ARCHIVUM MATHEMATICUM (BRNO)
Tomus 41 (2005), 59 – 69

GAP PROPERTIES OF HARMONIC MAPS AND SUBMANIFOLDS

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Abstract. In this article, we obtain a gap property of energy densities of harmonic maps from a closed Riemannian manifold to a Grassmannian and then, use it to Gaussian maps of some submanifolds to get a gap property of the second fundamental forms.

1. Introduction. Main Theorems

Let $f : (M^m, g) \to (N^n, h)$ be a smooth map between two Riemannian manifolds, $e(f) = \frac{1}{2}|df|^2$ be the energy density of f. f is called a harmonic map if it is a critical point of the energy functional

$$E(f) = \int_M e(f) dv_M \,. \tag{1}$$

It is known that (see [7]) if the Ricci curvature $\mathrm{Ric}^M \geq A > 0$ and the Riemannian sectional curvature $\mathrm{Riem}^N \leq B, B > 0$, and if f is harmonic, then $e(f) = 0$ or $e(f) = \frac{mA}{2(m-1)B}$ whenever $e(f) \leq \frac{mA}{2(m-1)B}$.

Let N be a Grassmannian, M a general closed Riemannian manifold, f a harmonic map from M to N. In this paper, we find some non-negative numbers A, B ($A < B$) such that if $A \leq e(f) \leq B$, then $e(f)$ equals to A or B.

We denote the Laplace-Beltrami operator on (M^m, g) by Δ_M. Then $-\Delta_M$ has a discrete spectrum:

$$\mathrm{spec}(\Delta_M) = \{0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots \to \infty\}. \tag{2}$$

Let

$$A(p,k) = \frac{p}{2(2p-1)}\left(\lambda_k + \lambda_{k+1} - \sqrt{\lambda_k^2 + \lambda_{k+1}^2 + \frac{4-6p}{p}\lambda_k\lambda_{k+1}}\right) \tag{3}$$

2000 *Mathematics Subject Classification*: 58E20, 53C43.

Key words and phrases: Grassmannian, Gaussian map, mean curvature, the second fundamental form.

*Research supported by National Natural Science Fundation of China No. 19901010, Fok Ying-Tung Education Fundation, and COCDM Project.

†Research supported by National Natural Science Fundation of China No. 10371047.

Received April 11, 2003.

and

$$B(p,k)=\frac{p}{2(2p-1)}\left(\lambda_k+\lambda_{k+1}+\sqrt{\lambda_k^2+\lambda_{k+1}^2+\frac{4-6p}{p}\lambda_k\lambda_{k+1}}\right). \tag{4}$$

Then $A(p,0)=0$, $B(p,0)=\frac{p}{2p-1}\lambda_1$; $A(1,k)=\lambda_k$, $B(1,k)=\lambda_{k+1}$. Let $G_{m,p}$ be the Grassmannian consisting of linear oriented m-subspaces of the Euclidean $m+p$-space. One can embedding it into the Euclidean space of m-wedge vectors. We denote the image of $G_{m,p}$ under this embedding still by $G_{m,p}$. We obtain

Theorem A. *Let $f : M^q \to G_{m,p}$ be harmonic. If $A(p,k) \le 2e(f) \le B(p,k)$ for some k, then $2e(f) = A(p,k)$ or $2e(f) = B(p,k)$. Especially, we have*

(1) *Let $f : M \to S^m(1)$ be harmonic. If $\lambda_k \le 2e(f) \le \lambda_{k+1}$ for some $k \ge 0$, then $2e(f) = \lambda_k$ or λ_{k+1}.*

(2) *Let $f : M \to G_{m,p}$ be harmonic. If $2e(f) \le \frac{p}{2p-1}\lambda_1$, then $2e(f) = \frac{p}{2p-1}\lambda_1$ or 0.*

As a corollary, we have

Theorem B. *Let M^m be a closed submanifold of E^{m+p} with parallel mean curvature, σ the square length of the second fundamental form. If $A(p,k) \le \sigma \le B(p,k)$ for some $k \ge 0$, then $\sigma = A(p,k)$ or $\sigma = B(p,k)$.*

