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Characterization of Grassmannians by one class of singular subspaces

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Abstract. Spaces related to Grassmann spaces are characterized in terms of the relation of any point to members of some class of singular subspaces of a parapolar space. It is not assumed that all singular subspaces have finite projective rank—only that at least one subspace in the specified class does so.

1 Introduction

The classical point-line geometries take as their points the coset space of a parabolic subgroup of a group of Lie type. In the case of Lie groups, these are most of the ruled manifolds that have concerned analysts, topologists and physicists for over a century. In general, however, they are defined relative to arbitrary fields and division rings where they are studied from the points of view of algebraic geometry and incidence geometry.

The point of view of incidence geometry begins with a geometry $(\mathscr{P}, \mathscr{L})$ of points and lines, and seeks to characterize a classical parabolic coset-space, by relatively simple axioms on these points and lines. The famous Veblen-Young theorem [18] characterizing all projective spaces of rank at least three is an example of this sort; so also is the characterization of all nondegenerate polar spaces of rank at least three (combined results of [17, 4, 12, 11]).

These two geometries make their appearance as proper convex subspaces of the remaining Lie-incidence geometries in the roles of singular subspaces and symplecta, respectively, and this fact has motivated the definition of *parapolar space* as well as the advancement of these spaces as a natural stage on which to characterize most of the remaining coset geometries for groups of spherical Lie type.

Of course the parapolar concept arose first with the early papers of Cooperstein [9, 10] and evolved to more adaptable forms in the work of F. Buekenhout [3, 2] and unpublished notes of A. Cohen. The first major characterization theorems depended on two break-through papers of A. Cohen [5, 6] and appeared in [8]. This theorem characterized at least one coset geometry for each group of exceptional Lie type but did so on the basis of a very restricted relation between a symplecton and any exterior point.

The present paper is part of a program to follow the other alternative: to characterize parapolar spaces by the relation of a point to a maximal singular subspace belonging to a limited class of such spaces. The basic hypothesis is that the set of points $x^{\perp} \cap M$ collinear with a point x exterior to a maximal singular space M is either empty, a single point, or has projective rank $d \geqslant 2$. So far, the Grassmann and half-spin geometries have been so characterized [15, 16], but the former characterization requires the hypothesis as M ranges over *all* maximal singular subspaces and when all of these have *finite* singular rank. Thus, as things stand, the characterization of Grassmannians in [15] rests upon an unreasonably strong hypothesis not at all in line with the second result [16] on half-spin geometries. The purpose of this paper is to achieve a singular characterization of Grassmannians that can be a companion to the half-spin characterization of [16]—that is, it must have these features: (1) the hypothesis is only on a class $\mathscr{M}$ of maximal singular subspaces rich enough to cover every line, and (2) it is not assumed that all singular subspaces possess finite projective rank.

Why study singular characterizations? Singular subspaces can be recognized in the point-residuals of any gamma space, while symplecta might not. In forthcoming work with S. Onefrei, the existing singular characterizations are used to obtain singular characterizations of several exceptional coset geometries—for example $E_{6,4}$, and $E_{7,7}$, where the points are the cosets of the maximal parabolic groups corresponding to the node at the end of the shortest and middle-length arms of the respective Dynkin diagrams.

2 The basic axioms and results

2.1 The hypotheses. We assume:

(D) 1. *$\Gamma = (\mathscr{P}, \mathscr{L})$ is a parapolar space.*

2. *There exists a class $\mathscr{M}$ of maximal singular subspaces of Γ such that*
   (a) *Every line $L \in \mathscr{L}$ lies in a member of $\mathscr{M}$.*
   (b) *Given a non-incident pair $(p, M) \in \mathscr{P} \times \mathscr{M}$, the set $p^{\perp} \cap M$ is empty, or is a line.*

We now describe the properties of a point-residual of a point-line geometry Γ satisfying the hypothesis (D) above.

(E) 1. Γ is a strong parapolar space.

2. Γ contains a class $\mathscr{M}$ of maximal singular subspaces with these properties:
   (a) Every member of $\mathscr{M}$ is a projective space (not necessarily of finite rank).
   (b) Every point lies in a member of $\mathscr{M}$.
   (c) If $M \in \mathscr{M}$, and x is a point not in M, then x is collinear with a unique point $\pi_M(x)$. (Thus $\pi_M : \mathscr{P} - M \to M$ is a well defined mapping—the *projection onto M*.)

3. Any symplecton is classical.

Remark. Conclusion 3 comes from the fact that the Γ discussed above is the point-residual of a strong parapolar space of symplectic rank at least three. In particicular the point-diameter is only two.

Theorem 1. *Suppose* $\Gamma = (\mathscr{P}, \mathscr{L})$ *is a point-line geometry satisfying the hypothesis* (E) *above. Then one of the following holds*:

1. Γ *is a generalized quadrangle.*
2. Γ *is the product geometry* $A \times B$ *of two maximal singular subspaces* A *and* B.
3. *Some symplecton is not a grid, and the members of* $\mathscr{M}$ *partition the points. Any line which is not contained in a element of* $\mathscr{M}$ *is a maximal singular subspace. No line intersects all members of* $\mathscr{M}$.

Theorem 2. *Assume* Γ *satisfies hypothesis* (D) *where some maximal singular subspace of* $\mathscr{M}$ *has finite projective rank. Suppose further that some line lies on at least two members of* $\mathscr{M}$. *Then one of the following holds*:

1. Γ *is a non-degenerate polar space of rank three.*
2. Γ *is the Grassmannian of all d-subspaces of a projective space* $\mathbf{P}(V)$, *possibly of infinite rank, and d is a finite integer greater than* 1.
3. *All maximal singular subspaces of* Γ *have finite rank and* Γ *is isomorphic to the factor geometry* $A_{2n-1,n}(D)/\langle\sigma\rangle$, *where* σ *is a polarity of Witt index at most* $n-5$.

Remark. If one wishes to omit the assumption that at least one line lies in at least two members of $\mathscr{M}$, the conclusion of Theorem 2 must be altered to include a third possiblity that is described in Theorem 22 in the last section of this paper. The above Theorem 2 is thus an immediate corollary of Theorem 22. No example of this third alternative to Theorem 22 is known to the author.

3 Review of basic concepts

3.1 The basic glossary. A point-line geometry is simply a pair of sets $(\mathscr{P}, \mathscr{L})$ with a symmetric incidence relation such that every member of $\mathscr{L}$ is incident with at least two members of $\mathscr{P}$. The elements of $\mathscr{P}$ are *points* while the elements of $\mathscr{L}$ are called *lines*. The set of points incident with a given line $L \in \mathscr{L}$ is called the *point-shadow of* L. A line incident with at least three points is said to be *thick*.

A subset S of $\mathscr{P}$ is called a *subspace* if and only if the point-shadow of any line intersects it at zero, exactly one, or all of its points. The intersection of an arbitrary collection of subspaces is a subspace (which might be empty). The intersection of all subspaces of Γ containing a subset X of $\mathscr{P}$ is called *the subspace generated by* X and is denoted $\langle X\rangle_\Gamma$.

Two points are *collinear* if and only if there exists a line incident with both of them.

The *point-collinearity graph* of $\Gamma = (\mathscr{P}, \mathscr{L})$ is the graph $\Delta(\Gamma)$ (or just Δ if Γ is understood) whose vertex set is $\mathscr{P}$ and whose edges are pairs of distinct points which are collinear. The geometry Γ is said to be *connected* if and only if the graph Δ is connected.

A subspace S is said to be a *singular* if any two of its points are collinear. If p is a point, the symbol $p^\perp$ denotes the set of all points which are either collinear with, or equal to p (this notation goes back to D. Higman and is standard). A point-line geometry $\Gamma = (\mathscr{P}, \mathscr{L})$ is called a *gamma space* if and only if $p^\perp$ is always a subspace, for each point p. This is equivalent to saying that for every point p and line L not incident with p, if p is collinear with at least two distinct points of the point-shadow of L, then p is collinear with all points of that point-shadow.

