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(σ,τ)-DERIVATIONS ON PRIME NEAR RINGS

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Abstract. There is an increasing body of evidence that prime near-rings with derivations have ring like behavior, indeed, there are several results (see for example [1], [2], [3], [4], [5] and [8]) asserting that the existence of a suitably-constrained derivation on a prime near-ring forces the near-ring to be a ring. It is our purpose to explore further this ring like behaviour. In this paper we generalize some of the results due to Bell and Mason [4] on near-rings admitting a special type of derivation namely (σ,τ)- derivation where σ,τ are automorphisms of the near-ring. Finally, it is shown that under appropriate additional hypothesis a near-ring must be a commutative ring.

1. Introduction

Throughtout the paper N will denote a zero symmetric left near-ring with multiplicative centre Z. An element x of N is said to be distributive if $(y+z)x = yx + zx$ for all $x, y, z \in N$. A near-ring N is called zero symmetric if $0x = 0$ for all $x \in N$ (recall that left distributivity yields $x0 = 0$). An additive mapping $d : N \longrightarrow N$ is said to be a derivation on N if $d(xy) = xd(y)+d(x)y$ for all $x, y \in N$ or equivalently, as noted in [8], that $d(xy) = d(x)\,y + x\,d(y)$ for all $x, y \in N$. Following [5], an additive mapping $d : N \longrightarrow N$ is called a σ-derivation if there exists an automorphism $\sigma : N \longrightarrow N$ such that $d(xy) = \sigma(x)\,d(y) + d(x)\,y$ for all $x, y \in N$. Further this as a motivation we define an additive mapping $d : N \longrightarrow N$ is called a (σ,τ)-derivation if there exists automorphisms $\sigma,\tau : N \longrightarrow N$ such that $d(xy) = \sigma(x)\,d(y) + d(x)\tau(y)$ for all $x, y \in N$. In case $\sigma = 1$, the identity mapping, d is called τ-derivation. Similarly if $\tau = 1, d$ is called σ-derivation. It is straightforward that an $(1,1)$-derivation is ordinary derivation. For $x, y \in N$, the symbol $[x, y]$ will denote the commutator $xy - yx$ while the symbol (x, y) will denote the additive commutator $x+y-x-y$. Following [5] for $x, y \in N$, the symbol $[x, y]_{\sigma,\tau}$ will denote the (σ,τ)-commutator $\sigma(x)y - y\tau(x)$ while (σ,τ)-derivation d will be called (σ,τ)-commuting if $[x, d(x)]_{\sigma,\tau} = 0$ for all $x \in N$. A near-ring N is

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said to be prime if $aNb = (0)$ implies that $a = 0$ or $b = 0$. Further an element $x \in N$ for which $d(x) = 0$ is called a constant.

Some recent results on rings deal with commutativity of prime and semi-prime rings admitting suitably constrained derivations. It is natural to look for comparable results on near-rings and this has been done in [1], [2], [3], [4], [5] and [8]. It is our purpose to extend some of these results on prime near-rings admitting suitably constrained (σ, τ)-derivation.

2. Preliminary results

We begin with the following lemmas which are useful in sequel.

Lemma 2.1. *An additive endomorphism d on a near-ring N is a (σ, τ)-derivation if and only if $d(xy) = d(x)\,\tau(y) + \sigma(x)\,d(y)$, for all $x, y \in N$.*

Proof. Let d be a (σ, τ)-derivation on a near-ring N. Since $x(y + y) = xy + xy$, we obtain

$$\begin{aligned} d(x(y+y)) &= \sigma(x)\,d(y+y) + d(x)\,\tau(y+y) \\ &= \sigma(x)\,d(y) + \sigma(x)\,d(y) + d(x)\,\tau(y) \\ &\quad + d(x)\,\tau(y)\,, \quad \text{for all} \quad x, y \in N\,. \end{aligned} \tag{2.1}$$

On the other hand, we have

$$\begin{aligned} d(xy + xy) &= d(xy) + d(xy) \\ &= \sigma(x)\,d(y) + d(x)\,\tau(y) + \sigma(x)\,d(y) + d(x)\,\tau(y) \\ &\quad \text{for all} \quad x, y \in N\,. \end{aligned} \tag{2.2}$$

Combining (2.1) and (2.2), we find that

$$\sigma(x)\,d(y) + d(x)\,\tau(y) = d(x)\,\tau(y) + \sigma(x)\,d(y)\,, \quad \text{for all} \quad x, y \in N\,.$$

Thus, we have

$$d(xy) = d(x)\,\tau(y) + \sigma(x)\,d(y)\,, \quad \text{for all} \quad x, y \in N\,. \tag{2.3}$$