Especially, we have

(1) *if $p = 1$ and $\lambda_k \le \sigma \le \lambda_{k+1}$, then $\sigma = \lambda_k$ or λ_{k+1};*

(2) *if $p \ge 2$ and $\sigma \le \frac{p}{2p-1}\lambda_1$, then $\sigma = 0$ or $\frac{p}{2p-1}\lambda_1$.*

S. S. Chern et al proved that if the square length σ of the second fundamental form of a minimal submanifold of spheres satisfies $\sigma \le \frac{mp}{2p-1}$, then $\sigma = 0$ or $\frac{mp}{2p-1}$. Our Theorem B shows that the similar gap phenomenon exists for submanifolds of the Euclidean space with parallel mean curvature. Our method is very different from theirs.

2. Preliminaries

Let M^m and N^n be two Riemannian manifolds, $f : M \to N$ be a smooth map. On M, we choose a local orthonormal field of frame around $x \in M$: $e = \{e_i, i = 1,\dots,m\}$. The dual is denoted by $\omega = \{\omega_i\}$. The corresponding fields around $f(x)$ are $e^* = \{e^*_\alpha, \alpha = 1,\dots,n\}$ and $\omega^* = \{\omega^*_\alpha\}$. We use the convention of summation. The ranges of indices in this section are:

$$i,j,\dots = 1,2,\dots,m\,; \qquad \alpha,\beta,\dots = 1,2,\dots,n\,. \tag{5}$$

Then the Riemann metrics of M and N can be written respectively as

$$ds^2_M = \sum \omega_i^2\,; \qquad ds^2_N = \sum \omega_\alpha^{*2}\,. \tag{6}$$

Let

$$f^*\omega^*_\alpha = \sum a_{\alpha i}\omega_i\,. \tag{7}$$

then

$$f^*ds^2_N = \sum a_{\alpha i}a_{\alpha j}\omega_i\omega_j\,. \tag{8}$$

Hence, the energy density of f is:

$$e(f) = \frac{1}{2}\mathrm{tr} f^* ds_N^2 = \frac{1}{2}\sum (a_{\alpha i})^2 . \tag{9}$$

The structure equations of M are:

$$\mathrm{d}\omega_i = \sum \omega_j \wedge \omega_{ji}, \omega_{ij} + \omega_{ji} = 0 , \tag{10}$$

$$\mathrm{d}\omega_{ij} = \sum \omega_{ik} \wedge \omega_{kj} + \Omega_{ij}, \Omega_{ij} = -\frac{1}{2}\sum R_{ijkl}\omega_k \wedge \omega_l , \tag{11}$$

where R_{ijkl} is the Riemannian curvature tensor of M. Take exterior differentiation in (7) and use the structure equations of M and N. we have

$$\sum Da_{\alpha i} \wedge \omega_i = 0 \tag{12}$$

where

$$Da_{\alpha i} := \mathrm{d}a_{\alpha i} + \sum a_{\alpha j}\omega_{ji} + \sum a_{\beta i}\omega^*_{\beta\alpha} \circ f =: \sum a_{\alpha ij}\omega_j . \tag{13}$$

By Cartan's Lemma, we have

$$a_{\alpha ij} = a_{\alpha ji} . \tag{14}$$

Define

$$b(f) = \sum a_{\alpha ij}\omega_i \otimes \omega_j \otimes e^*_\alpha \circ f \in \Gamma(T^*M \otimes T^*M \otimes f^{-1}TN) . \tag{15}$$

We call $b(f)$ the second fundamental form of f, $\tau(f) := \mathrm{tr}\, b(f) = \sum a_{\alpha ii} e^*_\alpha \circ f$ the tension field of f. Then $\tau(f) = 0$ if and only if f is harmonic. If $b(f) = 0$, we say that f is totally geodesic. Apparently,

$$\tau(f) = 0 \Longleftrightarrow \sum a_{\alpha ii} = 0 ; \quad b(f) = 0 \Longleftrightarrow a_{\alpha ij} = 0 . \tag{16}$$