If distinct lines always possess distinct point shadows, one may simply identify a line L with its own point-shadow. This certainly occurs when $(\mathscr{P}, \mathscr{L})$ is a *partial linear space*, that is point-line geometry in which two distinct points are incident with at most one line. A partial linear space that is a singular space is called a *linear space*. (It is a standard convention to denote the unique line on two distinct collinear points x and y of a partial linear space by the symbol xy.) A linear space $(\mathscr{P}, \mathscr{L})$ with all lines thick is called a *projective plane* if and only if any two distinct lines in it intersect at a point. A linear space $(\mathscr{P}, \mathscr{L})$ with all lines thick is called a *projective space* if and only if any two intersecting lines generate a projective plane. In this case, as is well known, the poset of all subspaces is the poset of flats of a matroid, and so the cardinality of any two minimal generating sets is the same—a number which when diminished by one, is called the *projective dimension* or *rank* of the projective space.

We remark that a gamma space, all of whose singular subspaces are linear spaces, must be a partial linear space.

A subspace S of $\Gamma = (\mathscr{P}, \mathscr{L})$ is said to be *convex* if and only if, for any two of its points—say x and y—the intermediate vertices of any path of minimal length connecting x to y in the point-collinearity graph Δ are also points of S.

The intersection of any collection of convex subspaces is also a convex subspace. The intersection of all convex subspaces of $\Gamma = (\mathscr{P}, \mathscr{L})$ containing a subset X of $\mathscr{P}$, is a convex subspace called the *convex closure* (*in* Γ) *of* X, and is denoted $\langle\langle X\rangle\rangle_\Gamma$.

3.2 Product geometries. Now suppose $\Gamma_i = (\mathscr{P}_i, \mathscr{L}_i)$ is a point-line geometry for $i = 1, 2$. We wish to describe the *product geometry* $\Gamma_1 \times \Gamma_2$. Its set of points is the Cartesian product $\mathscr{P}_1 \times \mathscr{P}_2$. There are two sorts of lines. A *vertical line* is one whose point shadow has the form $x_1 \times L_2 := \{(x_1, y_2) \mid y_2 \in L_2\}$ where $L_2 \in \mathscr{L}_2$. Similarly, a *horizontal line* is one whose point shadow has the form $L_1 \times y_2 := \{(x_1, y_2) \mid x_1 \in L_1\}$ where L_1 is a line of $\mathscr{L}_1$. Then the product geometry, $\Gamma_1 \times \Gamma_2$ becomes a point-line geometry $(\mathscr{P}_1 \times \mathscr{P}_2, \mathscr{L}_V \cup \mathscr{L}_H)$ where $\mathscr{L}_V$ and the $\mathscr{L}_H$ are respectively all the vertical and horizontal lines. It is usually not necessary to mention these sets explicitly since they are completely determined by the quartette $(\mathscr{P}_1, \mathscr{L}_1, \mathscr{P}_2, \mathscr{L}_2)$.

Suppose now, S_1 and S_2 are subspaces of a point-line geometry $\Gamma = (\mathscr{P}, \mathscr{L})$. If we write $\Gamma = S_1 \times S_2$ we intend to assert that Γ is isomorphic to the product geometry $S_1 \times S_2$ where the symbol S_i is interpreted to be the point-line geometry $(S_i, \mathscr{L}(S_i))$

where $\mathscr{L}(S_i)$ denotes those lines of $\mathscr{L}$ whose point-shadow lies entirely inside the subspace S_i, $i = 1, 2$.

3.3 Point residuals. Suppose $\Gamma = (\mathscr{P}, \mathscr{L})$ is a gamma space all of whose singular subspaces are projective spaces. Then, of course, Γ is a partial linear space. Given a point p, let $\mathscr{L}_p$ and Π_p be the collections of all lines and all projective planes, respectively, which are incident with point p. We say that $L \in \mathscr{L}_p$ is incident with $\pi \in \Pi_p$ if and only if $L \subseteq \pi$. With respect to this incidence relation, the geometry $\mathrm{Res}(p) = (\mathscr{L}_p, \Pi_p)$ is a also a gamma space of "points" and "lines" whose singular subspaces are projective spaces. The geometry $\mathrm{Res}(p)$ is called the *residue at point* p, or more generally, a *point-residue*.

3.4 Symplecta. A *non-degenerate polar space* is a point-line geometry $(\mathscr{P}, \mathscr{L})$ with these properties: (1) all lines are thick, (2) no point is collinear with all remaining points, and (3) given a point p and a line L not incident with p, p is collinear with exactly one or collinear with all of the points of the point-shadow of L. Obviously property (3) makes polar spaces a species of gamma space. If it never happens that some point p is collinear with more than one point of a line not incident with it, then the polar space is called a *non-degenerate polar space of rank* 2 or a *generalized quadrangle with thick lines*. For the rest of this paper, we use the term *polar space* to mean a non-degenerate polar space, and the term *generalized quadrangle* (or just "quadrangle" if the context is clear) to mean generalized quadrangle with thick lines. A generalized quadrangle in which each point is on just two lines is called a *grid*. Such a quadrangle is a product $L \times N$ of two lines.

It is well known that if a polar space is not a generalized quadrangle, then it is a partial linear gamma space whose singular subspaces are all projective spaces. If one of these maximal singular subspaces has finite projective rank $d \geqslant 2$, then all of its maximal singular subspaces possess the same finite projective rank d, and we say that *the polar space has rank* $d + 1$. Natural case-divisions make it convenient to distinguish those rank d polar spaces in which any second-maximal singular subspace lies in just two maximal singular subspaces. We call these *oriflamme polar spaces*.

If there exists a maximal singular subspace of infinite projective rank, then all maximal singular subspaces have infinite projective rank, but these ranks need not be the same cardinality. In this case we simply say that *the polar space has infinite rank*, and no particular infinite cardinal is specified as the rank.

A convex subspace S of Γ such that S, together with the lines contained in it forms a polar space, is called a *symplecton*. For example, if $\Gamma = A \times B$, where A and B are singular subspaces of Γ, then the convex closure $\langle\langle x, y\rangle\rangle$ of two points x and y which are not collinear is a convex subspace that is a grid, and hence is a symplecton.

A fundamental fact that we shall invoke many times is this:

If S is a symplecton of the gamma space $\Gamma = (\mathscr{P}, \mathscr{L})$, and x is a point not in S, then $x^{\perp} \cap S$ is a singular subspace.

3.5 Parapolar spaces. In this paper we adopt a definition of "parapolar space" equivalent to that introduced in A. Cohen's survey article in the *Handbook for Incidence Geometry* [7]. The reader should be warned that this definition is more general than that given in the literature preceeding the *Handbook*.[1]

A *parapolar space* is a connected point-line geometry with these properties:

(1) If $\{x, y\}$ is a non-collinear pair of distinct points, then either
  (a) $x^{\perp} \cap y^{\perp} = \varnothing$,
  (b) $|x^{\perp} \cap y^{\perp}| = 1$, or
  (c) $\langle\langle x, y\rangle\rangle$ is a symplecton.
  (In this case the pair (x, y) is called a *polar pair*.)

(2) Every line lies in at least one symplecton.

It easily follows that a parapolar space is a partial linear gamma space with every 4-circuit in a unique symplecton—which together with (2) is the definition of [7, page 688].

A parapolar space Γ is said to have *symplectic rank* (*at least*) k if and only if every symplecton of Γ has rank (at least) k as a polar space.[2] Some perfectly natural parapolar spaces (for example the Lie incidence geometries of type $C_{n,2}$, $n \geqslant 5$) possess symplecta of two different polar ranks.

