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# Switching of generalized quadrangles of order $s$ and applications

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*Dedicated to A. Barlotti at the occasion of his 80th birthday*

**Abstract.** Let $\mathcal{S} = (P, B, \mathrm{I})$ be a generalized quadrangle of order $s$, $s \neq 1$, having a flag $(x, L)$ with $x$ and $L$ regular. Then a generalized quadrangle $\mathcal{S}' = (P', B', \mathrm{I}')$ of order $s$ can be constructed. We say that $\mathcal{S}'$ is obtained by switching from $\mathcal{S}$ with respect to $(x, L)$. Examples are given where $\mathcal{S} \not\cong \mathcal{S}'$; e.g., starting from a $T_2(O)$ of Tits, with $O$ an oval of $\mathrm{PG}(2, q)$, $q$ even, with nucleus $n$, the GQ $T_2((O - \{x\}) \cup \{n\})$ with $x \in O$ can be obtained by switching from $T_2(O)$. Applications to translation generalized quadrangles of order $s$, $s$ even, and generalized quadrangles of order $(s, s^2)$, $s$ even, satisfying Property (G), are given.

## 1 Introduction

A (finite) *generalized quadrangle* (GQ) is an incidence structure $\mathcal{S} = (P, B, \mathrm{I})$ in which $P$ and $B$ are disjoint (nonempty) sets of objects called *points* and *lines*, respectively, and for which $\mathrm{I} \subseteq (P \times B) \cup (B \times P)$ is a symmetric point-line *incidence relation* satisfying the following axioms.

(i) Each point is incident with $1 + t$ lines ($t \geqslant 1$) and two distinct points are incident with at most one line.

(ii) Each line is incident with $1 + s$ points ($s \geqslant 1$) and two distinct lines are incident with at most one point.

(iii) If $x$ is a point and $L$ is a line not incident with $x$, then there is a unique pair $(y, M) \in P \times B$ for which $x \,\mathrm{I}\, M \,\mathrm{I}\, y \,\mathrm{I}\, L$.

Generalized quadrangles were introduced by Tits [15] in his celebrated work on triality.

The integers $s$ and $t$ are the *parameters* of the GQ and $\mathcal{S}$ is said to have *order* $(s, t)$; if $s = t$, then $\mathcal{S}$ is said to have *order* $s$. There is a point-line duality for GQ (of order $(s, t)$) for which in any definition or theorem the words “point” and “line” are interchanged and the parameters $s$ and $t$ are interchanged. Hence, we assume without further notice that the dual of a given theorem or definition has also been given.

Let $\mathcal{S} = (P, B, \mathrm{I})$ be a (finite) GQ of order $(s, t)$. Then $\mathcal{S}$ has $v = |P| =$

$(1+s)(1+st)$ points and $b = |B| = (1+t)(1+st)$ lines; see 1.2.1 of Payne and Thas [8]. Also, $s+t$ divides $st(1+s)(1+t)$, and, for $s \neq 1 \neq t$, we have $t \leqslant s^2$, and, dually, $s \leqslant t^2$; see 1.2.2 and 1.2.3 of Payne and Thas [8].

Given two (not necessarily distinct) points $x, y$ of $\mathscr{S}$, we write $x \sim y$ and say that $x$ and $y$ are *collinear*, provided that there is some line $L$ for which $x \mathrm{I} L \mathrm{I} y$. And $x \not\sim y$ means that $x$ and $y$ are not collinear. Dually, for $L, M \in B$, we write $L \sim M$ or $L \not\sim M$ according as $L$ and $M$ are *concurrent* or *non-concurrent*, respectively. The line which is incident with distinct collinear points $x, y$ is denoted by $xy$; the point which is incident with distinct concurrent lines $L, M$ is denoted by either $LM$ or $L \cap M$.

