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Foliations and contact structures

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Abstract. We introduce a notion of linear deformation of codimension one foliations into contact structures and describe some foliations which deform instantly into contact structures and some which do not. Restricting ourselves to closed smooth manifolds, we obtain a necessary and sufficient condition for a foliation defined by a closed nonsingular 1-form to be linearly deformable into contact structures.

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1 Introduction

A contact form on a $2n+1$-dimensional manifold M is a 1-form α such that $\alpha \wedge (d\alpha)^n$ is a volume form on M. The system of equations $\alpha(Z) = 1$ and $d\alpha(Z, X) = 0$ for arbitrary X uniquely determines a vector field Z called the Reeb vector field, or the characteristic vector field of α. The tangent subbundle $\xi = \operatorname{kern} \alpha$ of rank $2n$ is called the contact structure associated with α. In general, a contact structure on a $2n+1$-dimensional manifold is a rank $2n$ tangent subbundle which is locally determined by contact forms (see Blair's book [1] for more details about contact structures).

The manifolds in this paper will be assumed to be oriented and all planes fields considered herein are supposed to be transversely orientable. Let ξ be a hyperplane field on a manifold M. When ξ is a foliation, we say that ξ is deformable into contact structures if there exists a one parameter family ξ_t of hyperplane fields satisfying $\xi_0 = \xi$ and for all $t > 0$, ξ_t is contact. It is well known from Eliashberg–Thurston's work [2] that any oriented codimension 1 C^2-foliation on an oriented 3-manifold can be perturbed into contact structures, except the product foliation of $\mathbb{S}^2 \times \mathbb{S}^1$ by spheres $\mathbb{S}^2$. It was then unknown if this approximation can always be done through a deformation. In this note, we deal with particular deformations called "linear". For a foliation ξ defined by a 1-form α_0, a deformation ξ_t defined by 1-forms α_t is said to be linear if $\alpha_t = \alpha_0 + t\alpha$ where α is a 1-form on M (independent of t). We point out that our definition of linearity is weaker than that of Eliashberg–Thurston [2]. We construct some examples of deformations and prove the following results.

Theorem 1. *Let M be a closed, $2n+1$-dimensional manifold, α_0 a closed 1-form on M and α any 1-form on M. Then, the following two conditions are equivalent.*

(i) *The 1-forms $\alpha_t = \alpha_0 + t\alpha$ in a linear deformation of α_0 are contact for all $t > 0$.*

(ii) *The 1-form α is contact and $\alpha_0(Z) = 0$ where Z is the Reeb vector field of α.*

Theorem 2. *Let (M, α, Z) be a closed contact $2n+1$-dimensional manifold where Z is the Reeb vector field of the contact form α. Let α_0 be a closed 1-form such that $\alpha_0(Z)$ is not identically zero. Then there is a positive constant R such that the 1-form $\alpha_t = \alpha_0 + t\alpha$ is not contact for $0 \leqslant t \leqslant R$.*

A corollary of Theorem 1 is that besides the foliation of $\mathbb{S}^2 \times \mathbb{S}^1$ by 2-spheres, there are other foliations which are defined by closed nonsingular 1-forms and which cannot be linearly deformed into contact structures. Throughout this paper, the notation $\omega > 0$ for a differential form of top degree on M means that ω is equal to a fixed volume form multiplied by a positive function on M.

2 Examples in dimension 3

On the 3-dimensional torus $\mathbb{T}^3$ with coordinates θ, x and y, consider the nonsingular closed 1-form $d\theta$ with any one of the contact forms $\alpha_n = \cos n\theta\, dx + \sin n\theta\, dy$, where n is a positive integer. The one parameter family of "confoliations" [2] $\alpha_{n,t} = d\theta + t\alpha_n$ is a foliation for $t = 0$, but for $t > 0$, a direct calculation shows that $\alpha_{n,t}$ instantly becomes a contact form. Thus, the trivial foliation of the torus $\mathbb{T}^3$ by tori $\mathbb{T}^2$ is linearly deformable into contact structures.