A standard result is the following:

Lemma 3. *Suppose* Γ *is a parapolar space of symplectic rank at least three. Then the following holds.*

1. *Any singular subspace generated by a point and a line lies in a symplecton.*
2. *All singular subspaces of* Γ *are projective spaces.*

Proof. Suppose $A = \langle p, L\rangle_{\Gamma}$ is a singular subspace of Γ generated by a point p and a line L. We must show that A lies in a symplecton. By property (2) of a parapolar space, their exists a symplecton S containing L. If S contains p we are done, so we may assume p is not in S and so $p^{\perp} \cap S$ is a singular subspace of S. On the other hand, since S is a polar space of rank at least three, $L^{\perp} \cap S$ is not a singular subspace. Thus there is a point $x \in L^{\perp} \cap S - p^{\perp} \cap S$. Then $\{p, x\}$ is a pair of non-collinear points with a line L in $p^{\perp} \cap x^{\perp}$. Thus $R := \langle\langle p, x\rangle\rangle$ is a symplecton containing p and L and so contains A. So part 1 is proved.

Part 2 follows from the fact that all singular subspaces of a polar space are projective spaces—in particular the subspace A of the previous paragraph is a projective plane. From the general choice of p and L it follows that all singular subspaces are projective spaces.

[1] The shift in definitions can be justified on utilitarian grounds. Under the old definition, there was no real name for the geometry of the point-residual of a parapolar space with symplecta of rank three. This was a disadvantage since (like the current paper) most of the arguments take place at this level.

[2] In [7] the symplectic rank is called the *polar rank*, which to many students is the rank of a polar space, not a parapolar space.

The conclusions of Lemma 3 fail dramatically when there are symplecta of rank two—that is convex subspaces which are generalized quadrangles. For example, if A and B are two arbitrary linear spaces, each containing at least a line, then the product geometry $A \times B$ is a parapolar space of symplectic rank two with singular subspaces isomorphic to A and to B.

A parapolar space is called a *strong parapolar space* if the alternative 1(c) of the definition of parapolar space never occurs—that is, *every* pair of points at distance 2 in the collinearity graph is covered by a symplecton.

A particular example of a strong parapolar space of rank exactly three is the object being characterized by the Theorem 2 at the beginning of this paper. Let V be any right vector space over a division ring D, and let d be a positive integer properly bounded by the dimension of V, if the latter is finite. Let V_d be the full collection of all d-dimensional subspaces of V. Let $V_{d-1,d+1}$ be the full collection of pairs (A,B) where A and B are subspaces of dimensions $d-1$ and $d+1$ respectively and $A \subseteq B$. We say that a d-space C is *incident* with such a pair (A,B) if and only if $A \subseteq C \subseteq B$. Then the point-line geometry $\Gamma = (V_d, V_{d-1,d+1})$ subject to the described incidence is called the *Grassmannian of d-subspaces of V* and is denoted $A_d(V)$, or, if V has finite dimension $n+1$, it is denoted $A_{n,d}$ or $A_{n,d}(D)$ if the division ring D needs to be emphasized. When $d = 1$ it is a projective space. If $1 < d < \dim V - 1$, the Grassmannian is a strong parapolar space of symplectic rank three.

3.6 Two cited results. The following Lemma is due to A. Cohen. Although it might be described as a technical result, it is a very important one for virtually every purely local characterization of the classical Lie incidence geometries ultimately depends on this lemma.

The polar spaces of rank three were completely classified by J. Tits in [17]. Any generalized quadrangle that is isomorphic to a point residue in such a polar space is said to be a *classical quadrangle*.

Lemma 4. *Suppose $\Gamma = (\mathscr{P}, \mathscr{L})$ is a generalized quadrangle whose point-set is partitioned by a subcollection $\mathscr{S}$ of the line set (such a collection of lines $\mathscr{S}$ is usually called a* line spread*) with these properties*:

1. *Given any two distinct lines L and N of the spread, the subspace $\langle L, N\rangle_\Gamma$ that they generate is a grid, with all members $G(L,N)$ of the parallel class of the grid containing L and N belonging to the line spread $\mathscr{S}$.*

2. *Let $\mathscr{G}$ be the collection of all subsets of $\mathscr{S}$ of the form $G(L,N)$ for distinct lines L and N belonging to $\mathscr{S}$. Then $(\mathscr{S}, \mathscr{G})$ is a projective plane.*

Then Γ is not a classical generalized quadrangle.

This Lemma is proved in Cohen's fundamental paper [6].

We also require the following characterization of the Grassmannian of d-subspaces of a vector space.

Theorem 5. (Shult [14, Theorem 6.1, pp 173–4], Bichara and Tallini [1]) *Suppose* Γ *is a strong parapolar space of symplectic rank three. Suppose the full collection* $\mathscr{M}^*$ *of all maximal singular subspaces of* Γ *is partitioned into two subcollections,* $\mathscr{M}^* = \Pi + \Sigma$ *with these properties*:

1. *Any subspace in* Π *intersects any subspace of* Σ *in a line or the empty set.*
2. *Every line lies in exactly one member of* Π *and exactly one member of* Σ.
3. *Some singular subspace in* Π *contains a finite unrefinable chain of subspaces.*

Then Γ *is the Grassmannian* $A_d(V)$ *of d-subspaces of a vector space V over some division ring, where d is finite, but the dimension of V need not be.*

Proof. This is a corollary of Theorem 6.1 of [14, page 173]. We verify the hypotheses of that theorem. Condition (T1) follows from the fact that Γ is a gamma space. Conditions (T2)(i)–(ii) are simply hypotheses 1. and 2. given above. The intersection property (T3) is an easy consequence of the parapolar hypothesis and the fact that symplecta with hypothesis 2 are of type D_3 and any two maximal singular subspaces of a given class of such a symplecton always meet at a point. Finally, condition (T4) restates condition 3 above. The conclusion now follows from the cited Theorem 6.1.[3]

Corollary 6. *Suppose* Γ *is a strong parapolar space each of whose point-residuals* $Res_\Gamma(p)$ *is a product geometry* $A_p \times B_p$ *where* A_p *and* B_p *are projective spaces of positive rank, at least one of which has finite rank. Then one of the following holds*:

1. $\Gamma \simeq A_d(V)$ *for some vector space V and integer* $d > 2$.
2. Γ *is the quotient geometry* $A_{2n-1,n}/\langle\sigma\rangle$ *where* σ *is induced by a polarity of* $\mathbf{P}(V) = PG_{2n-1,1}(D)$ *of Witt index at most* $n-5$.

Remark. This would really be a Corollary of the beautiful theorem of A. Cohen [6] were it not for the small detail that the latter requires Γ to have finite singular rank—that is, *all* singular subspaces are projective spaces of finite rank (see [7, 6.3, p. 718]). We avoid this by using the preceding theorem where finiteness is required of only some maximal singular subspaces. The proof uses Cohen's two-fold covering construction [6].

Proof. Here, every point-residue $\mathrm{Res}_\Gamma(p)$ is a product geometry $A_p \times B_p$, where A_p and B_p are two distinct maximal singular subspaces of Γ which intersect at a line on p. In this case there are two classes of maximal singular subspaces $\mathscr{A}_p$ and $\mathscr{B}_p$ which contain p. Members of the same class pairwise intersect at exactly $\{p\}$, while members of different classes intersect at some line on p.

Thus each line $L \in \mathscr{L}$ lies in exactly two maximal singular subspaces A_L and B_L and at least one of these two is a member of $\mathscr{M}$, and so has finite projective rank.

Following the construction in Cohen [6], we now form a new geometry $\hat{\Gamma} = (\hat{\mathscr{P}}, \hat{\mathscr{L}})$,

[3] Note that the parameter n appearing in $A_{n,d}$ in the conclusion of Theorem 6.1 is any cardinal number not exceeded by d; there is no assumption there that it is finite.

where $\hat{\mathscr{P}}$ is the collection of all pairs $(p, \mathscr{X}_p)$ where $\mathscr{X}_p$ is either $\mathscr{A}_p$ or $\mathscr{B}_p$—that is, one of the two classes of maximal singular subspaces on p. For each line L of $\mathscr{L}$ let A_L and B_L be the two maximal singular subspaces containing L. Let $\hat{\mathscr{L}}$ be the set of all pairs (L, X_L) where $X_L \in \{A_L, B_L\}$. The pair $(p, \mathscr{X}_p)$ is said to be incident with (L, X_L) if and only if (1) p is incident with L in Γ and (2) $X_L \in \mathscr{X}_p$. There is an obvious geometry morphism $f : \hat{\Gamma} \to \Gamma$ which takes pair $(p, \mathscr{X}_p)$ to point p and takes pair (L, X_L) to line L. The point-mappings and line-mappings are both onto and all fibers have cardinality two.

