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CLUSTERING OF VAGUELY DEFINED OBJECTS

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Abstract. This paper is concerned with the clustering of objects whose properties cannot be described by exact data. These can only be described by fuzzy sets or by linguistic values of previously defined linguistic variables. To cluster these objects we use a generalization of classic clustering methods in which instead of similarity (dissimilarity) of objects, used fuzzy similarity (fuzzy dissimilarity) to define the clustering of fuzzy objects.

1. Introduction

The clustering analysis has been used for tens of years. Its aim is the formation or discovery of suitable or existing groups whose number is either predetermined or discovered. These groups must be as homogeneous as possible internally while being as different each from other as possible. Before starting the process of clustering, it is necessary to measure, weight,... the objects, i.e. we must choose a list of features describing satisfactory the object. Mostly we can describe the object using a list of quantitative and qualitative data. Such a list usually consists of a n-tuple of numbers. Clustering methods and algorithms can work with such described objects. Recently the clustering methods have been used also in branches, in which we cannot describe the objects with sufficient accuracy using only n-tuples of numbers, such objects are described mostly by vague terms. This expression occurs very often in medicine, biology, sociology, etc.

Classical clustering methods cannot work on this way define objects. To use them, we must use, instead of a vague expression, the exact one, but such a substitution leads to loosing information contained in the vague terms. Therefore it is useful to introduce a suitable description of this object and to define a clustering process for these objects.

I describe such "vague" objects using fuzzy sets (defined as fuzzy objects). I am trying to generalize as much as possible not only the objects, but all the clustering

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of "vague" objects, and to do it us similar as possible to clustering performed by man.

2. Base Notions of Classical Clustering

Let us have n objects, each object characterized by m parameters: $\mathbf{O} = \{O_1, \dots, O_n\}$, h-th object $O_h = (x_{h1}, \dots, x_{hm})$, where $x_{hj} \in \mathbf{R}$ for $h \in \{1, \dots, n\}$. It is possible to define the basic matrix of data: $\mathbf{X} = (x_{ij})_{n,m}$, where $x_{ij} \in \mathbf{R}$ and h-th raw is equal to h-th object O_h. The objects can be displayed as points in $\mathbf{R}^m$.

Important for clustering is the notion of **similarity measure** of two objects. We define a function $\mathbf{\Pi} : \mathbf{O} \times \mathbf{O} \to \mathbf{R}^+$, that satisfies

$$\begin{aligned} &\mathbf{\Pi}(O_h, O_s) \geq 0\,, \\ &\mathbf{\Pi}(O_h, O_s) = \mathbf{\Pi}(O_s, O_h)\,. \end{aligned}$$

A **dissimilarity measure** of objects is often used instead of a similarity in clustering methods. The dissimilarity of objects is denoted by $\mathbf{d} : \mathbf{O} \times \mathbf{O} \to \mathbf{R}^+$ and it must satisfy

$$\begin{aligned} &\mathbf{d}(O_h, O_s) = 0 \Leftrightarrow O_h = O_s\,, \\ &\mathbf{d}(O_h, O_s) \geq 0\,, \\ &\mathbf{d}(O_h, O_s) = \mathbf{d}(O_s, O_h)\,. \end{aligned}$$

$\mathbf{d}$ is often equal to any metric on $\mathbf{R}^m$ in real situations.

We try to divide objects into clusters. We call a **cluster** such a subset A of set $\mathbf{O}$, that satisfies

$$\max_{O_i,\, O_j \in A} \mathbf{d}(O_i,\, O_j) < \min_{O_k \in A,\, O_l \notin A} \mathbf{d}(O_k, O_l)\,.$$

Let us suppose we want to divide the set of objects $\mathbf{O}$ in to c clusters, where $1 < c < n$. We denote the set of clusters by $\mathbf{S}$: $\mathbf{S} = \{S_1, \dots, S_c\} \subseteq \mathcal{P}(\mathbf{O})$, $S_i \subseteq \mathbf{O}, p_i = |S_i|$ where the following conditions have to be met

$$\bigcup_{i=1}^{c} S_i = \mathbf{O}\,, \qquad S_i \cap S_j = \emptyset \quad \text{for} \quad i \neq j\,, \qquad \emptyset \subset S_i \subset \mathbf{O}\,.$$

This can be written in a dual representation. Let us define a matrix $U = (u_{ij})_{c,n}$ where $u_{ij} = 1$ if $O_j \in S_i$ and $u_{ij} = 0$ if $O_j \notin S_i$. Then we demand

$$\sum_{i=1}^{c} u_{ij} = 1\ \forall j = 1, \dots, n \quad \text{and} \quad 0 < \sum_{j=1}^{n} u_{ij} < n\ \forall i = 1, \dots, c\,.$$

We call the matrix U a c**-analysis** of $\mathbf{S}$. Let us denote M_c the set of all c-analysis, satisfying the following rules

$$\mathrm{M}_c = \Big\{ U \in \mathrm{V}_{cn}, u_{ij} \in \{0,1\}\, \forall i, j;\ \sum_{i=1}^{c} u_{ij} = 1\, \forall j;\ 0 < \sum_{j=1}^{n} u_{ij} < n\, \forall i \Big\}$$

where V_{cn} is a vector space of the dimension cn.

The **dissimilarity of clusters D** can be defined on the base of dissimilarity of objects **d**. Let us have clusters $A = \{A_1, \ldots, A_k\}$, $B = \{B_1, \ldots, B_t\}$, where $A_i \in \mathbf{O}\ \forall i = 1, \ldots, k$ and $B_j \in \mathbf{O}\ \forall j = 1, \ldots, t$. **D** must satisfy the following conditions

$$\mathbf{D}(A, A) = 0\,,$$
$$\mathbf{D}(A, B) \geq 0\,,$$
$$\mathbf{D}(A, B) = \mathbf{D}(B, A)\,.$$

The most frequently used method to define the dissimilarity of clusters is the **nearest neighbourhood method**:

$$\mathbf{D}(A, B) = \min_{A_i \in A,\, B_j \in B} \{\mathbf{d}(A_i, B_j)\}\,.$$

Clustering methods are defined on the basis of dissimilarity of objects and dissimilarity of clusters. These methods are divided into two basic groups: **hierarchical** clustering methods and **non-hierarchical** clustering methods. The fundamental difference between these methods is that hierarchical methods do not require the setting of the number of clusters and the resulting clusters form a hierarchy. More about methods of clustering see in [1-3, 7, 10, 11].

