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THE SYMMETRY OF UNIT IDEAL STABLE RANGE CONDITIONS

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Abstract. In this paper, we prove that unit ideal-stable range condition is right and left symmetric.

Let I be an ideal of a ring R. Following the first author(see [1]), (a_{11}, a_{12}) is an (I)-unimodular row in case there exists some invertible matrix $\mathbf{A} = (a_{ij})_{2\times 2} \in GL_2(R, I)$. We say that R satisfies unit I-stable range provided that for any (I)-unimodular row (a_{11}, a_{12}), there exist $u, v \in GL_1(R, I)$ such that $a_{11}u + a_{12}v = 1$. The condition above is very useful in the study of algebraic K-theory and it is more stronger than (ideal)-stable range condition. It is well known that $K_1(R, I) \cong GL_1(R, I)/V(R, I)$ provided that R satisfies unit I-stable range, where $V(R, I) = \{(1+ab)(1+ba)^{-1} \mid 1+ab \in U(R), (1+ab)(1+ba)^{-1} \equiv 1(\mod I)\}$(see [2, Theorem 1.2]). In [3], K_2 group was studied for commutative rings satisfying unit ideal-stable range and it was shown that $K_2(R, I)$ is generated by $\langle a, b, c\rangle_*$ provided that R is a commutative ring satisfying unit I-stable range. We refer the reader to [4-10], the papers related to stable range conditions.

In this paper, we investigate representations of general linear groups for ideals of a ring and show that unit ideal-stable range condition is right and left symmetric.

Throughout, all rings are associative with identity. $M_n(R)$ denotes the ring of $n \times n$ matrices over R and $GL_n(R, I)$ denotes the set $\{\mathbf{A} \in GL_n(R) \mid \mathbf{A} \equiv \mathbf{I}_n\big(\mod M_n(I)\big)\}$, where $GL_n(R)$ is the n dimensional general linear group of R and $\mathbf{I}_n = \operatorname{diag}(1, \dots, 1)_{n\times n}$. Write $\mathbf{B}_{12}(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$ and $\mathbf{B}_{21}(x) = \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}$. We always use $[u, v]$ to denote the matrix $\operatorname{diag}(u, v)$.

Theorem 1. *Let I be an ideal of a ring R. Then the following properties are equivalent:*

1. *R satisfies unit I-stable range.*
2. *For any $\boldsymbol{A} \in GL_2(R, I)$, there exist $u, v, w \in GL_1(R, I)$ such that $\boldsymbol{A} = [u, v]\boldsymbol{B}_{21}(*)\boldsymbol{B}_{12}(*)\boldsymbol{B}_{21}(-w)$.*

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Proof. (1) $\Rightarrow$ (2) Pick $\mathbf{A} = (a_{ij})_{2\times 2} \in GL_2(R,I)$. Then we have $u_1, v_1 \in GL_1(R,I)$ such that $a_{11}u_1 + a_{12}v_1 = 1$. So $a_{11} + a_{12}v_1u_1^{-1} = u_1^{-1}$; hence, $\mathbf{AB}_{21}(v_1u_1^{-1}) = \begin{pmatrix} u_1^{-1} & a_{12} \\ a_{21} + a_{22}v_1u_1^{-1} & a_{22} \end{pmatrix}$. Let $v = a_{22} - (a_{21} + a_{22}v_1u_1^{-1})u_1a_{12}$. Then $\mathbf{AB}_{21}(v_1u_1^{-1}) = \mathbf{B}_{21}\big((a_{21} + a_{22}v_1u_1^{-1})u_1\big)\begin{pmatrix} u_1^{-1} & a_{12} \\ 0 & v \end{pmatrix}$. It follows from $\mathbf{A}, \mathbf{B}_{21}(v_1u_1^{-1}), \mathbf{B}_{21}\big((a_{21} + a_{22}v_1u_1^{-1})u_1\big) \in GL_2(R)$ that $\begin{pmatrix} u_1^{-1} & a_{12} \\ 0 & v \end{pmatrix} \in GL_2(R)$. In addition, $\begin{pmatrix} u_1^{-1} & a_{12} \\ 0 & v \end{pmatrix} = \begin{pmatrix} u_1^{-1} & 0 \\ 0 & v \end{pmatrix}\begin{pmatrix} 1 & u_1a_{12} \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & u_1a_{12} \\ 0 & 1 \end{pmatrix} \in GL_2(R)$. This infers that $[u_1^{-1}, v] \in GL_2(R)$, and so $v \in U(R)$. Set $u = u_1^{-1}$, and $w = v_1u_1^{-1}$. Then $\mathbf{A} = [u,v]\mathbf{B}_{21}(*)\mathbf{B}_{12}(*)\mathbf{B}_{21}(-w)$. Clearly, $u, w \in GL_1(R,I)$. From $a_{22} \in 1 + I$ and $a_{12} \in I$, we have $v \in GL_1(R,I)$, as required.