Now each line $\hat{L} = (L, A_L)$ of $\hat{\Gamma}$ is the intersection of two maximal singular subspaces: (1) The first is the set $M_1(\hat{L}) := \{(x, \mathscr{X}_x) \mid x \in A_L, A_L \in \mathscr{X}_x\}$. (2) The second is $M_2(\hat{L}) := \{(x, \mathscr{X}_x) \mid x \in B_L, B_L \notin \mathscr{X}_x\}$. The intersection of these is $\hat{L}$ and the two spaces comprise all maximal singular spaces of $\hat{\Gamma}$ containing $\hat{L}$. Let $\mathscr{M}_i := \{M_i(\hat{L}) \mid \hat{L} \in \hat{\mathscr{L}}\}$ for $i = 1, 2$. As is evident from their definition, the two classes contain no space in common. In fact the two collections $\mathscr{M}_1$ and $\mathscr{M}_2$ obey the hypotheses on Σ and Π of Theorem 5, the intersection property resulting from the fact that the symplecta of Γ lift to $\hat{\Gamma}$ to make any connected component of the latter a strong parapolar space. (These details are in Cohen's fundamental paper [6].) It now follows from Theorem 5 that any connected component of $\hat{G}$ is isomorphic to the Grassmannian $A_d(V)$ of d-spaces of a vector space V.

If the geometry $\hat{\Gamma}$ is not connected, then each fiber of a point or line contains one object of each component geometry, and so the restriction of the morphism $f : \hat{\Gamma} \to \Gamma$ to one of the two connected components produces an isomorphism $\Gamma \simeq A_d(V)$. On the other hand, if Γ is connected then all maximal singular subspaces have the same projective dimension, say $n - 1$, and $f : \hat{\Gamma} = A_n(V) \to \Gamma$ is a two-fold covering defined by a deck-transformation σ of degree 2 exchanging the two classes $\mathscr{M}_1$ and $\mathscr{M}_2$. Thus σ is induced by a polarity. The condition that σ have Witt index at most $n - 5$ results from the fact that the image Γ is a parapolar space.

4 Immediate consequences of the Hypothesis (D)

Lemma 7. *The following statements hold*:

1. *If S is a symplecton and $M \in \mathscr{M}$ meets S in at least a line*, *then $S \cap M$ is a maximal singular subspace of S and is a plane*.

2. *Γ is a strong parapolar space of symplectic rank exactly three*.

Proof. Before proving the rest, let us first see why Γ is a strong parapolar space. Suppose x and y are a pair of non-collinear points collinear with a common point v. By hypothesis 2(a) there is a maximal singular subspace $M \in \mathscr{M}$ containing the line xv. Now $y^\perp \cap M$ contains the point v and so by hypothesis 2(b) $y^\perp \cap M$ contains a line A. But as M is singular, $x^\perp \cap y^\perp$ contains the line A, and so (x, y) is a polar pair. Thus all distance two point-pairs of $\mathscr{P}$ are polar pairs, whence Γ is a strong parapolar space.

Suppose S is a symplecton of Γ, L is a line in S, and M is a member of $\mathscr{M}$ containing L (such an M exists by hypothesis). Then $S \cap M$ is a singular subspace of the polar space S and so, for a point $s \in S - L^{\perp}$, $s^{\perp} \cap (S \cap M) = H$ is a hyperplane of $S \cap M$ not containing L. Since L is a line and H is a hyperplane of the same singular space, $H \cap L$ is non-empty.

Now as M is a singular subspace, $s^{\perp} \cap M - S \subseteq L^{\perp} \cap s^{\perp} \subseteq S$ (the second containment follows from the convexity of S). This asserts that a set disjoint from S is contained in S, and so the former is empty. Thus

$$s^{\perp} \cap M = s^{\perp} \cap (S \cap M) = H.$$

Since this is a non-empty set, hypothesis 2(b) forces the left-most term to be exactly a line. Thus $S \cap M$ is a plane.

If $S \cap M$ were not a maximal singular subspace of S then we could find a point in $(S \cap M)^{\perp} \cap S - M$ and that would also contradict part 2 (b) of the Hypothesis. Thus we see that $M \cap S$ is a plane as well as a maximal singular subspace of S, so all parts of the Lemma have been proved.

Lemma 8. *Suppose P is a plane and M is an element of $\mathcal{M}$ which meets P in exactly a point p. Suppose $P^{\perp} \cap M$ is not a line. Then there is a plane Q of M on p and a bijection*

$$\lambda : (\mathcal{L}(P))_p \to (\mathcal{L}(Q))_p,$$

from the lines of P on point p to the lines of Q on point p taking each line L to the line $L^{\perp} \cap M$.

Proof. We know how λ is defined. Suppose $P^{\perp} \cap M$ is not a line. Then of course it is just the point p. Thus for two distinct lines L and N belonging to the plane P and meeting at point p, $L^{\perp} \cap M := L'$ and $N^{\perp} \cap M := N'$ are distinct. Moreover, the intersection of the perps of L' and N share the line L and so their convex closure $S := \langle\langle N, L' \rangle\rangle$ is a symplecton which contains plane P and by Lemma 7 meets M at a plane Q containing line L'. Since P and Q are two planes of a non-degenerate rank three polar space S, we see that the lines of P on p are mapped bijectively onto the lines of Q on point p.

5 The Hypothesis (E)

Remark. This section, as well as Sections 6, 7, 8, 9, and 10 which follow, all assume hypothesis (E).

Lemma 9. *The following are easy consequences of Hypothesis* (E):

1. *Every symplecton meets a member of $\mathcal{M}$ at the empty set, or at a line.*
2. *If L is any line disjoint from $M \in \mathcal{M}$, then the mapping $L \to M$ induced by the projection mapping, has as its image, either a single point, or a line of M.*

3. *Every symplecton is a generalized quadrangle, consequently* Γ *is a strong parapolar space of symplectic rank exactly two.*

Proof. Of course, this is just a re-hash of Lemma 7, localized at a point, but lets prove it from (E) alone. Suppose the symplecton S intersected the subspace M of $\mathscr{M}$ non-trivially. If $S \cap M$ is not a maximal singular subspace of S, then there exists a point $p \in (S \cap M)^{\perp} - S$, at which point assumption E2(b) forces $S \cap M$ to contain exactly one point, say m. Choose $y \in S - m^{\perp}$. Again by part E2(b), y is collinear with a unique point m_y of M. Then $m_y \in m^{\perp} \cap y^{\perp} \subseteq S$. But that contradicts $S \cap M$ being a point.

So we may assume that $S \cap M$ is a maximal singular subspace of S. Then choosing $x \in S - M$, we see that E2(b) implies $x^{\perp} \cap (S \cap M)$ is a point. But the latter is a hyperplane of $S \cap M$. Thus the maximal singular subspace $S \cap M$ of S is a line, and so S is a generalized quadrangle.

Now this argument works for any symplecton S since any of its points lies in an element of $\mathscr{M}$. Parts 1 and 3 of the Lemma have been proved.

Suppose L is a line disjoint from a member of M and let $f : L \to M$ be the mapping which maps each point of L to the unique point of M with which it is collinear. Choose $x \in L$ and let R be the unique symplecton containing $f(x)$ and L. Then by Part 1, R meets M at a line L' opposite L in R. Then $L' = f(L)$, so $f(L)$ is a line as required.