Denoting by Z_n the characteristic vector field of α_n, one sees that $d\theta(Z_n) = 0$. Another way of expressing this is that $d\theta$ is a closed basic 1-form for the flow determined by Z_n. Now consider the other closed nonsingular 1-form $d\theta + df$ where df is a small differential of a nonbasic function f. A function f is said to be basic if its differential df is an exact basic 1-form. As a direct consequence of Theorem 2 in this paper, there exist an $R > 0$ such that the linear deformation $\alpha_{n,t} = d\theta + df + t\alpha_n$ of $d\theta + df$ is not contact for $0 \leqslant t \leqslant R$.

Another example of foliations which are deformable into contact structures is provided by the following proposition.

Proposition 1. *There exist transversely affine foliations with holonomy on a compact manifold which are deformable into contact structures.*

Proof. Let $f : \mathbb{S}^1 \to \mathbb{R}$, $\varphi \to \sin\varphi$. $f^{-1}(0) = \{\pi, 2\pi\}$. 0 is a regular value for f. Consider the foliation on $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$ defined by the 1-form $\omega = df + f\,d\theta$ where $d\theta$ is the volume form on $\mathbb{S}^1$. It is a transversely affine foliation since $d\omega = \omega \wedge d\theta$ and $d(d\theta) = 0$. This foliation has two compact leaves $\pi \times \mathbb{S}^1$, $2\pi \times \mathbb{S}^1$, each diffeomorphic to $\mathbb{S}^1$ and with holonomy since $\int_{\mathbb{S}^1} d\theta = 2\pi \neq 0$. Except for those two compact leaves, all the others are dense.

Multiplying by $\mathbb{S}^1$, one obtains on $\mathbb{T}^3$ a transversely affine foliation with holonomy and with exactly two compact toric leaves with nontrivial linear holonomy. All the other leaves are dense.

The above foliation can be deformed into contact structures. Indeed, there are two curves Γ_1 and Γ_2 contained respectively in the two compact leaves F_1 and F_2 with nontrivial linear holonomy. The other leaves contain Γ_1 and Γ_2 in their closures. Put $\Gamma = \Gamma_1 \cup \Gamma_2$. One can find a neighborhood U of Γ in $\mathbb{T}^3$ such that $\mathscr{F}_{|U}$ can be deformed into contact structures. Indeed, there is a 1-form β on a neighborhood U containing Γ. Put $U \subset \Gamma \times [-1,1] \times [-1,1]$ such that $\mathscr{F}_{|U}$ is defined by $\gamma = dz + u(z,x)\,dx$ where $\frac{\partial u}{\partial z} \geqslant C$, where C is a positive constant. Pick a differentiable monotone function

$$h : \mathbb{R} \to [0,1]$$

which is equal to a constant $k > 0$ near 0 and is positive on $[0,1]$, zero on $[1,+\infty[$. Let $\beta = h(x^2 + z^2)\,dy$; it can easily be verified that the 1-form β satisfies

$$\gamma \wedge d\beta + \beta \wedge d\gamma > 0.$$

Consider the family ξ_t of planes fields defined by $\mu_t = \gamma + t\beta$. ξ_t is a deformation of $\mathscr{F}_{|U}$; ξ_t is contact in U and coincides with $\mathscr{F}$ on $T^3 - U$. Thus, we have built a family of confoliations on the compact manifold T^3 beginning with the foliation $\mathscr{F}$. Take V and W other open sets such that $\overline{V} \subset W \subset \overline{W} \subset U$. We can find a family of confoliations $\tilde{\xi}_t$ such that $\tilde{\xi}_0$ is equal to $\mathscr{F}$, $\tilde{\xi}_t$ is contact for all $t > 0$ and $\tilde{\xi}_t = \xi_t$ on $\overline{V}$ [2], thus completing the proof. □

3 Generalisation of the above examples

In the example involving the 1-form $d\theta + df$, the contact form α_n and its characteristic vector field Z_n in the previous section, a crucial fact is that $df(Z_n)$ takes negative and positive values. More generally, one has the following lemma.