Conversely, let $d(xy) = d(x)\,\tau(y) + \sigma(x)\,d(y)$, for all $x, y \in N$. Then

$$\begin{aligned} d(x(y+y)) &= d(x)\,\tau(y+y) + \sigma(x)\,d(y+y) \\ &= d(x)\,\tau(y) + d(x)\,\tau(y) + \sigma(x)\,d(y) \\ &\quad + \sigma(x)\,d(y) \quad \text{for all} \quad x, y \in N\,. \end{aligned} \tag{2.4}$$

Also,

$$\begin{aligned} d(xy + xy) &= d(xy) + d(xy) \\ &= d(x)\,\tau(y) + \sigma(x)\,d(y) + d(x)\,\tau(y) + \sigma(x)\,d(y)\,, \\ &\quad \text{for all} \quad x, y \in N\,. \end{aligned} \tag{2.5}$$

Combining (2.4) and (2.5), we obtain

$$d(x)\,\tau(y) + \sigma(x)\,d(y) = \sigma(x)\,d(y) + d(x)\,\tau(y)\,, \quad \text{for all} \quad x, y \in N\,. \qquad \square$$

Lemma 2.2. *Let d be a (σ,τ)-derivation on the near-ring N. Then N satisfies the following partial distributive laws:*

(i) $(\sigma(x)\,d(y)+d(x)\,\tau(y))z=\sigma(x)\,d(y)z+d(x)\,\tau(y)z$, *for all* $x,y,z\in N$.

(ii) $(d(x)\,\tau(y)+\sigma(x)\,d(y))z=d(x)\,\tau(y)z+\sigma(x)\,d(y)z$, *for all* $x,y,z\in N$.

Proof. Note that for all $x,y,z\in N$,

$$d((xy)z)=\sigma(x)\sigma(y)\,d(z)+(\sigma(x)\,d(y)+d(x)\,\tau(y))\tau(z)\,. \tag{2.6}$$

On the other hand, we have

$$\begin{aligned} d(x(yz))=\sigma(x)\sigma(y)\,d(z)+\sigma(x)\,d(y)\tau(z)\\ +\,d(x)\,\tau(y)\tau(z)\,,\quad \text{for all}\quad x,y,z\in N\,. \end{aligned} \tag{2.7}$$

Equating (2.6) and (2.7), we find that

$$(\sigma(x)\,d(y)+d(x)\,\tau(y))z=\sigma(x)\,d(y)z+d(x)\,\tau(y)z\,,\quad \text{for all}\quad x,y,z\in N\,.$$

In the similar manner, (ii) can be proved. □

Lemma 2.3. *Let d be a (σ,τ)-derivation on N and suppose $u\in N$ is not a left zero divisor. If $[u,d(u)]_{\sigma,\tau}=0$, then $(x,\ u)$ is a constant for every $x\in N$.*

Proof. Since $u(u+x)=u^2+ux$, so we obtain

$$\sigma(u)\,d(x)+d(u)\,\tau(u)=d(u)\,\tau(u)+\sigma(u)\,d(x)\,,\quad \text{for all}\quad u\in N\quad \text{and}\quad x\in N\,.$$

Due to $[u,d(u)]_{(\sigma,\tau)}=0$, the above expression can be written as

$$\sigma(u)(d(x)+d(u))=\sigma(u)(d(u)+d(x))\,,\quad \text{for all}\quad u,x\in N$$

i.e.,

$$\sigma(u)(d(x,\ u))=0\,,\quad \text{for all}\quad x\in N\,.$$

Since σ is an automorphism of N, $\sigma(u)$ is not a left-zero divisor. Thus $d(x,u)=0$. Hence $(x,\ u)$ is constant, for all $x\in N$. □

Theorem 2.1. *Let N have no non-zero divisors of zero. If N admits a non-trivial (σ,τ)-commuting (σ,τ)-derivation d, then $(N,\ +)$ is abelian.*

Proof. Let c be any additive commutator. Then application of Lemma 2.3 yields that c is a constant. Moreover, for any $x\in N$, xc is also an additive commutator, hence a constant. Thus, $0=d(xc)=\sigma(x)\,d(c)+d(x)\,\tau(c)$ i.e. $d(x)\,\tau(c)=0$, for all $x\in N$ and additive commutators c. Since $d(x)\neq 0$ for some $x\in N$, so $\tau(c)=0$, and thus $c=0$ for all additive commutators c. Hence, $(N,\ +)$ is abelian. □