6 Fibred symplecta

We assume hypotheses (E). We suppose $M \in \mathscr{M}$ and that S is a symplecton sharing no point with M. Then the restriction of the projection mapping onto M produces a mapping $f : S \to M$ which takes a line of S either to a line of M, or to a point of M. Let $f(S)$ be the collection of all image points—that is, $f(S) = \{m \in M \mid m^{\perp} \cap S \neq \emptyset\}$.

Lemma 10. *Suppose* $m \in f(S)$. *Then its fibre* $f^{-1}(m)$ *is either a single point or is a line of* S.

Proof. By convexity of S, the fibre $f^{-1}(m)$ is a singular subspace of S. Since S is a generalized quadrangle, this fibre is either a point or is a line.

Lemma 11. *Let* F *be the set of points in* M *whose fibres are lines of* S. *Then* F *is a subspace of* M. *Moreover, if* E *is a line contained in* F, *then* $f^{-1}(E)$ *is a subquadrangle of* S *which is a grid. Then fibers of the points of* E *form one of the two line-spreads of this grid.*

Proof. Suppose a' and b' are two distinct points of F. Let A and B be their respective fibres. Then A and B are opposite lines of S. Then for each point x of A, there is a unique line T_x on x which intersects B at the unique point $b(x)$ of $x^{\perp} \cap B$. Then this transverse line maps by f to the unique line $E = a'b'$ of M. Thus, for each point $m \in L'$, the fibre $D(m) := f^{-1}(m)$ must intersect each transverse line T_x. This gives $L' \subseteq F$.

On the other hand if $z \in D(m)$, then either $D(m) = A$ and $z \in T_z$, or $D(m)$ is a line opposite A, so, if $\{x\} = z^{\perp} \cap A$, then $x^{\perp} \cap D(m)$ contains both z and the unique point of $T_x \cap D(m)$ while being a singleton set. This forces $z \in T_x$. Thus the fibre $f^{-1}(m)$ cannot contain a point not in $\bigcup_{x \in A} T_x$.

Thus we see that the fibres of points of L' form a line-spread of $G := \bigcup_{x \in A} T_x$ while the system $\{T_x \mid x \in A\}$ of transverse lines for A and B form another such system. Since the lines of each system are pairwise opposite, no further collinearities exist among the points. It follows that G is a subspace of S which is a grid with these two systems of line-spreads.

6.1 A particular situation. We suppose f is not injective. Then there are distinct points a and b of S collinear with a common point $f(a) = f(b) = a'$. Since $f^{-1}(a') := a'^{\perp} \cap S$ is a clique with at least two points, and since lines are maximal singular subspaces of S, we see that the fibre $f^{-1}(a')$ is a line A. Then any other line L on point a is mapped bijectively to a line $f(L) := L'$ of M. Next suppose B is a line meeting L at a point b distinct from A, chosen so that its image $f(B) := B'$ is also a line of M. Then B' meets L' at the point $b' = f(b)$. Then $\langle L', B' \rangle_M$ is a projective plane π.

Now A and B are opposite lines of the symplecton S. Thus each point a_i of line A is collinear with a unique point b_i of B, and we denote the full collection of lines $\{T_i := a_i b_i\}$ by $\mathscr{T}$ and call them the *transversal lines for A and B*. It is clear then that these transversal lines are mapped onto the full pencil of lines of π on point a', so in fact $\pi \subseteq f(S)$. This situation is illustrated in Figure 1.

Let $R := \bigcup \{f^{-1}(p) \mid p \in \pi\}$. Clearly R is a subspace of S since π is a subspace of M. Select any point u on line L so that u is distinct from both a and b. Let T_i be a transversal from A to B which is distinct from (and hence opposite to) L. Then u is collinear with a unique point v of T_i and v is not in B. It follows that line uv is opposite line B.

Now $f(u)$ and $f(v)$ are points of $\pi - (B' \cup \{a'\})$, on different members of the line pencil on a'. Then f maps the line uv of S to the line $f(u)f(v)$ of π. Since π is a

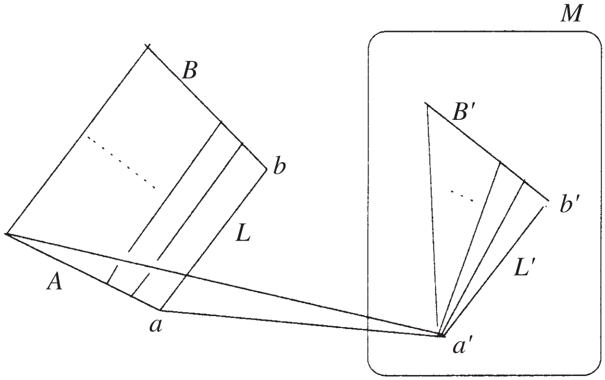

Figure 1. A special situation.

projective plane, $f(u)f(v)$ meets line $B' = f(B)$ at a point $f(b_i)$ for some $b_i \in B$. This means uv contains a point w with $f(w) = f(b_i)$. Since uv and B are opposite lines, $w \neq b_i$ and so are distinct points belonging to the fibre $C := f^{-1}(f(b_i))$. By Lemma 11 above, $G = f^{-1}(a'f(b_i)) = \langle A, C \rangle_S$ is a grid.

Now R contains $\langle A, B \rangle_S$ and so properly contains the grid G. Then any point of G lies on a line of R which is not in G, and these lines map onto lines of π. Let us choose distinct points x and y on a line T transversal to two of the line-fibres of G. Let L_x and L_y be lines of R which lie on x and y, respectively, but are not lines of G. $L_x \cap L_y = \emptyset$ since any point of their intersection would lie outside T while being collinear with distinct points x and y of T. Then $f(x) \neq f(y)$ and $f(L_x)$ and $f(L_y)$ are lines which intersect $L' = a'f(b_i)$ at distinct points $f(x)$ and $f(y)$ and so intersect each other at a point $p \in \pi - L'$. But that means $f^{-1}(p)$ contains a point of L_x and a point of L_y and since these are disjoint lines we see that $p \in F \cap \pi$.

It now follows from Lemmas 10 and 11 that R is a generalized quadrangle containing a line spread $\mathscr{F}$ (the fibres of $f : R \to \pi$), any two of which generate a grid, and, letting $\mathscr{G}$ be the collection of grids formed in this way, the incidence system $(\mathscr{F}, \mathscr{G})$ is isomorphic to $(\mathscr{P}_\pi, \mathscr{L}_\pi)$, the points and lines of the projective plane π.

By E(4) S is a classical quadrangle. Now Lemma 4 of Arjeh Cohen [6] shows that this is impossible.

We have proved the following:

Theorem 12. *Suppose S is a symplecton disjoint from a subspace M of $\mathscr{M}$. Then either*

1. *the projection into M induces a projective embedding $f : S \to M$, or*
2. *S is a grid and the projection on M induces a mapping $f : S \to M$ onto a line L of M. The fibres of the points of L form one of the line-spreads of the grid S.*

Corollary 13. *Suppose S is a symplecton which is not a grid. Suppose π is a plane which meets S at a line. Then for any point $x \in \pi - S$, x lies in a unique member M_x of $\mathscr{M}$ and the plane π itself lies in M_x.*

Proof. Suppose $\pi \cap S$ is a line L and choose x in $\pi - L$. Suppose x belongs to a singular subspace M' of $\mathscr{M}$ which does not contain π. Then $\pi \cap M' = \{x\}$. Also, convexity of S forces $M' \cap S = \emptyset$. But now projection on M' forces a mapping $f : S \to M'$ which possesses a non-trivial fibre. By the Theorem 12, S is a grid, an absurdity. So no such M' exists.

But by hypothesis x lies in some member M_x of $\mathscr{M}$, and so we see that we must have $\pi \subseteq M_x$ for any such singular space. This fact forces the uniqueness of M_x.

A line is called an *$\mathscr{M}$-line* if it is contained in one of the singular subspaces of $\mathscr{M}$. By axiom E2(a), the $\mathscr{M}$-lines cover all the points.