Lemma 1. *Let (M, α, Z) be a closed $2n+1$-dimensional contact manifold and β any closed 1-form on M. Then one has*

$$\int_M \beta(Z)\alpha \wedge (d\alpha)^n = 0.$$

Proof. First a general identity on contact manifolds. Given a contact manifold (M, α, Z), any 1-form β on M decomposes as $\beta = \beta(Z)\alpha + \gamma$ where γ is a 1-form satisfying the identity $\gamma(Z) = 0$. Therefore one has

$$\beta \wedge (d\alpha)^n = (\beta(Z)\alpha + \gamma) \wedge (d\alpha)^n = \beta(Z)\alpha \wedge (d\alpha)^n + \gamma \wedge (d\alpha)^n.$$

If $\{Z, E_1, \ldots, E_{2n}\}$ is a basis for the tangent space at a point $p \in M$ such that

$\alpha(E_i) = 0$ for $i = 1, \dots, 2n$, then since $\gamma(Z) = 0$ and $d\alpha(Z, E_i) = 0$ for $i = 1, \dots, 2n$, one has $\gamma \wedge (d\alpha)^n(Z, E_1, \dots, E_{2n}) = 0$ hence $\gamma \wedge (d\alpha)^n = 0$ which implies the identity:

$$\beta \wedge (d\alpha)^n = \beta(Z)\alpha \wedge (d\alpha)^n. \tag{1}$$

From

$$0 = i_Z[\beta \wedge \alpha \wedge (d\alpha)^n] = \beta(Z)\alpha \wedge (d\alpha)^n - \beta \wedge (d\alpha)^n,$$

and identity (1), we deduce that

$$\beta(Z)\alpha \wedge (d\alpha)^n = \beta \wedge (d\alpha)^n = -d[\beta \wedge \alpha \wedge (d\alpha)^{n-1}].$$

Therefore,

$$\int_M \beta(Z)\alpha \wedge (d\alpha)^n = -\int_M d(\beta \wedge \alpha \wedge (d\alpha)^{n-1}) = 0$$

by Stokes' Theorem. □

Remark. Lemma 1 implies that $\beta(Z)$ is either identically zero or takes negative and positive values on M.

4 Proof of Theorems 1 and 2

Proof of Theorem 1. In our proof of Theorem 1, we may assume without loss of generality that α_t being contact means $\alpha_t \wedge (d\alpha_t)^n > 0$, the case where $\alpha_t \wedge (d\alpha_t)^n < 0$ can be proven similarly.

In order to prove that (i) implies (ii), first observe that

$$\alpha_t \wedge d(\alpha_t)^n = t^n \alpha_0 \wedge (d\alpha)^n + t^{n+1}\alpha \wedge (d\alpha)^n \tag{2}$$

and therefore, if $\alpha_0 \wedge (d\alpha)^n < 0$ and $\alpha \wedge (d\alpha)^n \leqslant 0$, then $\alpha_t \wedge (d\alpha_t)^n < 0$.

Suppose that $\alpha_0 \wedge (d\alpha)^n < 0$ and $\alpha \wedge (d\alpha)^n > 0$ at some point $p \in M$. Let $\{E_0, E_1, \dots, E_{2n}\}$ be a positive tangent frame at p. Then evaluating (2) on the positive frame $\{E_0, \dots, E_{2n}\}$ and for any t such that

$$0 < t < \frac{|(\alpha_0 \wedge (d\alpha)^n)(E_0, \dots, E_{2n})|}{|(\alpha \wedge (d\alpha)^n)(E_0, \dots, E_{2n})|},$$

the right hand side of (2), $(t^n \alpha_0 \wedge (d\alpha)^n + t^{n+1}\alpha \wedge (d\alpha)^n)(E_0, \dots, E_{2n})$ is equal to

$$t^n(\alpha_0 \wedge (d\alpha)^n + t\alpha \wedge (d\alpha)^n)(E_0, \dots, E_{2n})$$

and satisfies

$$t^n(\alpha_0 \wedge (d\alpha)^n + t\alpha \wedge (d\alpha)^n)(E_0, \dots, E_{2n}) < t^n((\alpha_0 \wedge (d\alpha)^n)(E_0, \dots, E_{2n}) + |(\alpha_0 \wedge (d\alpha)^n)(E_0, \dots, E_{2n})|) = 0,$$

thus in this case also, one obtains $\alpha_t \wedge (d\alpha_t)^n < 0$. The above argument shows that if α_t is contact for $t > 0$, (i.e. $\alpha_t \wedge (d\alpha_t)^n > 0$), then $\alpha_0 \wedge (d\alpha)^n \geqslant 0$. But then,

$$\int_M \alpha_0 \wedge (d\alpha)^n = -\int_M d(\alpha_0 \wedge \alpha \wedge (d\alpha)^{n-1}) = 0,$$

so necessarily

$$\alpha_0 \wedge (d\alpha)^n = 0. \tag{3}$$

It follows from identity (2) that $\alpha \wedge (d\alpha)^n > 0$, that is, α is a contact form. Now, from identity (3) and using (1), one obtains

$$0 = \alpha_0 \wedge (d\alpha)^n = \alpha_0(Z)\alpha \wedge (d\alpha)^n,$$

which implies that $\alpha_0(Z) = 0$.