For $x \in P$, put $x^\perp = \{y \in P \mid y \sim x\}$, and note that $x \in x^\perp$. If $A \subseteq P$, then $A^\perp = \bigcap\{x^\perp \mid x \in A\}$. Hence, for $x, y \in P$, $x \neq y$, we have $\{x, y\}^\perp = x^\perp \cap y^\perp$; we have $|\{x, y\}^\perp| = s+1$ or $t+1$ according as $x \sim y$ or $x \not\sim y$. Further, $\{x, y\}^{\perp\perp} = \{u \in P \mid u \in z^\perp$ for all $z \in x^\perp \cap y^\perp\}$; we have $|\{x, y\}^{\perp\perp}| = s+1$ or $|\{x, y\}^{\perp\perp}| \leqslant t+1$ according as $x \sim y$ or $x \not\sim y$. The sets $\{x, y\}^\perp$ and $\{x, y\}^{\perp\perp}$ are respectively called the *trace* and the *span* of the pair $\{x, y\}$.

## 2 Regularity and nets

Let $\mathscr{S} = (P, B, \mathrm{I})$ be a finite GQ of order $(s, t)$. If $x \sim x'$, $x \neq x'$, or if $x \not\sim x'$ and $|\{x, x'\}^{\perp\perp}| = t+1$, where $x, x' \in P$, we say the pair $\{x, x'\}$ is *regular*. The point $x$ is *regular* provided $\{x, x'\}$ is regular for all $x' \in P$, $x' \neq x$. Regularity for lines is defined dually. A point $x$ is *coregular* provided each line incident with $x$ is regular. If $1 < s < t$, then one can show that no pair of distinct points is regular; see 1.3.6 of Payne and Thas [8]. If the GQ $\mathscr{S}$ has even order $s$ and the point $x$ is coregular, then $x$ is regular; see 1.5.2 (iv) of Payne and Thas [8]. A flag $(x, L)$, hence $x \mathrm{I} L$, is called *regular* if $x$ and $L$ are regular; for $s \neq 1 \neq t$ this implies that $s = t$.

A (finite) *net* of *order* $k$ ($\geqslant 2$) and *degree* $r$ ($\geqslant 2$) is an incidence structure $\mathscr{N} = (P, B, \mathrm{I})$ satisfying

(i) each point is incident with $r$ lines and two distinct points are incident with at most one line;

(ii) each line is incident with $k$ points and two distinct lines are incident with at most one point;

(iii) if $x$ is a point and $L$ is a line not incident with $x$, then there is a unique line $M$ incident with $x$ and not concurrent with $L$.

For a net of order $k$ and degree $r$ we have $|P| = k^2$ and $|B| = kr$. Also, $r \leqslant k+1$, with $r = k+1$ if and only if the net is an affine plane; see Dembowski [2].

**Theorem 2.1** (1.3.1 of Payne and Thas [8]). *Let $x$ be a regular point of the* GQ $\mathscr{S} = (P, B, \mathrm{I})$ *of order $(s, t)$, $s \geqslant 2$. Then the incidence structure with point set $x^\perp - \{x\}$, with line set the set of all spans $\{y, z\}^{\perp\perp}$, where $y, z \in x^\perp - \{x\}$, $y \not\sim z$, and with the natural incidence, is the dual of a net of order $s$ and degree $t+1$. If in particular $s = t > 1$, there arises a dual affine plane of order $s$. Also, in the case $s = t > 1$ the* *incidence structure* $\pi_x$ *with point set* $x^{\perp}$, *with line set the set of all spans* $\{y,z\}^{\perp\perp}$, *where* $y,z \in x^{\perp}$, $y \neq z$, *and with the natural incidence, is a projective plane of order* $s$.

## 3 Switching of generalized quadrangles

Let $\mathscr{S} = (P, B, \mathrm{I})$ be a GQ of order $s$, with $s \neq 1$, for which the flag $(x, L)$ is regular. Let $L_0, L_1, L_2, \ldots, L_s$ be the lines incident with $x$, where $L = L_0$.