Corollary 14. *Suppose S is a symplecton which is not a grid. Then every line of S is either an $\mathscr{M}$-line or is already a maximal singular subspace of Γ.*

Proof. This is immediate from Corollary 13.

7 Unfibered symplecta

7.1 M-projections which embedd a symplecton. Let S be a symplecton. We let $N^2(S)$ denote the set of points $x \in \mathscr{P} - S$ for which $x^\perp \cap S$ is a line. Similarly we write $N^1(S)$ for the set of points $x \in \mathscr{P} - S$ for which $x^\perp \cap S$ is a single point, and finally write $N^0(S)$ for the set of points $x \in \mathscr{P} - S$ for which $x^\perp \cap S = \emptyset$. Then we have the following partition of points:

$$\mathscr{P} = S + N^2(S) + N^1(S) + N^0(S).$$

The conclusion of Corollary 13 motivates another definition. The set U of all points $p \in \mathscr{P}$ which lie in a unique member of $\mathscr{M}$ will be called *the uniqueness set*.

Theorem 15. *Suppose $M \in \mathscr{M}$ and S is a symplecton which shares no point with M. Suppose further that the projection onto M induces an embedding $f : S \to M$ as in the first case of Theorem* 12. *Choose any pair (x, y) of non-collinear points of S, let M_y be a member $\mathscr{M}$ containing y, let L_y be the line $M_y \cap S$ and let t be the unique point of $x^\perp \cap L_y$. Let R be the unique symplecton containing t and $f(x)$. Then R is not a grid.*

The following uniqueness results hold:

1. *The subspace M_y is the unique element of $\mathscr{M}$ containing the point y.*
2. *Since y is arbitrarily chosen in S, we see that every point of S lies in a unique member of $\mathscr{M}$.*
3. *Every point of M lies in a unique member of $\mathscr{M}$.*
4. *Every line of S is either an $\mathscr{M}$-line or is itelf a maximal singular space.*

Proof. Let x, y, M_y, L_y and t be as chosen. Then $f(x)$ is the unique point of M collinear with x. We have $M \cap M_y = \emptyset$ since otherwise $f(L_y)$ is a single point, against f being an embedding. Then $f(x)$ is collinear to a unique vertex z in $M_y - L_y$. Now $(x, f(x), z, t, x)$ is a 4-circuit lying in a symplecton R. Now R meets S at line xt and R meets M_y at line tz. Also, since $S \cap M$ is non-empty, it too is a line—in fact it is the line $f(x)f(t)$. The configuration is illustrated in Figure 2.

Now we see that the point $f(x)$ sits on three distinct lines of R: $f(x)x$, $f(x)z$, and $f(x)f(t)$. Thus R cannot be a grid.

Now we note that $y \in M_y - R$ and M_y meets R at a line. Thus by Corollary 13 y lies in a unique member of $\mathscr{M}$.

Similarly every point of $M - (M \cap R)$ lies in a unique member of M. But R is (now, uniquely) determined by the choice of (x, y). But if we choose $u \in S \cap y^\perp - t^\perp$, then replacement of the pair (x, y) by (u, t) in the construction produces a new symplecton R' which is not a grid, and which meets M at the line $f(u)f(y)$. Note that line uy is opposite line xt in S and so $f(t)f(x)$ and $f(u)f(y)$ are disjoint lines in M since $f : S \to M$ is an embedding. But as before all points of line $f(z)f(t)$ lie in a unique member of $\mathscr{M}$, so all points of M have this uniqueness property.

There is more. Let x, y, u, t be as in the previous paragraph. Now $M_y \cap R = tz$ and $M_y \cap R' = yz'$ where z' is the unique point of M_y collinear with $f(u)$. These lines are

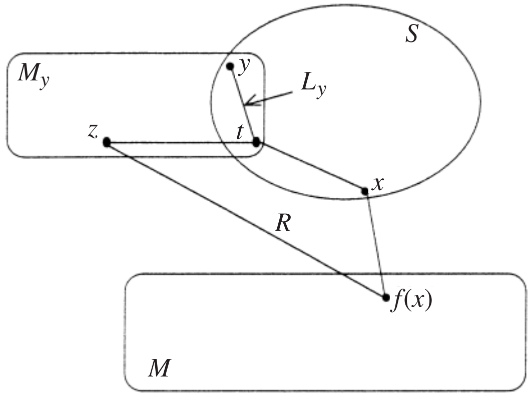

Figure 2. The configuration of Theorem 15.

disjoint since $f(u)f(y)$ and $f(t)f(x)$ are disjoint lines of M (as observed in the previous paragraph) and the perpendicular relation produces an isomorphism $M_y \to M$. But since the points outside either of these lines are uniqueness points, we have $M_y \subseteq U$.

Finally, suppose π is a plane not in an element of $\mathscr{M}$ meeting S at a line N. Then N is not an $\mathscr{M}$-line. Choose distinct points x and t of line N and let M_t be a member $\mathscr{M}$ on point t. Then $M_t \cap S$ is a line L_t on t. Clearly $L_t \neq N$. Now choose y in L_t distinct from t. Then y is not collinear with x. If we rename things, writing M_y for M_t and L_y for L_t we have exactly the construction of the symplecton R at the beginning of the Theorem. Now R is not a grid, but sits on the line $N = xt$ of the plane π. This is contrary to Corollary 5a. Thus no such π exists. It follows that all lines of S which are not $\mathscr{M}$-lines are already maximal singular subspaces. The proof is complete.

Corollary 16. *Suppose the symplecton S is not a grid and $M \cap S = \varnothing$, for some $M \in \mathscr{M}$. Then every point of $\mathscr{P}$ lies in a unique member of $\mathscr{M}$.*

Proof. Let U be the collection of uniqueness points—those points which lie in a unique member of $\mathscr{M}$. Our objective is to prove that $U = \mathscr{P}$.

By Theorem 12, since S is not a grid, the mapping $f : S \to M$ induced by the projection into M is an embedding. By Theorem 15 part 2, $S \subseteq U$. By Corollary 14, since S is not a grid, every point of $N^2(U)$ is a uniqueness point. Now suppose $y \in N^1(S) \cup N^0(S)$. Then by our hypothesis, y lies in a singular subspace $M_y \in \mathscr{M}$. Then $M_y \cap M = \varnothing$. Then, as S is not a grid, one obtains an embedding $f_y : S \to M_y$. Upon replacing M by M_y in Theorem 15 part 3, we see that $M_y \subseteq U$ and in particular $y \in U$.

The discoveries of the previous paragraph can be summarized by asserting

$$\mathscr{P} = S + N^2(S) + N^1(S) + N^0(S) \subseteq U,$$

which we were to prove.

8 The case that some member of $\mathcal{M}$ is a line

Theorem 17. *If a line L is a member of $\mathcal{M}$, then one of the following holds*:

1. Γ *is itself a generalized quadrangle.*
2. Γ *is a product geometry $L \times P$, where P is a maximal singular subspace. Every symplecton of* Γ *is a grid.*

Proof. Suppose L is a line in M. Since Γ is a strong parapolar space of rank two, L lies in a symplecton S. If $S = \mathcal{P}$, the first conclusion holds and we are done. So we assume $S \neq \mathcal{P}$.

Choose $y \in \mathcal{P} - S$. Then $y^{\perp} \cap L$ is a single point, say p. We claim that in S, there is only one further line on p besides L and that line is $y^{\perp} \cap S$. If not there would be a line N of S on p not in $y^{\perp}$. We could then form the symplecton $R := \langle\langle y, N \rangle\rangle$ and choose a point $z \in R - p^{\perp}$. Then z is collinear with a point q of L distinct from p. Then $q \subseteq p^{\perp} \cap z^{\perp} \subseteq R$, and so lies in $R \cap L = R \cap S \cap L = N \cap L = \{p\}$, an absurdity.