Let P be the set of all orthonormal frame of the $m+p$-dimensional Euclidean space E^{m+p} with the positive orientation. On P, we introduce an equivalent relation $\sim$: $e = (e_1, \ldots, e_{m+p}) \sim \overline{e} = (\overline{e}_1, \ldots, \overline{e}_{m+p})$ if and only if $(\overline{e}_1, \ldots, \overline{e}_m) = (e_1, \ldots, e_m) \cdot g$, if and only if $(\overline{e}_{m+1}, \ldots, \overline{e}_{m+p}) = (e_{m+1}, \ldots, e_{m+p}) \cdot h$ where $g \in SO(m)$ and $h \in SO(p)$. We denote $P/\sim$ by $G_{m,p}$. It can be identified with $\frac{SO(m+p)}{SO(m)\times SO(p)}$, also with the space consisting of oriented m-linear subspace of E^{m+p}. We call it a Grassmannian.

Let $V = \wedge^m E^{m+p}$ be the space of m-degree wedge product of E^{m+p}. There is a natural inner product in V:

$$\langle e_{i_1} \wedge \cdots \wedge e_{i_m}, e_{j_1} \wedge \cdots \wedge e_{j_m} \rangle = \delta^{i_1\ldots i_m}_{j_1\ldots j_m} , \tag{17}$$

with respect to which, V forms a $K = C^m_{m+p}$-dimensional Euclidean space, where $(e_1, \ldots, e_{m+p}) \in P$ and $i_k, j_k \in \{1, \ldots, m+p\}$, $k = 1, \ldots, m$.

We define a map $i : G_{m,p} \to V$ by:

$$X \mapsto e_1 \wedge \cdots \wedge e_m \tag{18}$$

for any $X = [e_1, \ldots, e_{m+p}] \in G_{m,p}$, the equivalent class of $(e_1, \ldots, e_{m+p}) \in P$ with respect to the relation $\sim$. Then i is an embedding (see [1]) from $G_{m,p}$ to V (precisely to S^{K-1}). We denote $i(G_{m,p})$ still by $G_{m,p}$.

In the rest of this section, our indice ranges are:

$$\begin{aligned} &i,j,k,l=1,\dots,m\,; \quad a,b,c,d=m+1,\dots,m+p\,;\\ &A,B,C,D=1,\dots,m+p\,. \end{aligned} \tag{19}$$

The motion equation of point x in E^{m+p} is:

$$\mathrm{d}x=\sum \omega_A e_A\,, \tag{20}$$

and the motion equation of the frame $\{e_A\}$ is:

$$\mathrm{d}e_A=\sum \omega_{AB} e_B\,. \tag{21}$$

Then the structure equations of E^{m+p} are:

$$\mathrm{d}\omega_A=\sum \omega_B\wedge\omega_{BA}, \omega_{AB}+\omega_{BA}=0\,, \tag{22}$$

$$\mathrm{d}\omega_{AB}=\sum \omega_{AC}\wedge\omega_{CB}\,. \tag{23}$$

For any $X\in G_{m,p}$, we can set $X=e_1\wedge\cdots\wedge e_m$. We have

$$\begin{aligned} \mathrm{d}X&=\mathrm{d}(e_1\wedge\cdots\wedge e_m)\\ &=\sum_i e_1\wedge\cdots\wedge e_{i-1}\wedge \mathrm{d}e_i\wedge e_{i+1}\wedge\cdots\wedge e_m\\ &=\sum_i e_1\wedge\cdots\wedge e_{i-1}\wedge(\sum_j \omega_{ij}e_j+\sum_a \omega_{ia}e_a)\wedge e_{i+1}\wedge\cdots\wedge e_m\\ &=\sum \omega_{ia}E_{ia} \end{aligned} \tag{24}$$

where $E_{ia}=e_1\wedge\cdots\wedge e_{i-1}\wedge e_a\wedge e_{i+1}\wedge\cdots\wedge e_m$. Hence, $\{E_{ia}\}$ forms a base of $T_XG_{m,p}$. Let $ds_G^2=\sum(\omega_{ia})^2$. Then it is a Riemannian metric making $\{E_{ia}\}$ orthonormal.