Define as follows the incidence structure $P(\mathscr{S}, x) = (P_x, B_x, \mathrm{I}_x)$. Points of $P(\mathscr{S}, x)$ are the points of $P - x^{\perp}$. Lines of $P(\mathscr{S}, x)$ are the lines of $B$ not incident with $x$ and the spans $\{x, y\}^{\perp\perp}$, with $y \not\sim x$. If $z \in P_x$ and $M \in B_x$ with $M \in B$, then $z \,\mathrm{I}_x\, M$ if and only if $z \,\mathrm{I}\, M$; if $z \in P_x$ and $N = \{x, y\}^{\perp\perp}$, $y \not\sim x$, then $z \,\mathrm{I}\, N$ if and only if $z \in N$. Then by Payne, see 3.1.4 of Payne and Thas [8], $P(\mathscr{S}, x)$ is a GQ of order $(s-1, s+1)$.

A *spread* of $P(\mathscr{S}, x)$ is a set $S$ of $s^2$ lines in $B_x$ such that each point of $P_x$ is incident (for $\mathrm{I}_x$) with exactly one element of $S$. Spreads of $P(\mathscr{S}, x)$ are the set $S_x$ of all lines of type $\{x, y\}^{\perp\perp}$, $y \not\sim x$, and any set $S_i$ consisting of all lines of $\mathscr{S}$ concurrent with $L_i$ but not incident with $x$, with $i = 0, 1, \ldots, s$. Clearly $B_x = S_x \cup S_0 \cup S_1 \cup \cdots \cup S_s$.

Let $x_1, x_2, \ldots, x_s, x$ be the points incident with $L$ and let $S_{x,i}$ be the set of all spans $\{x, y\}^{\perp\perp}$, with $y \sim x_i$, $y \not\sim x$. Then $S_x = S_{x,1} \cup S_{x,2} \cup \cdots \cup S_{x,s}$. Further, for $i \in \{1, 2, \ldots, s\}$, let $S_{i,j}$, with $j = 1, 2, \ldots, s$, be the sets $\{L_i, N\}^{\perp} - \{L\}$, with $N \sim L$, $x \,\not{\mathrm{I}}\, N$. Then $S_i = S_{i,1} \cup S_{i,2} \cup \cdots \cup S_{i,s}$. It is clear that the sets $S_x, S_0, \ldots, S_s$ and $S_{x,j}$, $S_{1,j}, \ldots, S_{s,j}$, with $j = 1, 2, \ldots, s$, satisfy the conditions of Theorem 1.1 in Payne [4], and so by Payne [4] a GQ $\mathscr{S}_{(x,L)}$ of order $s$ may be constructed as follows.

Points of $\mathscr{S}_{(x,L)}$ are of three kinds: points of type (i) are just the points of $P(\mathscr{S}, x)$; points of type (ii) are the $s^2 + s$ sets $S_{x,j}, S_{1,j}, \ldots, \mathscr{S}_{s,j}$, with $j = 1, 2, \ldots, s$; there is a unique point of type (iii) denoted by $x'$. Lines are of two types: lines of type (a) are the lines of $P(\mathscr{S}, x)$ not in the spread $S_0$; lines of type (b) are the spreads $S_x, S_1, \ldots, S_s$. The incidence is as follows. A point of type (i) is incident with a line of type (a) if and only if the two are incident in $P(\mathscr{S}, x)$. A point of type (i) is incident with no line of type (b). A point of type (ii) is incident with each line of type (a) which belongs to it and with the unique line of type (b) of which it is a subset. The unique point of type (iii) is incident with no line of type (a) and all lines of type (b).

It is an easy excercise to show that the flag $(x', S_x)$ of $\mathscr{S}_{(x,L)}$ is regular. Note that $\mathscr{S}_{(x,L)} \cong \mathscr{S}_{(L,x)}$ and that $(\mathscr{S}_{(x,L)})_{(x', \mathscr{S}_x)} \cong \mathscr{S}$. We say that $\mathscr{S}_{(x,L)}$ is obtained by *switching* from $\mathscr{S}$ (with respect to $(x, L)$) and that $\mathscr{S}$ and $\mathscr{S}_{(x,L)}$ are *switching equivalent*.