3. Prime near-rings

Lemma 3.1. *Let N be a prime near-ring.*

(i) *If z is a non-zero element in Z, then z is not a zero divisor.*

(ii) *If there exists a non-zero element z of Z such that $z+z\in Z$, then $(N,+)$ is abelian.*

(iii) *Let d be a non-trivial (σ,τ)-derivation on N. Then $xd(N)=(0)$ or $d(N)x=(0)$, implies $x=0$.*

(iv) *If N is 2-torsion free and d is a (σ,τ)-derivation on N such that $d^2=0$ and σ,τ commute with d, then $d=0$.*

(v) *If N admits a non-trivial (σ,τ)-derivation d for which $d(N)\subseteq Z$, then $c\in Z$ for each constant element c of N.*

Proof. (i) and (ii) are already proved in [4].

(iii) Let $xd(r)=0$, for all $r\in N$. Replace r by yz, to get $x\sigma(y)\,d(z)+x\,d(y)\,\tau(z)=0$, for all $y,z\in N$. Hence we have $x\sigma(y)\,d(z)=0$, for all $y,z\in N$. Since σ is an automorphism of N, $xNd(N)=(0)$. Again N is prime and $d(N)\neq 0$, we have $x=0$.

Arguing as above, we can show that $d(r)x=0$, for all $r\in N$, implies that $x=0$.

(iv) For arbitrary $x,y\in N$, we have $d^2(xy)=0$. After a simple calculation, we obtain $2d(\sigma(x))\,d(\tau(y))=0$. Since N is 2-torsion free, so $d(\sigma(x))\,d(N)=(0)$, for each $x\in N$. Hence $d=0$, by using (iii) and the fact that σ is an automorphisms.

(v) Let c be an arbitrary constant and let x be a non-constant element of N. Then $d(x)\,\tau(c)=d(xc)\in Z$ for each non-constant element x of N. This implies that $d(x)\,\tau(c)y=y\,d(x)\,\tau(c)$, for all $y\in N$. Since $d(x)\in Z\setminus\{0\}$, it follows that $d(x)\,\tau(c)y=d(x)\,y\tau(c)$, for all $y\in N$ and we conclude that $d(x)(yc-cy)=0$; for all $y\in N$ and additive commutator c. Hence, using (i), we get the required result. □

Theorem 3.1. *Let N be a prime near-ring admitting a non-trivial (σ,τ)-derivation d for which $d(N)\subseteq Z$. Then $(N,\ +)$ is abelian. Moreover, if N is 2-torsion free and σ, τ commute with d, then N is a commutative ring.*

Proof. Since $d(N)\subseteq Z$ and d is non-trivial, there exists a non-zero element x in N such that $z=d(x)\in Z\setminus\{0\}$ and $z+z=d(x+x)\in Z$. Hence $(N,\ +)$ is abelian by Lemma 3.1(ii).

Assume now that, N is 2-torsion free and σ, τ commute with d. Application of Lemma 2.2 (i) yields that,

$$(\sigma(x)\,d(y)+d(x)\tau(y))r=\sigma(x)\,d(y)\,r+d(x)\,\tau(y)r\,,\quad \text{for all}\quad x,y,r\in N\,. \tag{3.1}$$

Since $d(N)\subseteq Z$, it follows that $d(xy)\in Z$, for all $x,y\in N$. Thus, $d(xy)\,r=r\,d(xy)$, for all $x,y,r\in N$ and hence

$$\begin{aligned}(\sigma(x)\,d(y)+d(x)\,\tau(y))r&=r(\sigma(x)\,d(y)+d(x)\,\tau(y))\\&=r\sigma(x)\,d(y)+r\,d(x)\,\tau(y)\,,\\&\quad\text{for all}\quad x,y,r\in N\,.\end{aligned} \tag{3.2}$$

Combine (3.1) and (3.2) and use the fact that $(N,+)$ is abelian, to get

$$\sigma(x)\,d(y)r-r\sigma(x)\,d(y)=r\,d(x)\,\tau(y)-d(x)\,\tau(y)r\,,\quad \text{for all}\quad x,y,r\in N\,. \tag{3.3}$$

Since σ is an automorphism and $d(N)\subseteq Z$, the equation (3.3) can be rearranged to yield

$$d(y)\sigma(x)r - r\,d(y)\sigma(x) = d(x)r\tau(y) - d(x)\tau(y)r\,,\quad \text{for all}\ \ x,y,r\in N$$

or

$$d(y)(\sigma(x)r - r\sigma(x)) = d(x)(r\tau(y)-\tau(y)r)\,,\quad \text{for all}\ \ x,y,r\in N\,. \tag{3.4}$$