(2) $\Rightarrow$ (1) For any (I)-unimodular row (a_{11}, a_{12}), we get $\mathbf{A} = (a_{ij})_{2\times 2} \in GL_2(R,I)$. So there exist $u, v, w \in GL_1(R,I)$ such that $\mathbf{A} = [u,v]\mathbf{B}_{21}(*)\mathbf{B}_{12}(*)$ $\mathbf{B}_{21}(-w)$. Hence $\mathbf{AB}_{21}(w) = [u,v]\mathbf{B}_{21}(*)\mathbf{B}_{12}(*)$, and then $a_{11} + a_{12}w = u$. That is, $a_{11}u^{-1} + a_{12}wu^{-1} = 1$. As $u^{-1}, wu^{-1} \in GL_1(R,I)$, we are done. □

Let $\mathbb{Z}$ be the integer domain, $4\mathbb{Z}$ the principal ideal of $\mathbb{Z}$. Then $1 \in GL_1(\mathbb{Z}, 4\mathbb{Z})$, while $-1 \notin GL_1(\mathbb{Z}, 4\mathbb{Z})$. But we observe the following fact.

Corollary 2. *Let I be an ideal of a ring R. Then the following are equivalent:*

(1) *R satisfies unit I-stable range.*
(2) *For any $\boldsymbol{A} \in GL_2(R,I)$, there exist $u, v, w \in GL_1(R,I)$ such that $\boldsymbol{A} = [u,v]\boldsymbol{B}_{21}(w)\boldsymbol{B}_{12}(*)\boldsymbol{B}_{21}(*)$.*

Proof. (1) $\Rightarrow$ (2) Given any $\mathbf{A} = (a_{ij})_{2\times 2} \in GL_2(R,I)$, then $\mathbf{A}^{-1} \in GL_2(R,I)$. By Theorem 1, we have $u, v, w \in GL_1(R,I)$ such that $\mathbf{A}^{-1} = [u,v]\mathbf{B}_{21}(*)\mathbf{B}_{12}(*)$ $\mathbf{B}_{21}(-w)$. Thus $\mathbf{A} = \mathbf{B}_{21}(w)\mathbf{B}_{12}(*)\mathbf{B}_{21}(*)[u^{-1}, v^{-1}] = [u^{-1}, v^{-1}]\mathbf{B}_{21}(vwu^{-1})$ $\mathbf{B}_{12}(*)\mathbf{B}_{21}(*)$. Clearly, $u^{-1}, v^{-1}, vwu^{-1} \in GL_1(R,I)$, as required.

(2) $\Rightarrow$ (1) Given any $\mathbf{A} = (a_{ij})_{2\times 2} \in GL_2(R,I)$, we have $u, v, w \in GL_1(R,I)$ such that $\mathbf{A}^{-1} = [u,v]\mathbf{B}_{21}(w)\mathbf{B}_{12}(*)$ $\mathbf{B}_{21}(*)$, and so $\mathbf{A} = \mathbf{B}_{21}(*)\mathbf{B}_{12}(*)\mathbf{B}_{21}(-w)$ $[u^{-1}, v^{-1}] = [u^{-1}, v^{-1}]\mathbf{B}_{21}(*)\mathbf{B}_{12}(*)$ $\mathbf{B}_{21}(-vwu^{-1})$. It follows by Theorem 1 that R satisfies unit I-stable range. □