We emphasize that the construction of $\mathscr{S}_{(x,L)}$ from $\mathscr{S}$ is a combination of ideas of Payne.

Up to isomorphism the GQ $\mathscr{S}_{(x,L)}$ can also be described as follows. Let $\mathscr{S}' = (P', B', \mathrm{I}')$ be the following incidence structure. Points $(P')$ are of four types:

(i) the $s^3$ points of $P$ not in $x^{\perp}$;

(ii) the $s^2$ spans $\{L, N\}^{\perp\perp}$, with $L \not\sim N$;

(iii) the $s$ points $x_1, x_2, \ldots, x_s$ incident (for I) with $L$, but distinct from $x$;

(iv) the point $x$.

Lines ($B'$) are of four types:

(a) the $s^3$ lines of $B$ not in $L^\perp$;

(b) the $s^2$ spans $\{x, y\}^{\perp\perp}$, with $x \not\sim y$;

(c) the $s$ lines $L_1, L_2, \dots, L_s$ incident (for I) with $x$, but distinct from $L$;

(d) the line $L$.

Incidence ($\mathrm{I}'$) is defined as follows:

A point of type (i) is incident with the $s$ lines of type (a) incident (for I) with it and with the unique line of type (b) containing it. A point $\{L, N\}^{\perp\perp}$ of type (ii) is incident with the $s$ lines of type (a) contained in it, and with the unique line $L_i \in \{L, N\}^\perp$. A point $x_i$ of type (iii) is incident with the line $L$ and with the $s$ lines of type (b) contained in $x_i^\perp$. The point $x$ of type (iv) is incident with the $s+1$ lines $L, L_1, \dots, L_s$.

Then it is easily checked that $\mathscr{S}' \cong \mathscr{S}_{(x,L)}$.

The projective plane $\pi_x$ defined by the regular point $x$ of $\mathscr{S}$ is isomorphic to the dual of the projective plane $\pi'_L$ defined by the regular line $L$ of $\mathscr{S}'$; the projective plane $\pi_L$ defined by the regular line $L$ of $\mathscr{S}$ is isomorphic to the dual of the projective plane $\pi'_x$ defined by the regular point $x$ of $\mathscr{S}'$.

**Example.** Let $O = \{\pi_0, \pi_1, \dots, \pi_{q^n}\}$ be a *pseudo-oval* of $\mathrm{PG}(3n-1, q)$, that is, a set of $q^n + 1$ $(n-1)$-dimensional subspaces of $\mathrm{PG}(3n-1, q)$ every three of which generate $\mathrm{PG}(3n-1, q)$. Then $O$ defines a GQ $T(O)$ of order $q^n$; see 8.7 in Payne and Thas [8]. For $n = 1$ the GQ $T(O)$ is the Tits quadrangle $T_2(O)$ arising from the oval $O$ of $\mathrm{PG}(2, q)$. The lines $\pi_0, \pi_1, \dots, \pi_{q^n}$ of type (b) of $T(O)$ are regular, and so, for $q$ even, the point $(\infty)$ of type (iii) is also regular. For $q$ even the tangent spaces of $O$ all contain a common $(n-1)$-dimensional space $\pi$, called the *kernel* or *nucleus* of $O$; see Section 4.9 of Thas [9].

Assume that $q$ is even. Then it is easily checked that $T(O)_{((\infty),\pi_i)} \cong T(O_i)$, with $O_i = (O - \{\pi_i\}) \cup \{\pi\}$, $i = 0, 1, \dots, q^n$. The projective plane defined by the regular point $(\infty)$ of $T(O)$ is isomorphic to the dual of the projective plane defined by the regular line $\pi$ of $T(O_i)$, and the projective plane defined by the regular point $(\infty)_i$ of $T(O_i)$ is isomorphic to the dual of the projective plane defined by the regular line $\pi_i$ of $T(O)$.

## 4 Application to translation generalized quadrangles

Let $\mathscr{S} = (P, B, \mathrm{I})$ be a GQ of order $(s, t)$, $s \neq 1$, $t \neq 1$. A collineation $\theta$ of $\mathscr{S}$ is a *whorl* about the point $p$ provided $\theta$ fixes each line incident with $p$. Let $\theta$ be a whorl about $p$. If $\theta = \mathrm{id}$ or if $\theta$ fixes no point of $P - p^\perp$, then $\theta$ is an *elation* about $p$. If there is a group $G$ of elations about $p$ acting regularly on $P - p^\perp$, we say that $\mathscr{S}$ is an *elation generalized quadrangle* (EGQ) with *elation group* $G$ and *base point* $p$. Briefly, we say that $(\mathscr{S}^{(p)}, G)$ or $\mathscr{S}^{(p)}$ is an EGQ. Most known examples of GQ or their duals are EGQ, the notable exceptions being those of order $(s-1, s+1)$ and their duals. If the group $G$ is abelian, then we say that the EGQ $(\mathcal{S}^{(p)}, G)$ is a *translation generalized quadrangle* (TGQ) with *translation group* $G$ and *base point* $p$.

The following theorems are taken from Chapter 8 of Payne and Thas [8].

**Theorem 4.1.** *If* $(\mathcal{S}^{(p)}, G)$ *is a* TGQ *then the point* $p$ *is coregular and so* $s \leqslant t$. *Also,* $G$ *is the complete set of all elations about* $p$.

**Theorem 4.2.** *If* $(\mathcal{S}^{(p)}, G)$ *is a* TGQ *then the group* $G$ *is elementary abelian, and* $s$ *and* $t$ *must be powers of the same prime. If* $s < t$, *then there is a prime power* $q$ *and an odd integer* $a$ *for which* $s = q^a$ *and* $t = q^{a+1}$. *If* $s$ *(or* $t$*) is even then either* $s = t$ *or* $s^2 = t$.

The following result is easy to prove; see Theorem 4.4 in Thas [13].

**Theorem 4.3.** *Let* $(\mathcal{S}^{(p)}, G)$ *be a* TGQ *of order* $s$. *Then for any line* $L$ *incident with* $p$, *the projective plane* $\pi_L$ *defined by the regular line* $L$ *is a translation plane for which the translation line is the line of* $\pi_L$ *defined by the point* $p$.

In Section 5 of Thas [13] the following characterization of TGQ of order $s$ is obtained.

**Theorem 4.4.** *Let* $\mathcal{S} = (P, B, \mathrm{I})$ *be* GQ *of order* $s$, $s \neq 1$, *with coregular point* $p$. *If for at least one line* $L$ *incident with* $p$ *the corresponding projective plane* $\pi_L$ *is a translation plane with as translation line the set of all lines of* $\mathcal{S}$ *incident with* $p$, *then* $\mathcal{S}$ *is a* TGQ *with base point* $p$. *If in particular the plane* $\pi_L$ *is Desarguesian, then* $\mathcal{S}$ *is a* TGQ *with base point* $p$, *and also, for* $s$ *odd,* $\mathcal{S}$ *is isomorphic to the classical* GQ $Q(4, s)$ *arising from a non-singular quadric in* $\mathrm{PG}(4, s)$.

Next theorem illustrates how switching of GQ can be applied.

**Theorem 4.5.** *Let* $\mathcal{S} = (P, B, \mathrm{I})$ *be a* GQ *of order* $s$, $s$ *even, with coregular point* $x$. *Then* $\mathcal{S}$ *is a* TGQ *with base point* $x$ *if and only if the projective plane* $\pi_x$ *is a dual translation plane with translation point* $x$. *If in particular the plane* $\pi_x$ *is Desarguesian, then* $\mathcal{S}$ *is a* TGQ *with base point* $x$.

*Proof.* Let $\mathcal{S}$ be a TGQ of order $s$, $s$ even, with base point $x$. Then the point $x$ is coregular and regular. It is easy to check that $\pi_x$ is indeed a dual translation plane with translation point $x$.

Conversely, let $\mathcal{S} = (P, B, \mathrm{I})$ be a GQ of order $s$, $s$ even, with coregular point $x$ and assume that $\pi_x$ is a dual translation plane with translation point $x$. Let $L, L_1, \ldots, L_s$ be the lines incident with $x$. Now we consider the GQ $\mathcal{S}_{(x,L)}$ which is obtained by switching from $\mathcal{S}$ with respect to the flag $(x, L)$. Let $\mathcal{S}'$ be the GQ isomorphic with $\mathcal{S}_{(x,L)}$, described in Section 3. Then $(x, L)$ is a regular flag of $\mathcal{S}'$ and the projective plane $\pi_x$ is isomorphic to the dual of the projective plane $\pi'_L$. Also, all lines $L, L_1$, $L_2, \ldots, L_s$ are regular for $\mathcal{S}'$, hence $x$ is a coregular point of $\mathcal{S}'$. The plane $\pi'_L$ is a translation plane with $\{L, L_1, L_2, \ldots, L_s\}$ as translation line. By Theorem 4.4 $\mathscr{S}'$ is a TGQ with base point $x$. By 8.7 of Payne and Thas [8] $\mathscr{S}' \cong T(O)$, with $O$ a pseudo-oval in some $\mathrm{PG}(3n-1, q)$. Let $O = \{\pi, \pi_1, \ldots, \pi_s\}$, where $\pi$ corresponds to $L$, and let $\eta$ be the nucleus of $O$. By the example at the end of Section 4 the GQ $\mathscr{S} \cong \mathscr{S}'_{(x,L)}$ is isomorphic to $T(O')$, with $O' = (O - \{\pi\}) \cup \{\eta\}$. As $O'$ is again a pseudo-oval the GQ $T(O')$ is a TGQ, and so $\mathscr{S}$ is a TGQ with base point $x$. □

## 5 Application to generalized quadrangles satisfying Property (G)

Let $\mathscr{S} = (P, B, \mathrm{I})$ be a GQ of order $(s, s^2)$, $s \neq 1$. Then $|\{x, y, z\}^\perp| = s + 1$ for any triple $\{x, y, z\}$ of pairwise non-collinear points; see 1.2.4 of Payne and Thas [8]. We say $\{x, y, z\}$ is 3-*regular* provided $|\{x, y, z\}^{\perp\perp}| = s + 1$. The point $x$ is 3-*regular* if and only if each triple $\{x, y, z\}$ of pairwise non-collinear points is 3-regular.

Let $x_1, y_1$ be distinct collineair points. We say that the pair $\{x_1, y_1\}$ has *Property* (G), or that $\mathscr{S}$ has *Property* (G) *at* $\{x_1, y_1\}$, if every triple $\{x_1, x_2, x_3\}$ of points, with $x_1, x_2, x_3$ pairwise non-collinear and $y_1 \in \{x_1, x_2, x_3\}^\perp$, is 3-regular. The GQ $\mathscr{S}$ has *Property* (G) *at the line* $L$, or the line $L$ has *Property* (G), if each pair of points $\{x, y\}$, $x \neq y$ and $x \,\mathrm{I}\, L \,\mathrm{I}\, y$, has Property (G). If $(x, L)$ is a flag, then we say that $\mathscr{S}$ has *Property* (G) *at* $(x, L)$, or that $(x, L)$ has *Property* (G), if every pair $\{x, y\}$, $x \neq y$ and $y \,\mathrm{I}\, L$, has Property (G). It is clear that the point $x$ is 3-regular if and only if $(x, L)$ has Property (G) for each line $L$ incident with $x$.

Let $F$ be a *flock* of the quadratic cone $K$ with vertex $x$ of $\mathrm{PG}(3, q)$, that is, a partition of $K - \{x\}$ into $q$ disjoint irreducible conics. Then, relying on work of Payne [5, 6] and Kantor [3], Thas [10] proves that with $F$ corresponds a GQ $\mathscr{S}(F)$ of order $(q^2, q)$. In Payne [7] it was shown that $\mathscr{S}(F)$ satisfies Property (G) at its point $(\infty)$. The following fundamental result is taken from Thas [12].

**Theorem 5.1.** *A* GQ $\mathscr{S} = (P, B, \mathrm{I})$ *of order* $(q, q^2)$, *q odd and* $q \neq 1$, *satisfies Property* (G) *at some flag* $(x, L)$ *if and only if* $\mathscr{S}$ *is the dual of a flock* GQ.

In Thas [12] also several strong results for $q$ even are proved. It follows that Theorem 5.1 also holds for $q \in \{2, 4, 16\}$. Further it is conjectured that a GQ of order $(q, q^2)$, $q$ even, satisfies Property (G) at some line $L$ if and only if $\mathscr{S}$ is the dual of a flock GQ.

Let $\mathscr{S} = (P, B, \mathrm{I})$ be a GQ of order $(s, s^2)$, $s$ even, which satisfies Property (G) at some line $L$. Let $x_0, x_1, \ldots, x_s$ be the points incident with $L$. Further, let $\{x_i, y, z\}$ be a triple of pairwise non-collinear points for which $\{x_i, y, z\}^\perp$ contains an element incident with $L$, say $x_j$, with $i, j \in \{0, 1, \ldots, s\}$. Let $T = \{x_i, y, z\}$ and let $P'$ be the set of all points incident with lines of the form $uv$, $u \in T^\perp$, and $v \in T^{\perp\perp}$. If $B'$ is the set of all lines in $B$ which are incident with at least two points in $P'$, and if $\mathrm{I}'$ is the restriction of I to $(P' \times B') \cup (B' \times P')$, then $\mathscr{S}' = (P', B', \mathrm{I}')$ is a subquadrangle of $\mathscr{S}$ of order $s$; moreover $\{x_i, y\}$ is a regular pair of $\mathscr{S}'$, with $\{x_i, y\}^{\perp'} = T^\perp$ and $\{x_i, y\}^{\perp'\perp'} = T^{\perp\perp}$ (see 2.6.2 of Payne and Thas [8]). In this way there arise $s^3 + s^2$ subquadrangles of order $s$ in $\mathscr{S}$; see Theorem 3.1.5 of Thas [11]. This is the maximum number of subquadrangles of order $s$ of $\mathcal{S}$ containing the line $L$; see e.g. Lemma 2.4 of Brown and Thas [1].

**Theorem 5.2.** *Let* $\mathcal{S} = (P, B, \mathrm{I})$ *be a* GQ *of order* $(s, s^2)$, *s even, which satisfies Property* (G) *at some line L. Then the* $s^3 + s^2$ *subquadrangles of order s of* $\mathcal{S}$ *containing the line L are* TGQ *with base line L.*

*Proof.* Let $\mathcal{S}'$ be one of the subquadrangles of order $s$ of $\mathcal{S}$ containing the line $L$. By Theorem 3.2.1 of Thas [11] the line $L$ is a coregular line of $\mathcal{S}'$ and a regular line of $\mathcal{S}$. By Thas and Van Maldeghem [14] the dual net $\mathcal{N}_L^*$ defined by the regular line $L$ of $\mathcal{S}$ (see also Section 2) is isomorphic to the dual net $H_s^3$, with point set the set of all points of $\mathrm{PG}(3, s)$ not on a given line $M$ of $\mathrm{PG}(3, s)$, with line set the set of all lines of $\mathrm{PG}(3, s)$ skew to $M$, and where incidence is induced by $\mathrm{PG}(3, s)$. The dual affine plane $\mathcal{A}$ defined by the regular line $L$ of $\mathcal{S}'$ is a subgeometry of $\mathcal{N}_L^*$, hence is easily seen to be isomorphic to the dual of $\mathrm{AG}(2, s)$. Now by Theorem 4.5 the GQ $\mathcal{S}'$ is a TGQ with base line $L$. □

We hope Theorem 5.2 is a step towards the classification of all GQ of order $(s, s^2)$, $s$ even, satisfying Property (G) at some line $L$.

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