Conversely, if Condition (ii) is satisfied, then for $t > 0$, identity (2) implies

$$\alpha_t \wedge (d\alpha_t)^n = t^n\alpha_0(Z)\alpha \wedge (d\alpha)^n + t^{n+1}\alpha \wedge (d\alpha)^n = t^{n+1}\alpha \wedge (d\alpha)^n > 0,$$

that is, α_t is contact for $t > 0$. □

Corollary 1. *Let ξ be a fibration of a closed, odd-dimensional manifold M over the circle $\mathbb{S}^1$. If the leaves of ξ have nonzero Euler characteristic, then ξ cannot be linearly deformed into contact structures.*

Proof. Let α_0 be a closed 1-form defining the foliation ξ. If $\alpha_t = \alpha_0 + t\alpha$, $t \geqslant 0$ is a linear deformation of α_0 into contact forms, then by Theorem 1, α is a contact form whose characteristic vector field is tangent to the leaves of ξ. Therefore, each closed leaf of ξ has Euler characteristic zero. □

As a consequence of this corollary, we see that for a closed surface Σ_g of genus $g \neq 1$, the product foliation on $\Sigma_g \times \mathbb{S}^1$ cannot be linearly deformed into contact structures. It is known [2] that the foliation of $\mathbb{S}^2 \times \mathbb{S}^1$ by 2-spheres cannot even be perturbed into contact structures.

Proof of Theorem 2. Merging identities (1) and (2), one obtains the other identity:

$$\alpha_t \wedge (d\alpha_t)^n = t^n[\alpha_0(Z) + t]\alpha \wedge (d\alpha)^n.$$

By Lemma 1 and the remark in the previous section, there are constants a and b, $a < 0 < b$ such that the inequality $a \leqslant \alpha_0(Z) \leqslant b$ holds. It is clear that for $0 \leqslant t \leqslant -a$, the 1-form α_t fails to be contact on the subset $\Sigma_t \subset M$, where $\Sigma_t = \{p \in M; \alpha_0(Z)(p) = -t\}$. Therefore, one may take $R = |a|$. □

5 Further examples

5.1 K-contact geometry. A contact manifold M with contact form α, Reeb vector field Z, partial almost complex operator J and contact metric g is said to be K-contact [1] if Z is Killing relative to g. Suppose α_0 is a nonsingular harmonic (relative to g) 1-form, then α_0 is basic relative to the flow of Z [3]. This fact and Theorem 1 lead immediately to the following.

Corollary 2. *On a closed K-contact manifold M, any foliation defined by a nonsingular harmonic* 1*-form is linearly deformable into contact structures.*

If α is a contact form with Reeb vector field Z and α_0 a closed, relative to Z basic 1-form, then the characteristic vector field of $\alpha_0 + t\alpha$ is just $\frac{1}{t}Z$. Therefore, if α is a K-contact form with contact metric g, then each Z_t is a Killing vector field with respect to g. It follows from [4] that each of $\alpha_t = \alpha_0 + t\alpha$ is a K-contact form. We may rightly call the type of deformations in Corollary 2 "*linear K-contact deformations*".

5.2 Flat contact geometry. Going back to closed 3-dimensional manifolds, any contact metric structure with flat contact metric g carries a codimension one foliation which is determined by a parallel, hence harmonic, 1-form α_0 [5]. This foliation is parallelizable by two commuting orthogonal vector fields, each being the Reeb vector field of a contact form with contact metric g. Therefore, denoting the two contact forms by v and β, we see that the foliation determined by α_0 admits a 2-parameter family $\alpha_{t,s} = \alpha_0 + tv + s\beta$, $t \geqslant 0$, $s \geqslant 0$, of linear deformations into contact structures. These include and generalize examples presented in Section 2.

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