Let M be an m-dimensional submanifold of E^{m+p}. Identify the oriented tangent space at any point of M with an oriented m-dimensional linear subspace of E^{m+p} in the natural way. Suppose that $(e_1,\dots,e_m)$ is a frame of the tangent space with the positive orientation. Then, $\omega_a=0$. Therefore, $\omega_{ia}=\sum h^a_{ij}\omega_j$, $h^a_{ij}=h^a_{ji}$. We call (h^a_{ij}) the Weingarten matrix of M in E^{m+p}. We define the Gaussian map $g: M\to G_{m,p}$ of M by

$$g(x)=e_1\wedge\cdots\wedge e_m\,. \tag{25}$$

Then, by (24) we have, the tangent and the cotangent map g_* and g^* of g at x are

$$g_*e_i=dg(e_i)=\sum \omega_{ja}(e_i)E_{ja}=\sum h^a_{ji}E_{ja}\,, \tag{26}$$

$$g^*\omega_{ia}=\sum h^a_{ij}\omega_j\,. \tag{27}$$

By (7), (9) and (27) we know that the energy density of g is

$$e(g)=\frac{1}{2}\sum(h^a_{ij})^2=\frac{1}{2}\sigma\,, \tag{28}$$

where σ is the square length of the second fundamental form of M in E^{m+p}.

Hence we have

Lemma 2.1 *Let M^m be a submanifold of E^{m+p}, g the Gussian map of M^m, σ the square length of the second fundamental form of the submanifold. Then we have*

$$\sigma = 2e(g)\,. \tag{29}$$ □

Suppose that M^q is any q-dimensional closed manifold. Consider the following composition:

$$M \xrightarrow{f} G_{m,p} \xrightarrow{\iota} V\,, \tag{30}$$

where ι is the the inclusion of $G_{m,n}$ in V (noting that we have embedded $G_{m,n}$ into V). Let $F = \iota \circ f$. In the following, we calculate the Laplacian of F.

For any $x \in M$, set $f(x) = e_1 \wedge \cdots \wedge e_m \in G_{m,p}$, where $(e_1, \ldots e_{m+p}) \in P$. Then $F(x) \in V$. The ranges of indices in this section are the same as the above section. But $u \in \{1, \ldots, q\}$. Let $\{\epsilon_u, u = 1, \ldots, q\}$ be a local orthonormal field of frame around x, whose dual is $\{\theta_u\}$, and let

$$f^*\omega_{ia} = \sum a^a_{iu}\theta_u\,. \tag{31}$$

Then we have

Lemma 2.2

$$-\Delta_M F = \tau(f) + 2e(f)F + G\,, \tag{32}$$

where

$$G = \begin{cases} 2\sum_{i<j,a<b}\sum_u (a^a_{iu}a^b_{ju} - a^b_{iu}a^a_{ju})E_{ia,jb}\circ f\,, & m,p \geq 2\,;\\ 0\,, & \text{otherwise.}\end{cases} \tag{33}$$

Here $E_{ia,jb} = E_{jb,ia} = e_1 \wedge \cdots \wedge e_{i-1} \wedge e_a \wedge e_{i+1} \wedge \cdots \wedge e_{j-1} \wedge e_b \wedge e_{j+1} \wedge \cdots \wedge e_m$. It is a normal vector of $G_{m,p}$ in V.

Proof. Notice that $\{E_{ia}\}$ is an orthonormal base, whose dual is $\{\omega_{ia}\}$. By the structure equation (23) we have

$$\begin{aligned} \mathrm{d}\omega_{ia} &= \sum \omega_{ij} \wedge \omega_{ja} + \sum \omega_{ib} \wedge \omega_{ba}\\ &= \sum \omega_{jb} \wedge (-\omega_{ij}\delta_{ba} + \omega_{ba}\delta_{ij})\\ &\equiv \omega_{jb} \wedge \omega^*_{jb,ia} \circ f \end{aligned} \tag{34}$$

where $\omega^*_{jb,ia} \circ f = -\omega_{ij}\delta_{ba} + \omega_{ba}\delta_{ij}$ are the connection forms of $G_{m,p}$.

The tension field of f is

$$\tau(f) = \sum a^a_{iuu} E_{ia} \circ f \tag{35}$$

where (see (13))

$$\sum a^a_{iuv}\theta_v = \mathrm{d}a^a_{iu} - \sum a^a_{iv}\theta_{uv} + \sum a^b_{ju} f^*\omega^*_{jb,ia}\,. \tag{36}$$

Let $f_* = f_u\theta_u$. Then by (31) we have $f_u = \sum a^a_{iu}E_{ia} \circ f$.

Therefore

$$\sum f_{uv}\theta_v = \mathrm{d}f_u - \sum f_v\theta_{uv} = \sum \mathrm{d}a^a_{iu}\cdot E_{ia}\circ f \\ + \sum a^a_{iu}\mathrm{d}(E_{ia}\circ f) - \sum a^a_{iv}E_{ia}\circ f\theta_{uv}\,. \tag{37}$$

It is not difficult to check that if $m, p \geq 2$, we have

$$\mathrm{d}(E_{ia}\circ f) = -f^*\omega_{ji}E_{ja}\circ f + f^*\omega_{jb}E_{jb,ia}\circ f + f^*\omega_{ai}F + f^*\omega_{ab}E_{ib}\circ f\,,$$

and that if $m = 1$ or $p = 1$, we have

$$\mathrm{d}(E_{ia}\circ f) = -f^*\omega_{ji}E_{ja}\circ f + f^*\omega_{ai}F + f^*\omega_{ab}E_{ib}\circ f\,.$$

When $m, p \geq 2$,

$$\begin{aligned}\sum f_{uv}\theta_v &= \sum(a^a_{iuv}\theta_v + a^a_{iv}\theta_{uv} - a^b_{ju}f^*\omega^*_{jb,ia})E_{ia}\circ f \\ &\quad + \sum a^a_{iu}(-f^*\omega_{ji}E_{ja}\circ f + f^*\omega_{jb}E_{jb,ia}\circ f + f^*\omega_{ai}F + f^*\omega_{ab}E_{ib}\circ f) \\ &\quad - \sum a^a_{iv}E_{ia}\circ f\theta_{uv} \\ &= \sum(a^a_{iuv}\theta_v + a^a_{iv}\theta_{uv} - a^b_{ju}(-f^*\omega_{ij}\delta_{ba} + f^*\omega_{ba}\delta_{ij}))E_{ia}\circ f \\ &\quad + \sum a^a_{iu}(-f^*\omega_{ji}E_{ja}\circ f + f^*\omega_{jb}E_{jb,ia}\circ f + f^*\omega_{ai}F + f^*\omega_{ab}E_{ib}\circ f) \\ &\quad - \sum a^a_{iv}E_{ia}\circ f\theta_{uv} \\ &= \sum_{i,a,v} a^a_{iuv}E_{ia}\theta_v + \sum_{i\neq j, a\neq b} a^a_{iu}a^b_{jv}E_{ia,jb}\theta_v - \sum_{i,a,v} a^a_{iu}a^a_{iv}F\theta_v\,.\end{aligned} \tag{38}$$

Because $\Delta F = \Delta f = \sum f_{uu}$, we have

$$\Delta_M F = \tau(f) - 2e(f)F + 2\sum_{i<j, a<b}\sum_u (a^a_{iu}a^b_{ju} - a^b_{iu}a^a_{ju})E_{ia,jb}\circ f\,. \tag{39}$$

Similarly, When $m = 1$ or $p = 1$, we have

$$\Delta_M F = \tau(f) - 2e(f)F\,. \tag{40}$$

The lemma follows. □

The following theorem is well known:

Lemma 2.3 (Ruh-Vilms' Theorem) *Suppose that M is a submanifold of the Euclidean space. Then M has a parallel mean cavature if and only if its Gaussian map is harmonic.*

For the proofs, see [6] and [3]. Here we give another one.