3. The Insufficience of Classical Clustering

Objects clustered by the classical clustering are described by means of signs. The signs of objects may be of three fundamental types:

- **quantitative:** — the value of the sign represents quantity. Most often the sign of this type is described by numbers belonging to a numerable or innumerable set U_j. Often $\mathrm{U}_j = \mathbf{R}$ for every $j = 1, \ldots, m$ and the objects can be depicted as points in $\mathbf{R}^m$.
- **qualitative:** — the value of the sign is chosen from a finite set of possible states X_j. The values are regarded under disjoint values or disjoint intervals.
- **binary:** — the value of the sign is chosen from a two-element set. Most often this set is defined as {0,1}.

An object may also contain a combination of these types.

In practice, we often meet with objects that cannot be described by the above-mentioned types of signs. Such an object contains signs with values that cannot be defined precisely (i.e. there exists a sign of the object that may assume several values at the same time or, for a given sign, there exists "uncertainty" in representing the values of this sign). Then, the classical clustering cannot be applied directly to such types of objects. It is useful to include these objects in the clustering process, too. We use fuzzy sets to describe the "uncertainty". Objects defined in this way are called **fuzzy objects**.

My aim is to set up a clustering algorithm that approximates human activities as much as possible. This means the clustering of objects is described by fuzzy sets - fuzzy objects. In this case, generalized standard clustering methods are used.

Fuzzy dissimilarity is introduced, rather than dissimilarity, and, using this notion, the clustering of the fuzzy objects is defined.

4. Fuzzy Objects

In this paragraph the basic definitions and basic theorems are introduced without proofs. The more detailed description, see [15, 16, 18].

Definition 1. Let $\mathrm{U}_j, j = 1, \ldots, m$ be universal sets being linear spaces and let $fx_j = (\mathrm{U}_j, \mu_j^x), j = 1, \ldots, m$ be **normal** and **convex** fuzzy sets over universal set U_j. The values of the membership function μ_j^x are in lattice L= $(\langle 0, 1\rangle, \min, \max, 0, 1)$. We call m-tuple $fO = (fx_1, fx_2, \ldots, fx_m)$ the **fuzzy object** over universal sets $\mathrm{U}_1, \ldots, \mathrm{U}_m$(or just a fuzzy object or object if there is no danger of ambiguity) and we call $fx_j = (\mathrm{U}_j, \mu_j^x)$, j-th sign of fuzzy object fO. $\mathcal{FO}(\mathrm{U}_1, \mathrm{U}_2, \ldots, \mathrm{U}_m)$ is denoted the class of all fuzzy objects over universal sets $\mathrm{U}_1, \mathrm{U}_2, \ldots, \mathrm{U}_m$.
We denote the set of n fuzzy objects : $\mathbf{fO} = (fO_1, fO_2, \ldots, fO_n)$ and h-th fuzzy object is defined: $fO_h = (fx_{h1}, fx_{h2}, \ldots, fx_{hm})$ where $fx_{hk} = (U_k, \mu_{hk}^x), h = 1, \ldots, n,\ k = 1, \ldots, m$ are normal and convex fuzzy set.

Hierarchical and non-hierarchical clustering methods are used to compare dissimilarities (distances) of objects to find the smallest dissimilarity. The dissimilarity of fuzzy objects is defined in terms of fuzzy set on universal $\mathbf{R}$. Then we must define the comparison of fuzzy sets from $\mathcal{F}(\mathbf{R})$ ($\mathcal{F}(\mathbf{R})$ is the set of all fuzzy sets on the universal $\mathbf{R}$). We will define it using the extension principle.

Definition 2. Let $(\mathrm{U}, \leq)$ be a partial by ordered set and A, B fuzzy sets on the universal U: $\mathrm{A} = (\mathrm{U}, \mu_A), B = (\mathrm{U}, \mu_B)$. Then we define the partial ordering on the set of fuzzy sets

$$A \vee B = (U, \mu_{A \vee B}),\ \ \mu_{A \vee B}(z) = \sup_{z = \max\{a,b\}} \{\min\{\mu_A(a), \mu_B(b)\}\}$$
$$= 0 \quad \text{otherwise}$$

and we define $A \leq B \Leftrightarrow A \vee B = B$ and $A < B \Leftrightarrow A \leq B$ and $A \neq B$. If $A \vee B \neq B$ and $A \vee B \neq A$, the fuzzy sets A, B are called **incomparable**.

Definition 3. Let $fO_h = (fx_{h1}, \ldots, fx_{hm})$ and $fO_s = (fx_{s1}, \ldots, fx_{sm})$ be fuzzy objects. We call the mapping

$$\mathbf{f\Pi} : \mathcal{FO}(\mathrm{U}_1, \mathrm{U}_2, \ldots, \mathrm{U}_m) \times \mathcal{FO}(\mathrm{U}_1, \mathrm{U}_2, \ldots, \mathrm{U}_m)) \to \mathcal{F}(\mathbf{R}),$$

such that

$$\mathbf{f\Pi}(fO_h, fO_h) \geq \mathbf{0},$$
$$\mathbf{f\Pi}(fO_h, fO_s) = \mathbf{f\Pi}(fO_s, fO_h)$$

where $\mathbf{0} = \{(0, 1)\}$ is fuzzy set containing 0 with membership value 1 and $\geq$ is a partial ordering on the set of fuzzy sets $\mathcal{F}(\mathbf{R})$ a **fuzzy similarity** of the fuzzy objects. $\mathbf{f\Pi}(fO_h, fO_s) = (\mathbf{R}, \mu_{h,s}^{f\Pi})$ is a fuzzy set over universal set $\mathbf{R}$.

More we call the mapping $\mathbf{fd} : \mathcal{FO}(\mathrm{U}_1, \mathrm{U}_2, \ldots, \mathrm{U}_m) \times \mathcal{FO}(\mathrm{U}_1, \mathrm{U}_2, \ldots, \mathrm{U}_m)) \to \mathcal{F}(\mathbf{R})$, that satisfies

$$\begin{aligned}\mathbf{fd}(fO_h, fO_h) &\supseteq \mathbf{0}\,,\\ \mathbf{fd}(fO_h, fO_s) &\geq \mathbf{0}\,,\\ \mathbf{fd}(fO_h, fO_s) &= \mathbf{fd}(fO_s, fO_h)\end{aligned}$$

where $\mathbf{0} = \{(0,1)\}$ is fuzzy set containing 0 with membership value 1 and $\geq$ is a partial ordering on the set of fuzzy sets $\mathcal{F}(\mathbf{R})$ a **fuzzy dissimilarity** of the fuzzy objects. $\mathbf{fd}(fO_h, fO_s) = (\mathbf{R}, \mu^{fd}_{h,s})$ is a fuzzy set over universal set $\mathbf{R}$.

There are more ways of defining the dissimilarity of fuzzy objects. One of them uses the **extension principle**.

Definition 4. Let $fO_h = (fx_{h1}, \ldots, fx_{hm})$ and $fO_s = (fx_{s1}, \ldots, fx_{sm})$ be fuzzy objects. Then we define the **fuzzy dissimilarity fd** : $\mathcal{FO}(\mathrm{U}_1, \mathrm{U}_2, \ldots, \mathrm{U}_m) \times \mathcal{FO}(\mathrm{U}_1, \mathrm{U}_2, \ldots, \mathrm{U}_m)) \to \mathcal{F}(\mathbf{R})$ of fuzzy objects fO_h, fO_s by $\mathbf{fd}(fO_h, fO_s) = (\mathbf{R}, \mu^{fd}_{h,s})$ where

$$\begin{aligned}\mu^{fd}_{h,s}(z) &= \sup_{z=\mathbf{d}(O_h,O_s)} \{\min\{\mu^x_{h1}(x_{h1}), \ldots, \mu^x_{hm}(x_{hm}), \mu^x_{s1}(x_{s1}), \ldots, \mu^x_{sm}(x_{sm})\}\}\\ &= 0 \quad \text{otherwise}\end{aligned}$$

$\forall\, h, s = 1, 2, \ldots, n$ and $\mathbf{d}(O_h, O_s)$, $O_h = (x_{h1}, \ldots, x_{hm}), O_s = (x_{s1}, \ldots, x_{sm})$ is the dissimilarity of classical objects.

Theorem 1. *Let* $\mathbf{d}$ *be a dissimilarity of classical objects. Then* $\mathbf{fd}$ *(Def. 4) satisfies the conditions of fuzzy dissimilarity of fuzzy objects.*

Theorem 2. *Let a dissimilarity* $\mathbf{d} : (\mathrm{U}_1 \times \ldots \times \mathrm{U}_m) \times (\mathrm{U}_1 \times \ldots \times \mathrm{U}_m) \to \mathbf{R}$ *be a continuous mapping,* $\mathbf{fd}$ *from Def. 4,* $\alpha \in (0, 1\rangle$*,* fO_h, fO_s *be fuzzy objects. Then* $(\mathbf{fd}(fO_h, fO_s))_\alpha = \mathbf{d}((fO_h)_\alpha, (fO_s)_\alpha) = \{z; z = \mathbf{d}(x, y), x \in (fO_h)_\alpha, y \in (fO_s)_\alpha\}$ *where* $(A)_\alpha$ *is* α*-cut of fuzzy set* A*.*

Definition 5. Let us have clusters $\mathrm{A} = \{A_1, \ldots, A_k\}$, $\mathrm{B} = \{B_1, \ldots, B_t\}$, where A_i and B_j are fuzzy objects for $i = 1, \ldots, k$ and $j = 1, \ldots, t$. Any **fuzzy dissimilarity of clusters fD** must satisfy these conditions

$$\begin{aligned}\mathbf{fD}(\mathrm{A}, \mathrm{A}) &\supseteq \mathbf{0}\,,\\ \mathbf{fD}(\mathrm{A}, \mathrm{B}) &\geq \mathbf{0}\,,\\ \mathbf{fD}(\mathrm{A}, \mathrm{B}) &= \mathbf{fD}(\mathrm{B}, \mathrm{A})\end{aligned}$$

where $\mathbf{0} = \{(0,1)\}$ and $\geq$ is a partial ordering on the set of fuzzy sets $\mathcal{F}(\mathbf{R})$.

Definition 6. Let $fO_1, \ldots, fO_k$, $fO_i = (fx_{i1}, \ldots, fx_{im}), fx_{ij} = (\mathrm{U}_j, \mu^x_{ij})$ be fuzzy objects. The **centroid of fuzzy objects** $fO_1, \ldots, fO_k$ is the object $fT = (ft_1, ft_2, \ldots, ft_m)$ where

$$ft_j = (\mathrm{U}_j,\ \mu^t_j),\ \ \mu^t_j(z) = \sup_{z=\frac{x_1+\ldots+x_k}{k}} \{\min\{\mu^x_{1j}(x_1), \mu^x_{2j}(x_2), \ldots, \mu^x_{kj}(x_k)\}\}\,.$$

Theorem 3. *Let* $\mathrm{U}_1, \mathrm{U}_2, \ldots, \mathrm{U}_m$ *be linear spaces,* $fO_i = (fx_{i1}, \ldots, fx_{im})$, $fx_{ij} = (\mathrm{U}_j, \mu^x_{ij})$, $i = 1, \ldots, k$ *be fuzzy objects,* $\alpha \in (0, 1\rangle$ *and* $h(x_1, x_2, \ldots, x_k) = \frac{1}{k}\sum_{i=1}^{k}(x_i)$. *Then the centroid* fT *of fuzzy objects* $fO_1, \ldots, fO_k$ *is the fuzzy object (*$fT = (ft_1, ft_2, \ldots, ft_m) \in \mathcal{FO}(\mathrm{U}_1, \mathrm{U}_2, \ldots, \mathrm{U}_m)$ *and* ft_j *is normal and convex fuzzy set* $\forall j = 1, \ldots, m$*) and* $(ft_j)_\alpha = h((fO_1)_\alpha, \ldots, (fO_k)_\alpha)$.

If, moreover $(fx_{ij})_\alpha$ *is a closed interval* $\forall i = 1, \ldots, k$*, then* $(ft_j)_\alpha$ *is closed interval and* $(ft_j)_\alpha = \frac{1}{k}\sum_{i=1}^{k}(fx_{ij})_\alpha$ *where the multiplying by constants and the summing are arithmetic operations on intervals.*

Definition 7. We define a dissimilarity of clusters $\mathrm{fA} = \{fA_1, \ldots, fA_k\}, \mathrm{fB} = \{fB_1, \ldots, fB_t\}$ using the dissimilarity of fuzzy objects **fd**:

The nearest neighbourhood method:

$$\mathbf{fD}(\mathrm{fA}, \mathrm{fB}) = \min_{fA_i \in \mathrm{fA},\, fB_i \in \mathrm{fB}} \{\mathbf{fd}(fA_i, fB_j)\}\,.$$

The centroid method:

$$\mathbf{fD}(\mathrm{fA}, \mathrm{fB}) = \mathbf{fd}(\mathbf{fa}, \mathbf{fb})$$

where **fa** is the centroid of fuzzy objects from cluster fA and **fb** is the centroid of fuzzy objects from clusters fB.

Average dissimilarity method:

$$\mathbf{fD}(\mathrm{fA}, \mathrm{fB}) = \frac{1}{kt}\sum_{(i,j)}^{k*t} \mathbf{fd}(fA_i, fB_j)\,.$$

5. Clustering of Fuzzy Objects

With the dissimilarity of fuzzy objects and clusters defined, we can now proceed to methods dividing the set of fuzzy objects **fO** into clusters. For classical objects, there is a large number of these methods. For fuzzy objects, we shall focus on two major types: **hierarchical** and **non-hierarchical** methods. These types of methods are based on the dissimilarity of objects and dissimilarity of clusters. For fuzzy objects, the algorithm of these methods is the same, we only use fuzzy dissimilarity of fuzzy objects instead of dissimilarity of objects, and fuzzy dissimilarity of clusters instead of dissimilarity of clusters. In addition, in classical clustering methods, the dissimilarities are compared and, on the basis of this comparison, the least dissimilarities are chosen. For fuzzy objects, the dissimilarities are fuzzy sets above the universal set $\mathbf{R}_0^+$ and thus we must use the comparison of fuzzy sets to compare dissimilarities. The form of result of clustering of fuzzy objects will be similar to the clustering of classical objects where the membership to cluster of a fuzzy objects is an element of the set $\{0, 1\}$.

5.1 Hierarchical Clustering Methods of Fuzzy Objects.

Definition 8. The **hierarchy** on the set of fuzzy objects **fO** is a set $\mathbf{H} \subset \mathcal{P}(\mathbf{fO})$ that satisfies

1) $\mathbf{fO} \in \mathbf{H}$,

2) $\{fO_i\} \in \mathbf{H} \;\; \forall fO_i \in \mathbf{fO}$,

3) if $A_i \cap A_j \neq \emptyset$, then $A_i \subset A_j$ or $A_j \subset A_i$ $\forall A_i, A_j \in \mathbf{H}$.

For every set of fuzzy objects **fO** the **hierarchical** clustering method finds a sequence of its analyses $\Omega_0, \Omega_1, \ldots, \Omega_{n-1}$ into clusters and assigns each element A of the analysis Ω_i a fuzzy set $h(A) = (\mathbf{R}_0^+, \mu^A)$.

Algorithm 1.
1. In the first step of the algorithm we choose a fuzzy dissimilarity of clusters **fD** and create the analysis $\Omega_0 = \{A_{0,1}, A_{0,2}, \ldots, A_{0,n}\}$ where each cluster contains only one object $A_{0,j} = \{fO_j\}$. To each cluster $A_{0,j}$ fuzzy set $h(A_{0,j}) = (\mathbf{R}_0^+, \mu_0^A) = \{(0,1)\} = \mathbf{0}$ is assigned.
2. Suppose that in the i-th step of the algorithm $(0 < i \leq n-2)$ we have the analysis $\Omega_i = \{A_{i,1}, A_{i,2}, \ldots, A_{i,n-i}\}$. We choose one couple of clusters $(A_{i,u}, A_{i,v})$ satisfying

$$\mathbf{fD}(A_{i,u}, A_{i,v}) = \min_{A_{i,k} \in \Omega_i,\; A_{i,s} \in \Omega_i} \{\mathbf{fD}(A_{i,k}, A_{i,s})\}\,.$$

Let $\mathbf{fD}(A_{i,u}, A_{i,v}) = (\mathbf{R}_0^+, \mu_{i+1}^A)$. The analysis $\Omega_{i+1} = \{A_{i+1,1}, \ldots, A_{i+1,n-i-1}\}$ can be obtained from Ω_i this way: let us merge those clusters $A_{i,u}, A_{i,v}$ whose dissimilarity **fD** is the smallest to one cluster $A_{i,u} \cup A_{i,v} = A_{i+1,l}$ and assign a fuzzy set to cluster $A_{i+1,l}$ in the form $h(A_{i+1,l}) = (\mathbf{R}_0^+, \mu_{i+1}^A)$. The other clusters in analysis Ω_{i+1} are taken from analysis Ω_i.
3. The process is stopped by analysis Ω_{n-1}, which includes one cluster containing all the objects $\Omega_{n-1} = \{A_{n-1,1}\} = \mathbf{fO}$ where $h(A_{n-1,1}) = (\mathbf{R}_0^+, \mu_{n-1}^A)$. Fuzzy sets $(\mathbf{R}_0^+, \mu_0^A), (\mathbf{R}_0^+, \mu_1^A), \ldots, (\mathbf{R}_0^+, \mu_{n-1}^A)$ define clustering levels belonging to analyses $\Omega_0, \Omega_1, \ldots, \Omega_{n-1}$.

The result of a hierarchical clustering of classical objects (non-fuzzy objects) is often depicted in the form of a similarity tree. The **similarity tree** is a hierarchical clustering whose function $\mathbf{h} : \mathcal{P}(\mathbf{O}) \to \mathbf{R}^+$ satisfies: $A \subseteq B \Rightarrow \mathbf{h}(A) \leq \mathbf{h}(B)$ for $A, B \in \mathcal{P}(\mathbf{O})$.

We define $\mathbf{h}(A_{0,j}) = \text{defuzz}\,(h(A_{0,j})) = \mu_0, \ldots, \mathbf{h}(A_{n-1,1}) = \text{defuzz}\,(h(A_{n-1,1})) = \mu_{n-1}$, where defuzz is any defuzzification method. If $\mu_0, \mu_1, \ldots, \mu_{n-1}$ satisfy $\mu_i \leq \mu_{i+1} \;\; \forall i \in \{0, 1, \ldots, n-2\}$, then the analyses $\Omega_0, \Omega_1, \ldots, \Omega_{n-1}$ can be depicted as similarity tree, too.

Example 1. Example of the hierarchical clustering method for fuzzy objects. Let us choose $n = 10$, $m = 2$, $\mathrm{U}_1 = \mathbf{R}$, $\mathrm{U}_2 = \mathbf{R}$. Let the elements of fuzzy objects be in the form : $fO = (fx_1,\, fx_2)$, $fx_1 = (\mathbf{R},\, \mu_1^x)$, $fx_2 = (\mathbf{R},\, \mu_2^x)$. Each element of object have membership function in the form of two-sided Gaussian curve

$$\mu_1^x(x) = e^{\frac{-(x-C_x)^2}{2S_x^1}} \quad \text{for} \quad x \leq C_x \quad \text{and} \quad \mu_1^x(x) = e^{\frac{-(x-C_x)^2}{2S_x^2}} \quad \text{for} \quad x > C_x\,,$$

$$\mu_2^x(y) = e^{\frac{-(y-C_y)^2}{2S_y^1}} \quad \text{for} \quad y \leq C_y \quad \text{and} \quad \mu_2^x(y) = e^{\frac{-(y-C_y)^2}{2S_y^2}} \quad \text{for} \quad y > C_y\,.$$

To simplify this expression we used 6-tuple of parameters for fuzzy object: $fO = (C_x, C_y, S_x^1, S_x^2, S_y^1, S_y^2)$. Then

$$\begin{aligned} fO_1 &= (12, 3, 0.5, 1.5, 1.0, 1.0)\,, & fO_2 &= (16, 19, 2.3, 0.2, 0.9, 1.9)\,, \\ fO_3 &= (21, 13, 0.7, 1.9, 1.9, 3.0)\,, & fO_4 &= (20, 23, 2.7, 0.3, 1.7, 0.4)\,, \\ fO_5 &= (8, 20, 0.4, 3.1, 2.4, 0.5)\,, & fO_6 &= (16, 9, 0.9, 1.7, 1.7, 0.7)\,, \\ fO_7 &= (1, 23, 1.0, 1.3, 1.8, 2.3)\,, & fO_8 &= (25, 20, 0.3, 1.1, 3.3, 0.4)\,, \\ fO_9 &= (18, 6, 0.6, 0.6, 0.7, 1.7)\,, & fO_{10} &= (5, 28, 1.0, 1.0, 0.5, 0.5)\,. \end{aligned}$$

If we think fuzzy objects in the form $fO = (\mathbf{R} \times \mathbf{R}, \min\{\mu_1^x(x), \mu_2^x(y)\})$, then the fuzzy objects can be depicted in $\mathbf{R}^3$ as shown Fig. 1.

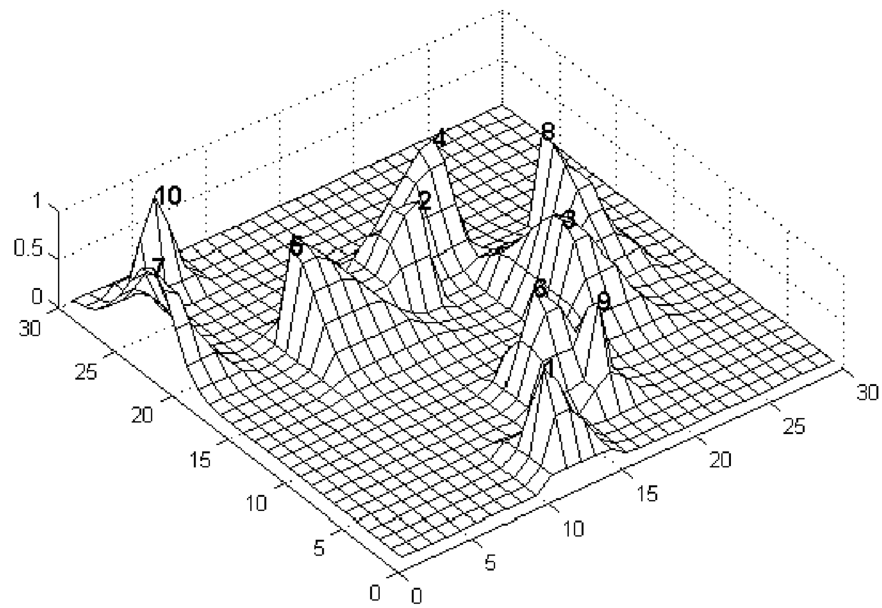

Figure 1.

Let us choose the extension of Euclidean metric $\mathbf{fd}_E(fO_h, fO_s)$, the nearest neighbourhood method for $\mathbf{fD}(S_i, S_j)$ and defuzz is equal to centroid defuzzication method. Then we obtain the result in the form of a similarity tree (Fig. 2.). Fuzzy objects are represented by numbers 1-10, numbers at nodes are clustering levels belonging to analyses $\Omega_0, \Omega_1, \ldots, \Omega_9$.

If we transform fuzzy objects into classical objects, for example, by taking only the points in which the membership function is equal to 1 (for example fO_3 is transformed to $O_3 = (21, 13)$) and use the hierarchical clustering of classical objects with **the same parameters**, then we get the tree shown at Fig. 3.

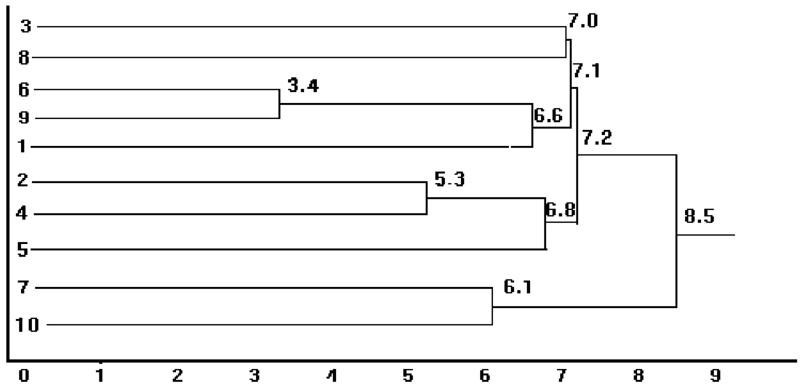

Figure 2.

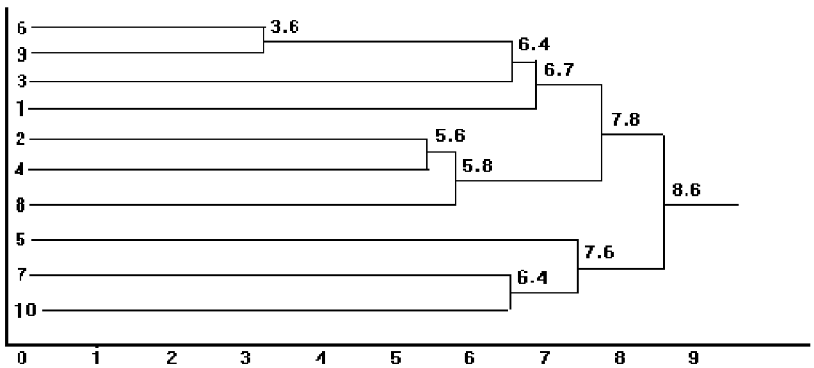

Figure 3.

5.2 **Non-Hierarchical Clustering Methods of Fuzzy Objects**

Non-hierarchical methods for classical objects try to find such an analysis of set O to clusters $\mathbf{S} = \{S_1, ..., S_c\}$, for which a previously chosen *functional of quality of the analysis* assumes extreme values. The **objective functional** method is one of the most frequent methods used in non-hierarchical clustering. Let us denote the functional $J_w : M_c \to \mathbf{R}^+$:

$$J_w(U) = \sum_{j=1}^{n} \sum_{i=1}^{c} u_{ij}(D_{ij})^2$$

where $U = (u_{ij}) \in M_c$ and $D_{ij} = \mathbf{D}(\{O_j\}, S_i)$ is the dissimilarity of clusters $\{O_j\}$ and S_i. For $\mathbf{D}(O_j, S_i)$ is often used the centroid method. Then $J_w : M_c \times \mathbf{R}^{cm} \to \mathbf{R}^+$ is defined as

$$J_w(U, v) = \sum_{j=1}^{n} \sum_{i=1}^{c} u_{ij} (d_{ij})^2$$

where $U = (u_{ij}) \in M_c$, $d_{ij} = \mathbf{d}(O_j, v_i)$ and $v = (v_1, \ldots, v_c)$, v_i is the centroid of cluster S_i.

For fuzzy objects, we define the non-hierarchical method the same way. We try to find such an analysis of the set $\mathbf{fO}$ to clusters $\mathbf{S} = \{S_1, \ldots, S_c\}$, for which a previously chosen functional of quality of analysis assumes extreme values.

Definition 9. Let $\mathbf{fO}$ be fuzzy objects. We define functional $fJ_w : M_c \to \mathcal{F}(\mathbf{R})$ as

$$fJ_w(U) = \sum_{j=1}^{n} \sum_{i=1}^{c} u_{ij} (fD_{ij})^2$$

where $U = (u_{ij}) \in M_c$ and $fD_{ij} = \mathbf{fD}(\{fO_j\}, S_i)$ is the fuzzy dissimilarity of clusters $\{fO_j\}$ and S_i. If we choose the centroid method for the dissimilarity of clusters, then $fJ_w : M_c \times (\mathcal{F}O(U_1, \ldots, U_m))^c \to \mathcal{F}(\mathbf{R})$,

$$fJ_w(U, fv) = \sum_{j=1}^{n} \sum_{i=1}^{c} u_{ij} (fd_{ij})^2$$

where $U = (u_{ij}) \in M_c$, $fd_{ij} = \mathbf{fd}(fO_j, fv_i)$ and $fv = (fv_1, \ldots, fv_c)$, fv_i is the centroid of fuzzy objects from cluster S_i.

In these definitions we define the sum and square of fuzzy set using the extension principle.

Remark 1. We define the function fJ_w using the extension principle: Let $fD_{ij} = \mathbf{fD}(\{fO_j\}, S_i) = (\mathbf{R}, \mu_{ij}^{fD})$. Then $fJ_w(U) = (\mathbf{R}, \mu^{fJ})$,

$$\mu^{fJ}(z) = \sup_{z=J(D)} \min\{\mu_{ij}^{fD}(D_{ij}), \quad j = 1, \ldots, n, \ i = 1, \ldots, c\},$$

$$J(D) = \sum_{j=1}^{n} \sum_{i=1}^{c} u_{ij} (D_{ij})^2, \quad D = (D_{ij}) \in \mathbf{R}^{cn}.$$

Similarly we define centroid method: Let $fd_{ij} = \mathbf{fd}(fO_j, fv_i) = (\mathbf{R}, \mu_{ij}^{fd})$. Then $fJ_w(U, fv) = (\mathbf{R}, \mu^{fJ})$,

$$\mu^{fJ}(z) = \sup_{z=J(d)} \min\{\mu_{ij}^{fd}(d_{ij}), \quad j = 1, \ldots, n, \ i = 1, \ldots, c\},$$

$$J(d) = \sum_{j=1}^{n} \sum_{i=1}^{c} u_{ij} (d_{ij})^2, \quad d = (d_{ij}) \in \mathbf{R}^{cn}.$$

A local minimum of an objective functional is defined as the optimal analysis of fuzzy objects **fO** to clusters. We find a minimum $fJ_w(U, fv)$ on $M_c \times (\mathcal{F}O(U_1, U_2, \ldots, U_m))^c$ by an iterative method.

Algorithm 2.
1. In the first step, we choose a number of clusters c, the initial clusters $\Omega_0 = \{S_{0,1}, \ldots, S_{0,c}\}$ and calculate their centroids $fv^0 = (fv_1^0, \ldots, fv_c^0)$.
2. Let us have in the k-th step analysis **fO** to c clusters: $\Omega_k = \{S_{k,1}, \ldots, S_{k,c}\}$ and their centroids $fv^k = (fv_1^k, \ldots, fv_c^k)$. Then for the following analysis of the set of fuzzy objects **fO** to c clusters $\Omega_{k+1} = \{S_{k+1,1}, \ldots, S_{k+1,c}\}$ holds: for each fuzzy object $fO_i \in \mathbf{fO}$, we have

$$fO_i \in S_{k+1,h} \Leftrightarrow \mathbf{fd}(fO_i, fv_h^k) = \min_{j=1,2,\ldots,c} \{\mathbf{fd}(fO_i, fv_j^k)\} \quad \forall i = 1, 2, \ldots, n.$$

Then for each cluster $S_{k+1,j}$ we calculate its centroid fv_j^{k+1}.
3. We compare analyses Ω_k and Ω_{k+1}. Then:
a) There exists $S_{k,h}$ of analysis Ω_k satisfying: $S_{k,h} \neq S_{k+1,j}$ for $j = 1, \ldots, c$. Then we go to step 2.
b) No $S_{k,h}$ of analysis Ω_k satisfying: $S_{k,h} \neq S_{k+1,j}$ for $j = 1, \ldots, c$ exists. Then the analyses Ω_k and Ω_{k+1} are made up of the same subsets and we stop the algorithm. The analysis $\Omega_k = \{S_{k,1}, \ldots, S_{k,c}\}$ is the resulting analysis and its subset $S_{k,1}, \ldots, S_{k,c}$ are the resulting clusters.

Remark 2.
a) Step 2 can be changed: Let us have in the k-th step, analysis of the set of objects to c clusters $\Omega_k = \{S_{k,1}, \ldots, S_{k,c}\}$ and their centroids $fv^k = (fv_1^k, \ldots, fv_c^k)$. Then we create the following analysis of the set of objects **fO** to c clusters $\Omega_{k+1} = \{S_{k+1,1}, \ldots, S_{k+1,c}\}$ in this way: we choose one object $fO_r \in S_{k,q}$ so, that

$$\mathbf{fd}(fO_r, fv_h^k) = \min_{\substack{i=1,\ldots,n \\ j=1,\ldots,c}} \{\mathbf{fd}(fO_i, fv_j^k)\} \text{ and } h \neq q.$$

Then $S_{k+1,h} = S_{k,h} \cup \{fO_r\}, S_{k+1,q} = S_{k,q} - \{fO_r\},\ S_{k+1,j} = S_{k,j}$ for $j \in \{1, \ldots, c\} - \{h, q\}$ and we recalculate the centroids of clusters that have been changed. If we cannot choose such objects, we define $\Omega_{k+1} = \Omega_k$.
b) If there exists no minimum (i.e. fuzzy sets are incomparable) we get it using some defuzzification method.
c) We can choose some parameters for fuzzy objects that have effect upon the clustering algorithm.
Splitting parameter is the fuzzy set $\mathbf{P} = (\mathbf{R},\ \mu_P)$: if "diameter" of cluster A is greater then $\mathbf{P}$ (for example $\max_{fO_i, fO_j \in A}\{\mathbf{fd}(fO_i, fO_j)\} > \mathbf{P}$), then the cluster A is split into two clusters $\mathrm{A}_1, \mathrm{A}_2 : \mathrm{A}_1 \cap \mathrm{A}_2 = \emptyset, \mathrm{A}_1 \cup \mathrm{A}_2 = \mathrm{A}$ and "diameters" of clusters $\mathrm{A}_1, \mathrm{A}_2$ are less then $\mathbf{P}$.
Merging parameter is the fuzzy set $\mathbf{L} = (\mathbf{R}, \mu_L)$: if distance of centroids of clusters $\mathrm{A}_1, \mathrm{A}_2$ is less than $\mathbf{L}$ (for example $\mathbf{fd}(fv_1, fv_2) < \mathbf{L}$), then clusters are merged into one cluster: $\mathrm{A}_1 \cup \mathrm{A}_2 = A$.

We can also define other parameters: **the maximum of the number of clusters, the minimum of the number of clusters, the minimum of the number of objects in cluster**,... These parameters have an influence the clustering algorithm and we must choose them conveniently.

Example 2. An example of the non-hierarchical clustering method for fuzzy objects. Let us choose $n = 10$, $m = 2$, $U_1 = \mathbf{R}$, $U_2 = \mathbf{R}$. Similar Example 1, every fuzzy object is defined by two fuzzy set (two-sided Gaussian curve) in form $fO = (fx_1,\ fx_2)$, $fx_1 = (\mathbf{R},\ \mu_1^x)$, $fx_2 = (\mathbf{R},\ \mu_2^x)$, where

$$\mu_1^x(x) = e^{\frac{-(x-C_x)^2}{2S_x^1}} \quad \text{for} \quad x \leq C_x \quad \text{and} \quad \mu_1^x(x) = e^{\frac{-(x-C_x)^2}{2S_x^2}} \quad \text{for} \quad x > C_x\,,$$

$$\mu_2^x(y) = e^{\frac{-(y-C_y)^2}{2S_y^1}} \quad \text{for} \quad y \leq C_y \quad \text{and} \quad \mu_2^x(y) = e^{\frac{-(y-C_y)^2}{2S_y^2}} \quad \text{for} \quad y > C_y\,.$$

To simplify the expression we write it as 6-tuple $fO = (C_x, C_y, S_x^1, S_x^2, S_y^1, S_y^2)$, too. Then

$$\begin{aligned}
fO_1 &= (10, 19, 1.7, 0.9, 2.9, 0.3)\,, & fO_2 &= (20, 23, 1.3, 1.7, 2.1, 1.8)\,,\\
fO_3 &= (22, 16, 1.4, 0.9, 0.2, 2.7)\,, & fO_4 &= (10, 12, 2.5, 0.3, 1.1, 2.1)\,,\\
fO_5 &= (20, 11, 1.9, 0.9, 2.4, 0.4)\,, & fO_6 &= (16, 25, 0.7, 1.8, 2.4, 0.3)\,,\\
fO_7 &= (14, 10, 0.1, 2.7, 1.3, 1.7)\,, & fO_8 &= (7, 9, 0.7, 1.2, 0.4, 2.5)\,,\\
fO_9 &= (15, 20, 0.5, 2.8, 0.3, 2.1)\,, & fO_{10} &= (23, 8, 2.9, 0.3, 2.3, 0.5)\,.
\end{aligned}$$

The fuzzy objects can be depicted at $\mathbf{R}^3$ as shown in Fig. 4.

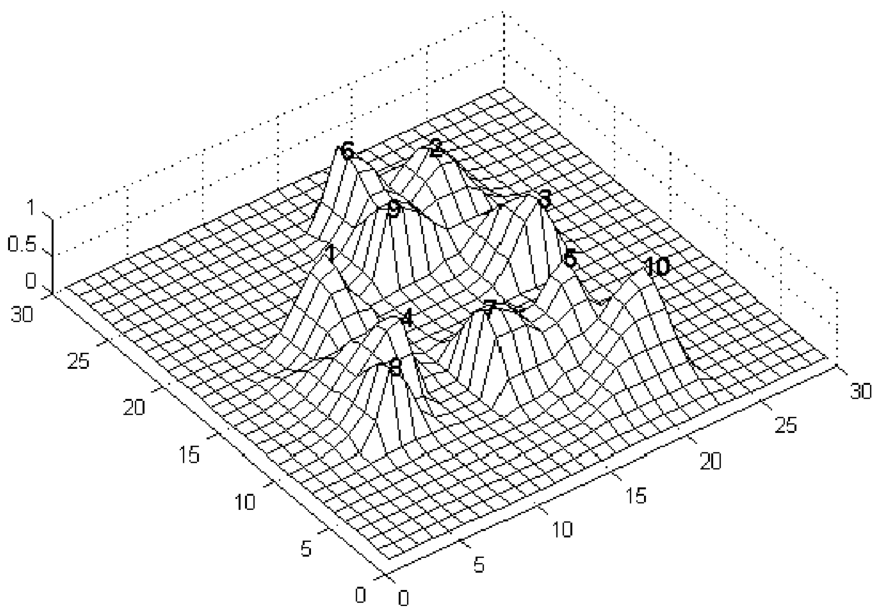

Figure 4.

We will cluster these objects using the non-hierarchical clustering method with the functional $fJ_w(U,\ fv) = \sum_{j=1}^{n}\sum_{i=1}^{c} u_{ij}(fd_{ij})^2$ where $fd_{ij} = \mathbf{fd}_E(fO_j, fv_i)$, fv_i is the centroid of cluster S_i. Let us choose $c = 3$ and the initial clusters $S_1 = \{fO_1, fO_2, fO_3, fO_4, fO_5\}$, $S_2 = \{fO_6, fO_7, fO_8\}$, $S_3 = \{fO_9, fO_{10}\}$. We will get the clusters $S_1 = \{fO_2, fO_3, fO_6, fO_9\}$, $S_2 = \{fO_5, fO_7, fO_{10}\}$, $S_3 = \{fO_1, fO_4, fO_8\}$ after 6 iterations. If we transform the fuzzy objects into the classical objects (in the same way as in Example 1.) and use the non-hierarchical clustering of classical objects with **the same parameters**, then we get the clusters $S_1 = \{O_1, O_2, O_6, O_9\}, S_2 = \{O_3, O_5, O_{10}\}, S_3 = \{O_4, O_7, O_8\}$ after 5 iterations. After comparing results of both the examples (for example fuzzy object fO_7, in this example, belongs to the same cluster with objects fO_5, fO_{10} and does not belong to cluster with fuzzy objects fO_4, fO_8) we can say, that the clustering using fuzzy objects correspond better to the idea of the clusters of vaguely defined objects.

6. Conclusion

Since computers are more employed nowadays, they are also used for decisions previously made by man. The use of fuzzy sets enables them to solve problems on which the prior mathematical methods failed. This approach also includes the clustering of objects by their properties. The clustering of fuzzy objects can be broadly used instead of human decisions in case where classical methods fail or cause difficulties. This is mainly in areas dealing with objects that cannot be easily described by qualitative or quantitative values. Such a typical area can be medicine with the object "patient". The description of such an object contains a great deal of non-precisely defined data - for example his condition, quality of sleep, intensity of pain, etc. On the basis of these "vague" terms the physician has to determine the correct right diagnosis, which means that he must put the patient in some cluster. In engineering we usually work with exactly described objects, but the inherent difficulty of such a description can be so great, that it is practically unusable. For example a description of the structure of material may include the number of grains, their size, shape, distance, etc. In such a case, it would be better to describe the structure by less exact but more convenient means. This way, we transform the classical objects into fuzzy ones and can use the methods for clustering fuzzy objects.

This paper is a continuation of [12, 13, 17] dealing with other various definition of the similarity (dissimilarity) of fuzzy objects are dealt with.

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