Thus, for each $y \in p^{\perp} - S$ we have $y^{\perp} \cap S = N$, for any line N in S which contains p and is distinct from L. Since $y^{\perp} \cap S$ is a clique, the line N is unique. Thus S is a grid. But then we have $p^{\perp} - S \subseteq N^{\perp}$, which must be a singular subspace (otherwise Γ would not have symplectic rank two). Thus, for any p of the line L, we have

$$p^{\perp} = L \cup A_p, \quad \text{and} \quad L \cap A_p = \{p\},$$

where $A_p = N_p^{\perp}$ and N_p is the unique line of the grid S on p such that N_p is distinct from the line L. Clearly each A_p is a singular subspace which is not a line. Moreover we have a partition into maximal singular subspaces:

$$\mathcal{P} = \biguplus \{A_p \mid p \in L\}. \tag{1}$$

Now choose any point x in $\mathcal{P}$. If $x \in L$ set $p(x) = x$ and note that L is the unique line on x not in $A_{p(x)}$. If $x \in \mathcal{P} - L$, let $p(x)$ be the unique point of $x^{\perp} \cap L$. Then $x \in A_{p(x)}$. We claim that there is a unique line L_x on x which is not in $A_{p(x)}$. First there is at least one such line, since the symplecton $R_x := \langle\langle xp(x), L \rangle\rangle$ contains one. On the other hand, if L' were such a line, then the symplecton $\langle\langle xp(x), L' \rangle\rangle$ would intersect L non-trivially and hence would contain L (Lemma 9, part 1), forcing it to coincide with the R_x. But R_x satisfies the hypothesis that we had for S above, and so R_x is a grid. That means there is only one line L_x on x which does not lie in the singular space $A_{p(x)}$, and that line is opposite L.

Thus all lines of Γ which are not in one of the singular subspaces A_p form a spread of lines transverse to the components A_p of the partition in equation (1). Every symplecton on such a transverse line is a grid with its intersections with the A_p forming a spread. It follows that for any two distinct points x and y of the line L, the point-bijection $A_x \to A_y$ induced by the system of transverse lines takes lines of A_x

to lines A_y. This is the last step needed to conclude that Γ is the product geometry $L \times A_p$.

9 Symplecta disjoint from no member of $\mathscr{M}$

Theorem 18. *Suppose S is a symplecton that is not disjoint from any singular subspace of $\mathscr{M}$. Then either* (a) *Γ is a generalized quadrangle*, (b) *S is a grid, or* (c) *all points are uniqueness points.*

Proof. We suppose that S is a symplecton disjoint from no member of $\mathscr{M}$. By way of contradiction we assume that S is not a grid and that M_1 and M_2 are two distinct members of $\mathscr{M}$ which meet at a point p in the symplecton S.

If either M_i were a line, we could apply Theorem 17 to conclude that Γ is a generalized quadrangle, or that $\Gamma = L \times P$ for some singular subspace P. But in the latter case S would be a grid, contrary to assumption. So Γ is a generalized quadrangle, which is one of our conclusions. Thus we may assume that neither M_1 nor M_2 are lines.

For $i = 1, 2$, let $L_i := S \cap M_i$ and suppose N_i is a line of M_i on p distinct from the line L_i (this is possible since M_i is not a line). Let $R := \langle\langle N_1, N_2 \rangle\rangle$, the unique symplecton on the N_i. Choose a point $z \in R - p^{\perp}$ and let M_z be a member of $\mathscr{M}$ on point z. Now by our hypothesis, M_z cannot be disjoint from S, and so $M_z \cap S$ is an $\mathscr{M}$-line A_z of S which is not no point p. Then the unique point a_z of $p^{\perp} \cap A_z$ is collinear with both p and z and so belongs to R—that is $R \cap S = pa_z$. Also a_z is on neither line L_i since the unique point m_i of $z^{\perp} \cap M_i$ lies in $N_i - \{p\}$, $i = 1, 2$. Thus the three lines za_z, zm_1 and zm_2 on point z are all distinct and belong to R. It follows that R is not a grid.

Now choose any point $w \in S - p^{\perp}$ and let M_w be a member of $\mathscr{M}$ containing point w. If $M_w \cap R = \varnothing$, then Corollary 16 (applied with the non-grid R and singular space M_y replacing S and M respectively) would not allow M_1 and M_2 to intersect at p. Thus we must conclude that $M_w \cap R$ is a line B of R not containing p. Then the point t of $p^{\perp} \cap B$ lies in R and also lies in $p^{\perp} \cap w^{\perp} \subseteq S$, and so is a point of $R \cap S = pa_z$ as well as a point of the $\mathscr{M}$-line $L_w = M_w \cap S$.

But in this configuration we can replace R by a new symplecton $R' := \langle\langle N_1, N_2' \rangle\rangle$ where N_2' is another line of M_2 on p distinct from both N_2 and L_2. Then $R' \cap R = N_1$. But just as we argued for R above, R' is not a grid, and meets S at a line. We note that the two lines $R \cap S$ and $R' \cap S$ are distinct since $R \cap R' = N_1$. But in the last line of the previous paragraph we saw that the $\mathscr{M}$-line $L_w = M_w \cap S$ intersected $R \cap S$ at a point (we called t). So similarly, L_w intersects line $R' \cap S$ at a point s distinct from t. The gamma space property of S then forces $L_w \subseteq p^{\perp}$, which is absurd since $w \in L_w - p^{\perp}$.

Thus, if S is not a grid, we have shown that all points of S are uniqueness points. Now consider any point y of $\mathscr{P} - S$. Then by assumption, $y^{\perp} \cap S$ is a line L_y. But if M_y is any element of $\mathscr{M}$ on y, then by assumption, $M_y \cap S$ is a line, and so must be L_y. Thus any element of $\mathscr{M}$ on y contains the plane $\langle y, L_y \rangle$, and this forces it to be M_y. Thus y is also a uniqueness point.

10 When all symplecta are grids

In this section we add to (E) the extra hypothesis:

(G) *Every symplecton of* Γ *is a grid.*

Theorem 19. *If hypothesis* (G) *is assumed, then* $\Gamma \simeq M \times A$ *where* $M \in \mathscr{M}$ *and* A *is a singular subspace.*

Proof. By way of contradiction suppose the conclusion is false. Then by Theorem 11, we may assume that no member of $\mathscr{M}$ is a line.

Choose any singular subspace M of $\mathscr{M}$. Suppose, for some point p in M, L_1 and L_2 are two distinct lines on p which are not in M. If L_1 is not in $L_2^{\perp}$, then the symplecton R on L_1 and L_2 must meet M at a third line (see Lemma 7), and so is not a grid, against (G). Thus, always we must have

Step 1: *For any point* p *in a singular subspace* M *of* $\mathscr{M}$, *there exists another maximal singular subspace* $A(p)$ *such that*

$$p^{\perp} = M \cup A(p) \quad \textit{where } M \cap A(p) = \{p\}.$$

Suppose, for the moment that p and M are fixed as in Step 1. Choose any point y not in $p^{\perp}$. Then by hypothesis (E), y is collinear with a unique point y_1 of $M - \{p\}$. Then there is a symplecton G which is a grid on the intersecting lines yy_1 and y_1p. Then p lies on a unique line L of this grid distinct from the line py_1. Then by Step 1, L is a line of the singular space $A(p)$. Moreover y is collinear with a unique point y_2 of line L, since all of this occurs within the grid G.

Now suppose y were collinear with another point y_2' of $A(p)$. Then y_2' would lie in $y^{\perp} \cap p^{\perp} \subseteq G$, and so would lie in $G \cap A(p) = L$. But that forces $y_2' = y_2$. Thus y_2 is the unique point of $A(p)$ which is collinear with such a point y.

Now y lies in some member, say M_y, of $\mathscr{M}$, and so applying Step 1 with M_y replacing M and y replacing p, we see that $y^{\perp} = M_y \cup A$, the union of two maximal singular subspaces which intersect at point y. Since yy_1 and yy_2 are distinct lines of grid G on y, then yy_1 lies in one of the maximal singular subspaces (M_y or A) and yy_2 lies in the other.