Suppose on contrary that N is not commutative and choose $r,y\in N$ with $r\tau(y)-\tau(y)r\neq 0$. Let $x=d(a)$, $a\in N$. This yields that $\sigma(x)=\sigma(d(a))=d(\sigma(a))\in Z$. Now (3.1) becomes $d(y)(d(\sigma(a))r - rd(\sigma(a)) = d^2(a)(r\tau(y)-\tau(y)r)$, i.e., $d^2(a)(r\tau(y)-\tau(y)r)=0$, for all $a\in N$. By Lemma 3.1 (i), we see that the central element $d^2(a)$ can not be a divisor of zero, we conclude that $d^2(a)=0$, for all $a\in N$. But by Lemma 3.1 (iv), this can not happen for non-trivial derivation d. Thus, $r\tau(y)-\tau(y)r=0$, for all $r,y\in N$. Since τ is an automorphism of N, the above expression implies that $rz-zr=0$, for all $r,z\in N$. Hence N is a commutative ring.

Theorem 3.2. *Let N be a prime near-ring admitting a non-trivial (σ,τ)-derivation d such that $d(x)d(y)=d(y)d(x)$, for all $x,y\in N$. Then $(N,\ +)$ is abelian. Moreover, if N is 2-torsion free and σ,τ commute with d, then N is a commutative ring.*

Proof. In view of our hypothesis, we have $d(x+x)\,d(x+y)=d(x+y)\,d(x+x)$, for all $x,y\in N$. This implies that $d(x)\,d(x)+d(x)\,d(y)=d(x)\,d(x)+d(y)\,d(x)$, for all $x,y\in N$ and hence $d(x)\,d(x,y)=0$, for all $x,y\in N$ i.e., $d(x)\,d(c)=0$, for all $x\in N$ and additive commutator c. Now, application of Lemma 3.1 (iii) yields that $d(c)=0$, for all additive commutators c. Since N is a left near-ring and c is an additive commutator, xc is also an additive commutator for any $x\in N$. Hence $d(xc)=0$, for all $x\in N$ and additive commutator c. Thus by Lemma 3.1 (iii), $c=0$ and hence $(N,\ +)$ is abelian. □

Assume now that N is 2-torsion free and σ, τ commute with d. Then applications of Lemmas 2.1 and 2.2 (i) yield that,

$$\begin{aligned} d(d(x)y)\,d(z) &= (d^2(x)\tau(y)+\sigma(d(x))d(y))\,d(z)\\ &= d^2(x)\tau(y)\,d(z)+\sigma(d(x))\,d(y)\,d(z)\\ &\quad \text{for all}\quad x,y,z\in N\,. \end{aligned}$$

This implies that

$$d^2(x)\tau(y)\,d(z) = d(d(x)y)\,d(z) - \sigma(d(x))\,d(y)\,d(z)\,,\quad \text{for all}\ x,y,z\in N\,. \tag{3.5}$$

Also, since $d(x)\,d(y)=d(y)\,d(x)$, for all $x,y\in N$, we find that

$$(3.6) \qquad \begin{aligned} d(d(x)y)\,d(z) &= d(z)\,d(d(x)y) \\ &= d(z)\,(d^2(x)\tau(y) + \sigma(d(x))d(y)) \\ &= d(z)\,d^2(x)\tau(y) + d(z)d(\sigma(x))\,d(y) \\ &= d^2(x)\,d(z)\tau(y) + \sigma(d(x))\,d(y)\,d(z) \\ &\quad \text{for all } x, y, z \in N\,. \end{aligned}$$

Combine (3.5) and (3.6), to get

$$(3.7) \qquad d^2(x)((\tau(y)d(z) - d(z)\tau(y)) = 0\,, \quad \text{for all} \quad x, y, z \in N\,.$$

Now replacing y by yr in (3.7), we get

$$d^2(x)\tau(y)(\tau(r)d(z) - d(z)\tau(r)) = 0\,, \quad \text{for all} \quad r, x, y, z \in N\,.$$

Thus, $d^2(x)N(\tau(r)d(z) - d(z)\tau(r)) = (0)$, for all $r, x, z \in N$. Since N is prime and τ is an automorphism, $rd(z) - d(z)r = 0$ or $d^2(x) = 0$, for all $x \in N$. But the last conclusion is impossible by Lemma 3.1 (iv). Hence, we have $rd(z) - d(z)r = 0$, for all $r, z \in N$. This implies that $d(N) \subseteq Z$. Hence N is a commutative ring by Theorem 3.1.

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