Theorem 3. *Let I be an ideal of a ring R. Then the following are equivalent:*

(1) *R satisfies unit I-stable range.*
(2) *For any $\boldsymbol{A} \in GL_2(R,I)$, there exist $u, v, w \in GL_1(R,I)$ such that $\boldsymbol{A} = [u,v]\boldsymbol{B}_{12}(*)\boldsymbol{B}_{21}(*)\boldsymbol{B}_{12}(-w)$.*
(3) *For any $\boldsymbol{A} \in GL_2(R,I)$, there exist $u, v, w \in GL_1(R,I)$ such that $\boldsymbol{A} = [u,v]\boldsymbol{B}_{12}(w)\boldsymbol{B}_{21}(*)\boldsymbol{B}_{12}(*)$.*

Proof. (1) $\Rightarrow$ (2) Observe that if $\mathbf{A} \in GL_2(R,I)$, then the matrix $P^{-1}AP$ belongs to $GL_2(R,I)$, where $P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Thus the formula in Theorem 1 can be replaced by

$$\mathbf{A} = (\mathbf{P}[u,v]\mathbf{P}^{-1})(\mathbf{P}\mathbf{B}_{21}(*)\mathbf{P}^{-1})(\mathbf{P}\mathbf{B}_{12}(*)\mathbf{P}^{-1})(\mathbf{P}\mathbf{B}_{21}(-w)\mathbf{P}^{-1})\,.$$

That is, $\mathbf{A} = [v,u]\mathbf{B}_{12}(*)\mathbf{B}_{21}(*)\mathbf{B}_{12}(-w)$, as required.

(2) $\Rightarrow$ (1) For any (I)-unimodular (a_{11}, a_{12}) row, $\begin{pmatrix} * & * \\ a_{12} & a_{11} \end{pmatrix} \in GL_2(R,I)$. So we have $u,v,w \in GL_1(R,I)$ such that $\begin{pmatrix} * & * \\ a_{12} & a_{11} \end{pmatrix} = [u,v]\mathbf{B}_{12}(*)\mathbf{B}_{21}(*)\ \mathbf{B}_{12}(-w)$. Thus $a_{11} + a_{12}w = v$; hence, $a_{11}v^{-1} + a_{12}wv^{-1} = 1$. Obviously, $v^{-1}, wv^{-1} \in GL_1(R,I)$, as required.

(2) $\Leftrightarrow$ (3) is obtained by applying (1) $\Leftrightarrow$ (2) to the inverse matrix of an invertible matrix $\mathbf{A}$. $\square$

Let I be an ideal of a ring R. We use R^{op} to denote the opposite ring of R and use I^{op} to denote the corresponding ideal of I in R^{op}.

Corollary 4. *Let I be an ideal of a ring R. Then the following are equivalent:*

(1) *R satisfies unit I-stable range.*
(2) *R^{op} satisfies unit I^{op}-stable range.*

Proof. (2) $\Rightarrow$ (1) Construct a map $\varphi : M_2(R^{\mathrm{op}}) \to M_2(R)^{\mathrm{op}}$ by $\varphi\big((a_{ij}^{\mathrm{op}})_{2\times 2}\big) = \big((a_{ij})_{2\times 2}^T\big)^{\mathrm{op}}$. It is easy to check that φ is a ring isomorphism.

Given any $\mathbf{A} \in GL_2(R,I)$, $\varphi^{-1}\big(P^{\mathrm{op}}(\mathbf{A}^{-1})^{\mathrm{op}}(P^{-1})^{\mathrm{op}}\big) \in GL_2(R^{\mathrm{op}}, I^{\mathrm{op}})$, where $P = [1,-1]$. By Theorem 1, there exist $u^{\mathrm{op}}, v^{\mathrm{op}}, w^{\mathrm{op}} \in GL_1(R^{\mathrm{op}}, I^{\mathrm{op}})$ such that $\varphi^{-1}\big(P^{\mathrm{op}}(\mathbf{A}^{-1})^{\mathrm{op}}(P^{-1})^{\mathrm{op}}\big) = [u^{\mathrm{op}}, v^{\mathrm{op}}]\mathbf{B}_{21}(*^{\mathrm{op}})\ \mathbf{B}_{12}(*^{\mathrm{op}})\mathbf{B}_{21}(-w^{\mathrm{op}})$, whence $P^{-1}\mathbf{A}^{-1}P = \mathbf{B}_{12}(-w)\mathbf{B}_{21}(*)\mathbf{B}_{12}(*)[u,v]$. This means that $P^{-1}\mathbf{A}P = [u^{-1}, v^{-1}]$ $\mathbf{B}_{12}(*)\mathbf{B}_{21}(*)\mathbf{B}_{12}(w)$. So $A = (P[u^{-1}, v^{-1}]P^{-1})(P\mathbf{B}_{12}(*)P^{-1})(P\mathbf{B}_{21}(*)P^{-1})$ $(P\mathbf{B}_{12}(w)P^{-1})$. Hence $A = [u^{-1}, v^{-1}]\mathbf{B}_{12}(*)\mathbf{B}_{21}(*)B_{12}(-w)$. Clearly, u^{-1}, v^{-1}, $uwv^{-1} \in GL_1(R,I)$. According to Theorem 3, R satisfies unit I-stable range.

(1) $\Rightarrow$ (2) is symmetric. $\square$

Theorem 5. *Let I be an ideal of a ring R. Then the following are equivalent:*

(1) *R satisfies unit I-stable range.*
(2) *For any (I)-unimodular (a_{11}, a_{12}) row, there exist $u,v \in GL_1(R,I)$ such that $a_{11}u - a_{12}v = 1$.*
(3) *For any $\boldsymbol{A} \in GL_2(R,I)$, there exist $u,v,w \in GL_1(R,I)$ such that $\boldsymbol{A} = u,v]\boldsymbol{B}_{21}(*)\boldsymbol{B}_{12}(*)\boldsymbol{B}_{21}(w)$.*

Proof. (1) $\Leftrightarrow$ (2) Observe that $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in GL_2(R,I)$ if and only if $\begin{pmatrix} a_{11} & -a_{12} \\ -a_{21} & a_{22} \end{pmatrix} \in GL_2(R,I)$. Thus $(a_{11}, -a_{12})$ is an (I)-unimodular row if and only if so is (a_{11}, a_{12}), as required.

(2) $\Leftrightarrow$ (3) is similar to Theorem 1. $\square$

Let I be an ideal of a ring R. As a consequence of Theorem 5, we prove that R satisfies unit I-stable range if and only if for any $\mathbf{A} \in GL_2(R,I)$, there exist

$u, v, w \in GL_1(R, I)$ such that $\mathbf{A} = [u, v]\mathbf{B}_{12}(*)\mathbf{B}_{21}(*)\mathbf{B}_{12}(w)$. We say that $\begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix}$ is an (I)-unimodular column in case there exists $A = (a_{ij})_{2\times 2} \in GL_2(R, I)$. By the symmetry, we can derive the following.

Corollary 6. *Let I be an ideal of a ring R. Then the following are equivalent:*

(1) *R satisfies unit I-stable range.*

(2) *For any (I)-unimodular column $\begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix}$, there exist $u, v \in GL_1(R, I)$ such that $ua_{11} + va_{21} = 1$.*

(3) *For any (I)-unimodular column $\begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix}$, there exist $u, v \in GL_1(R, I)$ such that $ua_{11} - va_{21} = 1$.*

Suppose that R satisfies unit I-stable range. We claim that every element in I is an difference of two elements in $GL_1(R, I)$. For any $a \in I$, we have $\begin{pmatrix} 1 & a \\ a & 1+a^2 \end{pmatrix} = \mathbf{B}_{21}(a)\mathbf{B}_{12}(a) \in GL_2(R, I)$. This means that $(1, a)$ is an (I)-unimodular. So we have some $u, v \in GL_1(R, I)$ such that $u + av = 1$. Hence $a = v^{-1} - uv^{-1}$, as asserted.

Let I be an ideal of a ring R. Define $QM_2(R) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a + c = b + d, a, b, c, d \in R \right\}$ and $QM_2(I) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a + c = b + d, a, b, c, d \in I \right\}$. Define $Q^T M_2(R) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a + b = c + d, a, b, c, d \in R \right\}$ and $Q^T M_2(I) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a+b = c+d, a, b, c, d \in I \right\}$. As an application of the symmetry of unit ideal-stable range condition, we derive the following.

Theorem 7. *Let I be an ideal of a ring R. Then the following are equivalent:*

(1) *R satisfies unit I-stable range.*

(2) *$QM_2(R)$ satisfies unit $QM_2(I)$-stable range.*

(3) *$QM_2^T(R)$ satisfies unit $QM_2^T(I)$-stable range.*

Proof. (1) $\Rightarrow$ (2) Let $TM_2(R)$ denote the ring of all 2×2 lower triangular matrices over R, and let $TM_2(I)$ denote the ideal of all 2×2 lower triangular matrices over I. If $(\mathbf{A}_{11}, \mathbf{A}_{12})$, where $\mathbf{A}_{11} = \begin{pmatrix} a_{11} & 0 \\ a_{21} & a_{22} \end{pmatrix}$ and $\mathbf{A}_{12} = \begin{pmatrix} b_{11} & 0 \\ b_{21} & b_{22} \end{pmatrix}$, is a unimodular row, then (a_{11}, b_{11}) and (a_{22}, b_{22}) are unimodular rows, and so $a_{11}u_1 + b_{11}v_1 = 1$ and $a_{22}u_2 + b_{22}v_2 = 1$ for some u_1, u_2, v_1, $v_2 \in GL_1(R, I)$. Then there are matrices $\mathbf{U} = \begin{pmatrix} u_1 & 0 \\ ** & u_2 \end{pmatrix}, \mathbf{V} = \begin{pmatrix} v_1 & 0 \\ ** & v_2 \end{pmatrix}$ such that $\mathbf{A}_{11}\mathbf{U} + \mathbf{A}_{12}\mathbf{V} = \mathbf{I}$. Now we construct a map $\psi : QM_2(R) \to TM_2(R)$ given by $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \to \begin{pmatrix} a+c & 0 \\ c & d-c \end{pmatrix}$ for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in QM_2(R)$. For any $\begin{pmatrix} x & 0 \\ z & y \end{pmatrix} \in TM_2(R)$, we have $\psi\left(\begin{pmatrix} x-z & x-y-z \\ z & y+z \end{pmatrix} \right) =$

$\begin{pmatrix} x & 0 \\ z & y \end{pmatrix}$. Thus it is easy to verify that ψ is a ring isomorphism. Also we get that $\psi\mid_{QM_2(I)}$ is an isomorphism from $QM_2(I)$ to $TM_2(I)$. Therefore $QM_2(R)$ satisfies unit $QM_2(I)$-stable range.

(2) $\Rightarrow$ (1) As $QM_2(R)$ satisfies unit $QM_2(I)$-stable range, we deduce that $TM_2(R)$ satisfies unit $TM_2(I)$-stable range. Given any $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(R,I)$, then

$$\begin{pmatrix} \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} b & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} c & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} d & 0 \\ 0 & 1 \end{pmatrix} \end{pmatrix} \in GL_2\big(TM_n(R), TM_n(I)\big)\,.$$

Thus we have $\begin{pmatrix} u & 0 \\ ** & v \end{pmatrix} \in GL_1\big(TM_2(R), TM_2(I)\big)$ such that $\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} b & 0 \\ 0 & 0 \end{pmatrix}$ $\begin{pmatrix} u & 0 \\ ** & v \end{pmatrix} \in GL_1\big(TM_2(R), TM_2(I)\big)$. Therefore $a + bu \in GL_1(R,I)$ and $u \in GL_1(R,I)$, as desired.

(1) $\Leftrightarrow$ (3) Clearly, we have an anti-isomorphism $\psi : Q^TM_2(R) \to QM_2(R^{\mathrm{op}})$ given by $\psi\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{pmatrix} a^{\mathrm{op}} & c^{\mathrm{op}} \\ b^{\mathrm{op}} & d^{\mathrm{op}} \end{pmatrix}$ for any $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Q^TM_2(R)$. Hence $Q^TM_2(R) \cong \big(QM_2(R^{\mathrm{op}})\big)^{\mathrm{op}}$. Likewise, we have $Q^TM_2(I) \cong \big(QM_2(I^{\mathrm{op}})\big)^{\mathrm{op}}$. Thus we complete the proof by Corollary 4. □

It follows by Theorem 7 that R satisfies unit 1-stable range if and only if so does $QM_2(R)$ if and only if so does $QM_2^T(R)$.

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