We have established

Step 2: *Let* p *and* M *be as in Step* 1. *If* $y \in \mathscr{P} - p^{\perp}$, *then* y *is collinear with a unique point* y_1 *of* M *and with a unique point* y_2 *of* $A(p)$. *Then* $y^{\perp}$ *is the union of two maximal singular spaces; one is* $A(y_1)$ *and the other contains* yy_2. *Thus every line on point* y *is either in* $y_1^{\perp}$ *or is in* $y_2^{\perp}$.

We next show

Step 3: *Let* p, M, y, y_1 *and* y_2 *be as in* (*Step* 2). *If a point* y' *is collinear with both* y_1 *and* y_2, *it is either* y *or* p.

Clearly, any such y' lies in the grid G which contains them, and the result follows.

Now we can complete the proof of the Theorem. First we can uniquely assign coordinates from $M \times A(p)$ to each point at follows. If y is not in $p^{\perp}$, we assign coordinates (y_1, y_2) to y. If m is a point of M, we assign coordinates (m, p) to m. If a is a point of $A(p)$, we assign the coordinates (p, a) to a. Note that p has coordinates (p, p) and by (Step 3) each point of $\mathscr{P}$ receives a unique coordinate by this device. Conversely if (m, a) is arbitrarily chosen with neither coordinate equal to p, then there is a unique grid $G(m, a)$ on $\{m, a\}$ containing a unique point $y \in G(m, a) - \{p\}$ collinear with both m and a. Thus introducing coordinates produces a complete bijection $\mathscr{P} \to M \times A(p)$.

Now let us consider the collection of coordinates of the points on an arbitrary line of L. If L is in M or in $A(p)$, then one of the coordinates is constantly p while the other coordinates range through a line of M or $A(p)$, respectively. Similarly, if L contains a point $y = (y_1, y_2)$ not in $p^{\perp}$, then by Step 2, this line is either in $y_1^{\perp}$ or in $y_2^{\perp}$ but not both. In the former case the left coordinates of the points of L are constantly y_1 while the right-hand coordinates range over the line py_2 at which the grid $G' = \langle\langle y_1, y_2 \rangle\rangle$ intersects $A(p)$. In the latter case the right coordinates of all points of L are constantly y_2 while the right coordinates range over the line py_1.

Thus all lines have the form $m \times L$ where m is a point of M and L is a line of $A(p)$, or else have the form $N \times a$ where N is a line of M and a is a point of $A(p)$.

The geometry on $M \times A(p)$ with this collection of lines is precisely the product geometry, $M \times A(p)$, and the desired isomorphism follows.

This contradicts the assumption the theorem was false, completing the proof.

11 The proof of Theorem 1

We assume Hypothesis (E). We are to prove the following conclusion:

(C) One of the following holds:

1. Γ is a generalized quadrangle.
2. $\Gamma = A \times B$, the product of two maximal singular subspaces.
3. Γ properly contains a symplecton which is not a grid. The members of $\mathscr{M}$ partition the points of Γ. Moreover, if $\mathscr{M}$ is the full collection of all maximal singular subspaces A of Γ with the property that $|p^{\perp} \cap A| = 1$ for all $p \in \mathscr{P} - A$, then in this case, every line that is not an $\mathscr{M}$-line, is itself a maximal singular subspace of Γ. Finally, no line that is not an $\mathscr{M}$-line intersects every member of $\mathscr{M}$.

As before, let U be the set of points of Γ which lie in a unique member of $\mathscr{M}$.

Step 1: *If $\mathscr{P} \neq U$, then either* (i) *Γ is a generalized quadrangle, or* (ii) *every symplecton of Γ is a grid.*

Proof. Suppose $\mathscr{P} \neq U$. Suppose by way of contradiction that Γ is not a generalized quadrangle and that there exists a symplecton S which is not a grid.

Now by Theorem 18, if S is disjoint from no member of $\mathscr{M}$, then either (1) Γ is a generalized quadrangle, (2) S is a grid or (3) $\mathscr{P} = U$. But any of the conclusions (1), (2) or (3) goes against the suppositions of the previous paragraph.

Thus, we must assume that there is a singular subspace $M \in \mathscr{M}$ such that $S \cap M = \varnothing$. Then, since S is not a grid, restriction of the projection mapping into M produces an embedding $f : S \to M$. In that case the hypotheses of Corollary 16 are in place, forcing us to conclude that $\mathscr{P} = U$, contrary to the hypothesis of the theorem.

This contradiction completes the proof of this Step 1.

Step 2: *If* $\mathscr{P} \neq U$, *then either* (i) Γ *is a generalized quadrangle or* (ii) Γ *is a product geometry* $A \times B$ *of two singular subspaces* A *and* B.

Proof. By Step 1, either (i) holds or all symplecta are grids. But by Theorem 19, the latter case forces conclusion (ii) above.

Now we can complete the proof of the theorem. Suppose, that the first two conclusions of (C) fail—that is, Γ is not a generalized quadrangle nor is it a product of two maximal singular subspaces. Then, from contrapositive of the statement of Step 2, $\mathscr{P} = U$.

Let us assume now that $\mathscr{M}$ is the full collection of all maximal singular subspaces A with the property that every point outside A is collinear with exactly one point of A. Since Γ is not a generalized quadrangle, nor a product of two maximal singular subspaces, the elements of $\mathscr{M}$ partition the points. If L is not an $\mathscr{M}$-line, then L must be disjoint from some member of $\mathscr{M}$, otherwise $L \in \mathscr{M}$, by maximality of $\mathscr{M}$, and so Γ is a product of two singular spaces, contrary to assumption. Choose $M \in \mathscr{M}$ so that $L \cap M = \varnothing$. Then the set of points of M which are collinear with a point of L themselves form a line L' of M. Choose point y in L, and let M_y be the (now unique) member of $\mathscr{M}$ containing y. Since L is not an $\mathscr{M}$-line, $M_y \cap L = \{y\}$. Since $M_y \cap M = \varnothing$, there is a bijection $f : M_y \to M$ taking each point of M_y to the unique point of M collinear with it. This mapping f is an isomorphism of linear spaces. Note that $f(y) \in L'$. Thus there is a line N in M_y such that $f(N) = L'$. Now let $R = \langle\langle y, L'\rangle\rangle$, the symplecton on y and L'. Now each point of $N \cup L - \{y\}$ is collinear with y and a point of $L' - \{f(y)\}$, forcing $N \cup L \subseteq R$. Thus y lies on at least three lines of R, namely L, N, and $yf(y)$, and so R is not a grid.

If L were properly contained in a singular subspace B, then $B \notin \mathscr{M}$ as L is not an $\mathscr{M}$-line. Then, choosing $x \in B - L$, and $M_x \in \mathscr{M}$ containing x we see that the convexity of R forces $R \cap M_x = \varnothing$. But the mapping $g : R \to M_x$ induced by projection on M_x is not an embedding since g maps the line L to x. By Theorem 12 R must be a grid. This contradicts the conclusion of the previous paragraph.

Thus L cannot properly lie in another singular subspace—that is, it is itself a maximal singular subspace.

11.1 Local recognition of the three conclusions of Theorem 1. Our purpose here is to identify one of the three alternatives of conclusion (C) above, by a property of any one of its points. The reason for doing this will emerge in the next section.

Theorem 20. *Suppose* $\Gamma = (\mathscr{P}, \mathscr{L})$ *satisfies hypothesis* (E) *so that one of the three conclusions listed in* (C) *above holds.*

1. *Suppose* Γ *contains a point* x *such that every maximal singular subspace on* x *is a line. Then* Γ *is a generalized quadrangle.*
2. *Suppose* Γ *contains a point* x *that lies in exactly two maximal singular subspaces. Then* Γ *is a product geometry.*
3. *Suppose* Γ *contains a point* x *which lies in at least three maximal singular subspaces, only one of which is a member of* $\mathscr{M}$ *and it is not a line. Then* Γ *contains a proper symplecton which is not a grid, and the elements of* $\mathscr{M}$ *partition the points of* Γ.

Proof. