geq t_0$, *and*

$$\int_{\pm\epsilon}^{\pm\infty} \frac{dx}{G(x)} < \infty\,,$$

for every $\varepsilon > 0$, *then every solution* x *of Eq.* (1.1) *is either oscillatory or satisfies* $\lim_{t\to\infty} x(t) = 0$.

Proof. Proceeding as in the proof of Theorem 6, we obtain x has property V_2 for large t. Then $y(t) = L_1 x(t)$ is the nonoscillatory solution of the equation

$$\left(r_2(t)\, y'(t)\right)' + b(t)\, G\left(y\left(g(t)\right)\right) = 0\,,$$

where $b(t) = q(t)\, f\left(\dfrac{R_{12}\left(g(t), t_1\right)}{R_2\left(g(t), t_1\right)}\right) y\left(g(t)\right)$ for $t \geq t_1$. Then by Remark 2

$$\int_{t_1}^{\infty} q(s)\, R_2\left(g(s), t_1\right) f\left(\frac{R_{12}\left(g(s), t_1\right)}{R_2\left(g(s), t_1\right)}\right) y\left(g(s)\right)\, ds < \infty\,. \tag{2.17}$$

In the same way as in the proof of Theorem 6 from (2.14) we have

$$\begin{aligned} r_2(t)\, y'(t) &\geq f\left(y\left(g(t)\right)\right) \int_t^{\infty} q(s)\, f\left(\frac{R_{12}\left(g(s), t_1\right)}{R_2\left(g(s), t_1\right)}\right) ds \\ &\geq f\left(y\left(g(t_2)\right)\right) \int_t^{\infty} q(s)\, f\left(\frac{R_{12}\left(g(s), t_1\right)}{R_2\left(g(s), t_1\right)}\right) ds \end{aligned}$$

for $t \geq t_2$. Dividing this inequality by $r_2(t)$ and integrating it from t_2 to $t \geq t_2$ we get

$$\begin{aligned} y(t) &\geq f\left(L_1 x\left(g(t_2)\right)\right) \int_{t_2}^{t} \frac{1}{r_2(s)} \left(\int_s^{\infty} q(u)\, f\left(\frac{R_{12}\left(g(u), t_1\right)}{R_2\left(g(u), t_1\right)}\right) du\right) ds \\ &\geq f\left(L_1 x\left(g(t_2)\right)\right) \int_{t_2}^{t} \frac{1}{r_2(s)} \left(\int_t^{\infty} q(u)\, f\left(\frac{R_{12}\left(g(u), t_1\right)}{R_2\left(g(u), t_1\right)}\right) du\right) ds \\ &= f\left(L_1 x\left(g(t_2)\right)\right) \left(R_2(t, t_2) - R_2(t_0, t_2)\right) \int_t^{\infty} q(s)\, f\left(\frac{R_{12}\left(g(s), t_1\right)}{R_2\left(g(s), t_1\right)}\right) ds\,. \end{aligned}$$

Then there exists a $t_3 \geq t_2$ such that

$$y\left(g(t)\right) \geq \frac{1}{2} f\left(L_1 x\left(g(t_2)\right)\right) R_2\left(g(t), t_2\right) \int_{g(t)}^{\infty} q(s)\, f\left(\frac{R_{12}\left(g(s), t_1\right)}{R_2\left(g(s), t_1\right)}\right) ds$$

for $t \geq t_3$. This inequality and (2.17) contradict the condition (2.16).

Let $x(t) > 0$, $L_1 x(t) < 0$, $t \geq t_1$. By Remark 1 we have $\lim_{t\to\infty} x(t) = 0$. The proof is complete. □

Example 7. The equation

$$x'''(t) + t^{-3} x'(t) + t^{-5/2} x^3\left(t^{1/3}\right) = 0\,, \quad t \geq 1\,,$$

satisfies the assumptions of Theorem 7 but the condition (2.13) of Theorem 6 does not hold.

There are many sufficient conditions for the oscillation of equation (2.10) in the literature. The reader can refer to [1]–[2], [14] for them.

Theorem 8. *Let the hypotheses of Lemmas 1–3 and* $g(t) \le t$, (1.5), (1.9), (1.11) *hold. If*

$$\int^{\infty} q(s)\, f\left(R_{12}\left(g(s), T\right)\right) ds = \infty \quad \textit{for} \quad T \ge t_0\,, \tag{2.18}$$

then every solution x *of Eq.* (1.1) *is either oscillatory or satisfies* $\lim_{t\to\infty} x(t) = 0$.

Proof. Proceeding as in the proof of Theorem 1, we obtain x has property V_2 for large t. From Eq. (1.1), we have

$$\begin{aligned} -\frac{d}{dt} L_2 x(t) &= \frac{p(t)}{r_1(t)} L_1 x(t) + q(t)\, f\left(x\left(g(t)\right)\right) \\ &\ge q(t)\, f\left(R_{12}\left(g(t), t_1\right) L_2 x\left(g(t)\right)\right) \\ &\ge q(t)\, f\left(R_{12}\left(g(t), t_1\right)\right) f\left(L_2 x(t)\right) \end{aligned}$$

or

$$\frac{-\dfrac{d}{dt}\left(L_2 x(t)\right)}{f\left(L_2 x(t)\right)} \ge q(t)\, f\left(R_{12}\left(g(t), t_1\right)\right) \qquad \text{for} \qquad t \ge t_2 \ge t_1\,.$$

Integrating the above inequality from t_2 to t, we have

$$\int_{L_2 x(t)}^{L_2 x(t_2)} \frac{du}{f(u)} \ge \int_{t_2}^{t} q(s)\, f\left(R_{12}\left(g(s), t_1\right)\right) ds\,.$$

Taking lim of both sides of the above inequality as $t \to \infty$, we obtain at a contraction to condition (2.18).

Let $x(t) > 0$, $L_1 x(t) < 0$, $t \ge t_1$. By Remark 1 we have $\lim_{t\to\infty} x(t) = 0$. The proof is complete. □

Example 8. Consider

$$x'''(t) + \frac{1}{4t^2} x'(t) + \frac{25}{4} \frac{(\lambda t)^{\alpha}}{t^4} \left|x(\lambda t)\right|^{\alpha - 1} x(\lambda t) = 0,\ t \ge 1\,, \quad 0 < \alpha\,,\ \lambda < 1\,. \tag{2.19}$$

By choosing $\rho_2(t) = t^2$, it is easy to check that all conditions of Theorem 8 are satisfied. Then every solution $x(t)$ of Eq. (2.19) is either oscillatory or satisfies $\lim_{t\to\infty} x(t) = 0$. Observe that $x(t) = \dfrac{1}{t}$ is a solution of Eq. (2.19).

Theorem 9. *Let* $g(t) \le t$ *and the function* f *satisfy the condition*

$$\liminf_{|u|\to\infty} |f(u)| > 0\,. \tag{2.20}$$

If

$$\int^{\infty} q(t)\, dt = \infty\,, \tag{2.21}$$

then every solution x *of Eq.* (1.1) *is either oscillatory or satisfies* $\lim_{t\to\infty} x(t) = 0$.

Proof. Proceeding as in the proof of Theorem 1, we obtain x has property V_2 for large t. Since x has property V_2, $\lim_{t\to\infty} x(t)$ exists. If $\lim_{t\to\infty} x(t) = \infty$, then from (2.20) and (2.21) we obtain

$$\int^{\infty} q(t) f(x(g(t)))\, dt = \infty . \tag{2.22}$$

If $\lim_{t\to\infty} x(t) = K < \infty$, then from (2.21) and the continuity f (2.22) holds, too. Integrating the inequality $L_3 x(t) + q(t) f(x(g(t))) \le 0$ from t_1 to $t \ge t_1$ and using (2.22) we get $L_2 x(t) < 0$ for all sufficiently large t, a contradiction.

Let $x(t) > 0$, $L_1 x(t) < 0$, $t \ge t_1$. By Remark 1 ($\rho_2(t) = 1$) we have $\lim_{t\to\infty} x(t) = 0$. The proof is complete. □

Example 9. Consider the third order equation

$$\left(\frac{1}{t}x'(t)\right)'' + \frac{1}{4t^3}x'(t) + \frac{1}{t}x(t-\ln t)\left(1 + \frac{1}{1+x^2(t-\ln t)}\right) = 0 , \tag{2.23}$$

for $t \ge 1$. It is easy to check that all conditions of Theorem 9 are satisfied. Then every solution $x(t)$ of Eq. (2.23) is either oscillatory or satisfies $\lim_{t\to\infty} x(t) = 0$.

Now, we consider

$$R_2(t, t_0) < \infty . \tag{1.4}$$

Theorem 10. *Let the hypotheses of Lemmas 1–2 and* (1.4), (1.5), (1.11) *hold. In addition to the first order delay equation*

$$y'(t) + \frac{p(t)}{r_1(t)} R_2(g(t), T)\, y(g(t)) + q(t) f(R_{12}(g(t), T)) f(y(g(t))) = 0 \tag{2.1}$$

for every $T \ge t_0$ is oscillatory. If

$$\int_T^{\infty} \left(\frac{1}{r_2(u)} \int_T^u \left(Dq(s) f(R_1(g(s), T)) f(R_2(\infty, g(s))) + \frac{p(s)}{r_1(s)} R_2(\infty, g(s)) \right) ds \right) du = \infty \tag{2.24}$$

for every $D > 0$ and any $T \ge t_0$, then every solution x of Eq. (1.1) *is either oscillatory or satisfies* $\lim_{t\to\infty} x(t) = 0$.

Proof. Let x be a nonoscillatory solution of (1.1) on $[T, \infty)$, $T \ge t_0$. Without loss of generality, we may assume that $x(t) > 0$ and $x(g(t)) > 0$ for $t \ge T_1 \ge T$. From Lemma 1 it follows that $L_1 x(t) > 0$ or $L_1 x(t) < 0$ for $t \ge t_1 \ge T_1$. There are three possibility to consider:

(i) $L_1 x(t) > 0$, $L_2 x(t) > 0$, $L_3 x(t) \le 0$ for $t \ge t_1$;

(ii) $L_1 x(t) > 0$, $L_2 x(t) < 0$, $L_3 x(t) \le 0$ for $t \ge t_1$; and

(iii) $L_1 x(t) < 0$ for $t \ge t_1$.

Case (i): The proof is exactly the same as that Theorem 1 – Case (i).

Case (ii): There exists a $t_2 \geq t_1$ such that

$$x\left(t\right) \geq R_1\left(t, t_1\right) L_1 x\left(t\right) \quad \text{for} \quad t \geq t_2$$

and so there exists a $t_3 \geq t_2$ such that

$$x\left(g\left(t\right)\right) \geq R_1\left(g\left(t\right), t_1\right) L_1 x\left(g\left(t\right)\right) := R_1\left(g\left(t\right), t_1\right) v\left(g\left(t\right)\right) \quad \text{for} \quad t \geq t_3\,, \tag{2.25}$$

where $v\left(t\right) = L_1 x\left(t\right)$. Using (2.25) and (1.5) in Eq. (1.1), we find

$$\left(r_2\left(t\right) v'\left(t\right)\right)' + \frac{p\left(t\right)}{r_1\left(t\right)} v\left(g\left(t\right)\right) + q\left(t\right) f\left(R_1\left(g\left(t\right), t_1\right)\right) f\left(v\left(g\left(t\right)\right)\right) \leq 0 \tag{2.26}$$

for $t \geq t_3$. Clearly, $v\left(t\right) > 0$ and $v'\left(t\right) < 0$ for $t \geq t_3$. Now, for $s \geq t \geq t_3$ one can easily see that

$$-r_2\left(s\right) v'\left(s\right) \geq -r_2\left(t\right) v'\left(t\right) \quad \text{for} \quad s \geq t \geq t_3\,. \tag{2.27}$$

Dividing (2.27) by $r_2\left(s\right)$ and integrating from t to $u \geq t \geq t_3$, we have

$$v\left(t\right) \geq v\left(t\right) - v\left(u\right) \geq -r_2\left(t\right) v'\left(t\right) R_2\left(u, t\right)\,.$$

Letting $u \to \infty$ in the above inequality, we get

$$v\left(t\right) \geq -r_2\left(t\right) v'\left(t\right) R_2\left(\infty, t\right) \quad \text{for} \quad t \geq t_3\,. \tag{2.28}$$

Combining (2.28) with the inequality

$$-r_2\left(t\right) v'\left(t\right) \geq -r_2\left(t_3\right) v'\left(t_3\right) \quad \text{for} \quad t \geq t_3\,,$$

which implied by (2.27), we find

$$v\left(t\right) \geq -r_2\left(t_3\right) v'\left(t_3\right) R_2\left(\infty, t\right) \quad \text{for} \quad t \geq t_3\,.$$

Thus, there exists a constant $b > 0$ and a $t_4 \geq t_3$ such that

$$v(g(t)) \geq b R_2(\infty, g(t)) \quad \text{for} \quad t \geq t_4\,. \tag{2.29}$$

Integrating inequality (2.26) from t_3 to t, we have

$$\int_{t_3}^{t} \Big(\frac{p(s)}{r_1(s)} v(g(s)) + q(s) f(R_1(g(s), t_1)) f(v(g(s)))\Big)\, ds$$
$$\leq r_2(t_3) v'(t_3) - r_2(t) v'(t)\,.$$

Using Eq. (2.29) and (1.5) in the above inequality, we get

$$\frac{1}{r_2\left(t\right)} \int_{t_3}^{t} \Big(f\left(b\right) q\left(s\right) f\left(R_1\left(g\left(s\right), t_1\right)\right) f\left(R_2\left(\infty, g\left(s\right)\right)\right)$$
$$+ b \frac{p\left(s\right)}{r_1\left(s\right)} R_2\left(\infty, g\left(s\right)\right)\Big)\, ds \leq -v'\left(t\right)\,, \quad t \geq t_4\,.$$

Integrating the above inequality from t_4 to t, we find

$$b\int_{t_4}^{t}\Big(\frac{1}{r_2(\tau)}\int_{t_3}^{\tau}\Big(Dq(s)\,f(R_1(g(s),t_1))\,f(R_2(\infty,g(s))) + \frac{p(s)}{r_1(s)}R_2(\infty,g(s))\Big)\,ds\Big)\,d\tau \le v(t_4) < \infty\,,$$

where $D = \dfrac{f(b)}{b}$ is a constant. This inequality implies

$$\int_{t_4}^{\infty}\Big(\frac{1}{r_2(\tau)}\int_{t_3}^{\tau}\Big(Dq(s)\,f(R_1(g(s),t_1))\,f(R_2(\infty,g(s))) + \frac{p(s)}{r_1(s)}R_2(\infty,g(s))\Big)\,ds\Big)\,d\tau < \infty\,,$$

which contradictions condition (2.24).

Case (iii): Let $x(t) > 0$, $L_1x(t) < 0$, $t \ge t_1$. By Remark 1 we have $\lim_{t\to\infty} x(t) = 0$. The proof is complete. □

Corollary 4. *Let the hypotheses of Lemmas 1–2 and* (1.4) *hold. In addition to the first order delay equation*

$$y'(t) + \Big(Kq(t)\,R_{12}(g(t),T) + \frac{p(t)}{r_1(t)}R_2(g(t),T)\Big)\,y(g(t)) = 0 \tag{2.2}$$

for some $K > 0$ and every $T \ge t_0$ is oscillatory. If

$$\int_{T}^{\infty}\Big(\frac{1}{r_2(u)}\int_{T}^{u}R_2(\infty,g(s))\Big(Kq(s)\,R_1(g(s),T) + \frac{p(s)}{r_1(s)}\Big)\,ds\Big)\,du = \infty \tag{2.30}$$

for some $K > 0$ and any $T \ge t_0$, then every solution x of Eq. (1.1) *is either oscillatory or satisfies* $\lim_{t\to\infty} x(t) = 0$.

Theorem 11. *Let the hypotheses of Lemmas 1–2 and* (1.4) *hold. Then every solution x of Eq.* (1.1) *is either oscillatory or satisfies* $\lim_{t\to\infty} x(t) = 0$ *if one of the following conditions holds:*

(I_1) *Condition* (2.30) *and*

$$\limsup_{t\to\infty}\int_{g(t)}^{t}\Big(Kq(s)\,R_{12}(g(s),T) + \frac{p(s)}{r_1(s)}R_2(g(s),T)\Big)\,ds > 1 \tag{2.4}$$

for some $K > 0$ and every $T \ge t_0$.

(I_2) *Condition* (2.30) *and*

$$\liminf_{t\to\infty}\int_{g(t)}^{t}\Big(Kq(s)\,R_{12}(g(s),T) + \frac{p(s)}{r_1(s)}R_2(g(s),T)\Big)\,ds > \frac{1}{e} \tag{2.6}$$

for some $K > 0$ and any $T \ge t_0$.

(I_3) *Conditions* (1.5)*,* (1.7)*,* (1.11)*,* (2.24)*, and*

$$\limsup_{t\to\infty} P(t) \int_{g(t)}^{t} q(s) f(R_{12}(g(s), T))\, ds > 0 \tag{2.8}$$

for any $T \geq t_0$*.*

(I_4) *Conditions* $g(t) \leq t$*,* (1.5)*,* (1.8)*,* (1.11)*,* (2.24)*, and*

$$\int_{T}^{\infty} q(s) R_2(g(s), T) f\left(\frac{R_{12}(g(s), T)}{R_2(g(s), T)}\right) ds = \infty \tag{2.13}$$

for $T \geq t_0$*.*

(I_5) *Conditions* $g(t) \leq t$*,* (1.5)*,* (1.9)*,* (1.11)*,* (2.24)*, and*

$$\int^{\infty} q(s) f(R_{12}(g(s), T))\, ds = \infty \tag{2.19}$$

for $T \geq t_0$*.*

Remark 3. We note that conditions of theorems can be changed when the conditions are satisfied both (1.5) and (1.6) at the same time (see Corollary 2).

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