Proof. Let $g_* = \sum A_{(ja)i}\omega_i\otimes E_{ja}\circ g \in \Gamma(T^*M\otimes g^{-1}(TG_{m,p}))$. Then by (26), we have $A_{(ka)i} = h^a_{ki}$. The latter is in $\Gamma(T^*M\otimes T^*M\otimes NM)$ where NM is the normal bundle of M. We denote the covariant derivative of h^a_{ki} in $\Gamma(T^*M\otimes g^{-1}(TG_{m,p}))$

by $h^a_{ki;j}$, and that in $\Gamma(T^*M \otimes T^*M \otimes NM)$ by $h^a_{ki|j}$. Then

$$\begin{aligned}\sum h^a_{ki;j}\omega_j &= \mathrm{d}h^a_{ki} + \sum h^a_{kj}\omega_{ji} + \sum h^b_{li}\omega^*_{(lb)(ka)} \circ g \\ &= \mathrm{d}h^a_{ki} + \sum h^a_{kj}\omega_{ji} + \sum h^b_{li}(-\omega_{kl}\delta_{ba} + \omega_{ba}\delta_{kl}) \\ &= \mathrm{d}h^a_{ki} + \sum h^a_{kj}\omega_{ji} - \sum h^a_{li}\omega_{kl} + \sum h^b_{ki}\omega_{ba} \\ &= \sum h^a_{ki|j}\omega_j\,. \end{aligned} \tag{41}$$

Hence $\tau(g)_{(ka)} = h^a_{ki;i} = h^a_{ki|i} = h^a_{ik|i} = h^a_{ii|k}$. The lemma follows. □

Let A be a $m \times n$ matrix, A' its transport. Define $N(A) = \mathrm{tr}(AA')$. Then, we have

Lemma 2.4 *$N(AB' - BA') \le 2N(A)N(B)$ for $m \times n$ matrices A and B*

This inequality is proved by G. R. Wu and W. H. Chen in [9]. For completeness, we prove it in the following.

Proof. $N(A)$ is invariant under orthogonal transformations. Put $C = AB' - BA'$. It is anti-symmetric. By the theory of linear algebra, $\exists U \in O(m)$ such that

$$UCU' = \tilde{C} = \mathrm{diag}\left(\begin{pmatrix} 0 & \lambda_1 \\ -\lambda_1 & 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 & \lambda_p \\ -\lambda_p & 0 \end{pmatrix}, 0\right) \tag{42}$$

where $2p = \mathrm{rank}C$, $\lambda_1, \dots, \lambda_p$ are non-zero real numbers, the last 0 is a zero matrix of $(m-2p) \times (m-2p)$. Let $\tilde{A} = UA = (\xi^\alpha_i)$ and $\tilde{B} = UB = (\eta^\alpha_i)$. Then we have

$$\tilde{C}_{2r-1,2r} = \sum_\alpha (\xi^\alpha_{2r-1}\eta^\alpha_{2r} - \xi^\alpha_{2r}\eta^\alpha_{2r-1}) = \lambda_r\,, \quad 1 \le r \le p\,. \tag{43}$$

Hence we have

$$\begin{aligned} N(C) = N(\tilde{C}) &= 2\sum_{r=1}^p \left(\sum_\alpha (\xi^\alpha_{2r-1}\eta^\alpha_{2r} - \xi^\alpha_{2r}\eta^\alpha_{2r-1})\right)^2 \\ &= 2\sum_{r=1}^p (X_r \cdot Y_r)^2 \end{aligned} \tag{44}$$

where $X_r = (\xi^1_{2r-1}, \dots, \xi^n_{2r-1}, \xi^1_{2r}, \dots, \xi^n_{2r})$, $Y_r = (\eta^1_{2r}, \dots, \eta^n_{2r}, -\eta^1_{2r-1}, \dots, -\eta^n_{2r-1})$, $X_r \cdot Y_r$ stands for the euclidean inner product. By Schwarz inequality we have

$$\begin{aligned} N(C) &= 2\sum_{r=1}^p (X_r \cdot Y_r)^2 \le 2\sum_{r=1}^p |X_r|^2|Y_r|^2 \\ &\le 2\sqrt{\sum_{r=1}^p |X_r|^4}\sqrt{\sum_{r=1}^p |Y_r|^4} \le 2\sum_{r=1}^p |X_r|^2 \sum_{r=1}^p |Y_r|^2 \\ &\le 2N(\tilde{A})N(\tilde{B}) = 2N(A)N(B) \end{aligned} \tag{45}$$

as desired. □

3. Proofs of Theorems A and B

Proof of Theorem A

Expand F as $F = F_0 + \sum_{s\geq 1} F_s$, where F_0 is a constant vector called the mass center of F or f, F_s, $s \geq 0$ are eigenfunctions of Δ_M with respect to the eigenvalues λ_s, i.e.

$$\Delta_M F_s = -\lambda_s F_s \,. \tag{46}$$

If $F_0 = 0$, we say that F or f is mass-symmetric. If $\exists u_i \geq 1, i = 1, \dots, k$, such that $F = F_0 + \sum_{i=1}^{k} F_{u_i}$, then F or f is called of k-type and $\{u_1, \dots, u_k\}$ is by definition the order of F or f. For example, if f is a minimal isometric immersion of M^q into S^{q+p}, then $F = i \circ f$ is mass symmetric, of 1-type and its order is $\{k\}$ for some $k \geq 1$ by Takahashi theorem([8]):

$$\Delta_M F = HF - qF \tag{47}$$

where H is the mean curvature of f.

Denote

$$\Psi_k = -\int_M \langle \Delta_M F, F\rangle dv_M - \lambda_k \int_M \langle F, F\rangle dv_M \,, \tag{48}$$

$$\Theta_k = \int_M \langle \Delta_M F, \Delta_M F\rangle dv_M + \lambda_k \int_M \langle \Delta_M F, F\rangle dv_M \,. \tag{49}$$

Then

$$\begin{aligned} \Psi_k &= \int_M \langle \sum \lambda_s F_s, \sum F_s\rangle dv_M - \lambda_k \int_M \langle \sum F_s, \sum F_s\rangle dv_M \\ &= \sum \lambda_s \int_M \langle F_s, F_s\rangle dv_M - \sum \lambda_k \int_M \langle F_s, F_s\rangle dv_M \\ &= \sum \lambda_s a_s - \sum \lambda_k a_s \end{aligned} \tag{50}$$

where $a_s = \int_M \langle F_s, F_s\rangle dv_M$. Similarly

$$\Theta_k = \sum \lambda_s^2 a_s - \lambda_k \sum \lambda_s a_s \,. \tag{51}$$

Accordingly

$$\begin{aligned} \Theta_k - \lambda_{k+1}\Psi_k = \lambda_k \lambda_{k+1} a_0 + \sum_{s\geq 1} (\lambda_s - \lambda_k)(\lambda_s - \lambda_{k+1}) a_s \geq 0 \,, \\ \forall k \geq 0 \,, \end{aligned} \tag{52}$$

and the equality holds if and only if F is

(a) of 1-type and its order is $\{1\}$ when $k = 0$;

(b) of 2-type and its order is $\{k, k+1\}$ when $k \geq 1$.

On the other hand, by (32), and noting that $E_{ia,jb}$ is normal to $G_{m,p}$ at $f(x)$, and also normal to $F(x)$ (as a vector in V), we have:

$$\int_M \langle F, F\rangle dv_M = V_M \text{ the volume of } M^q\,; \tag{53}$$

$$\int_M \langle \Delta_M F, F\rangle dv_M = -2E(f)\,, \quad \text{by Lemma 2.2 and noting that } \tau(f)(x)\perp F(x)\,; \tag{54}$$

$$\int_M \langle \Delta_M F, \Delta_M F\rangle dv_M = \int_M \langle \tau(f), \tau(f)\rangle dv_M + \int_M |df|^4 dv_M + \int_M |G|^2 dv_M\,. \tag{55}$$

Hence,

$$\Psi_k = 2E(f) - \lambda_k V_M\,; \tag{56}$$

$$\Theta_k = \int_M \langle \tau(f), \tau(f)\rangle dv_M + \int_M |df|^4 dv_M + \int_M |G|^2 dv_M - 2\lambda_k E(f)\,. \tag{57}$$

From (52), (56) and (57) we get:

$$\int_M \langle \tau(f), \tau(f)\rangle dv_M + \int_M |G|^2 dv_M + \int_M (|df|^2 - \lambda_k)(|df|^2 - \lambda_{k+1}) dv_M \geq 0\,. \tag{58}$$

So, when p is 1, we have

$$\int_M \langle \tau(f), \tau(f)\rangle dv_M + \int_M (|df|^2 - \lambda_k)(|df|^2 - \lambda_{k+1}) dv_M \geq 0\,, \tag{59}$$

whence, if $\tau = 0$, we have

$$\int_M (|df|^2 - \lambda_k)(|df|^2 - \lambda_{k+1}) dv_M \geq 0\,,$$

i.e.

$$\int_M (2e(f) - \lambda_k)(2e(f) - \lambda_{k+1}) dv_M \geq 0\,. \tag{60}$$

When $m, p \geq 2$, we put $A_a = (a^a_{iu})$ be $m \times q$ matrices. From Lemma 2.4, we have

$$|G|^2 = 2\sum_{i<j,a<b}\Big(\sum_u (a^a_{iu}a^b_{ju} - a^b_{iu}a^a_{ju})\Big)^2 = \sum_{a<b}\sum_{i,j}\Big(\sum_u (a^a_{iu}a^b_{ju} - a^b_{iu}a^a_{ju})\Big)^2$$

$$= \sum_{a<b} N(A_aA'_b - A_bA'_a) \leq 2\sum_{a<b} N(A_a)N(A_b)$$

$$= \bigg(\Big(\sum_a N(A_a)\Big)^2 - \sum_a (N(A_a))^2\bigg) \leq \frac{p-1}{p}\Big(\sum_a N(A_a)\Big)^2$$

$$= \frac{(p-1)}{p}|df|^4 \,. \tag{61}$$

Insert it into (58), we have

$$\int_M \langle \tau(f), \tau(f)\rangle dv_M + \int_M \Big(\frac{2p-1}{p}|df|^4 - (\lambda_k + \lambda_{k+1})|df|^2 + \lambda_k\lambda_{k+1}\Big) dv_M \geq 0\,, \tag{62}$$

i.e.

$$\int_M \langle \tau(f), \tau(f)\rangle dv_M + \frac{2p-1}{p}\int_M (|df|^2 - A(p,k))(|df|^2 - B(p,k)) dv_M \geq 0\,. \tag{63}$$

If f is harmonic, then $\tau(f) = 0$. Therefore (63) becomes

$$\int_M (|df|^2 - A(p,k))(|df|^2 - B(p,k)) dv_M \geq 0\,, \tag{64}$$

i.e.

$$\int_M (2e(f) - A(p,k))(2e(f) - B(p,k)) dv_M \geq 0\,. \tag{65}$$

This inequality is also valid for $p = 1$ by (60). Hence if $A(p,k) \leq 2e(f) \leq B(p,k)$ for some $p \geq 1$ and some $k \geq 0$, then the integrand in (65) is non-positive, hence vanishing. So $2e(f) = A(p,k)$ or $2e(f) = B(p,k)$. Theorem A follows. □

Proof of Theorem B

By Theorem A, Ruh-Vilms' Theorem (Lemma 2.3) and Lemma 2.1, Theorem B follows. □

Remark 3.1. The order of the map in Theorem A must be $\{1\}$ when $k = 0$ or $\{k, k+1\}$ when $k \geq 1$.

Remark 3.2. When $p = 1$, $G_{m,p} = S^m$. From (60) we conclude that

(i) If f is mass symmetric and of order $\{k, k+1\}$, and $2e(f) \leq \lambda_k$ or $2e(f) \geq \lambda_{k+1}$ for some $k \geq 1$, then f is harmonic, and $2e(f) = \lambda_k$ or $2e(f) = \lambda_{k+1}$.

(ii) If f is of order $\{1\}$ and $2e(f) \geq \lambda_1$, then f is harmonic and $2e(f) = \lambda_1$.

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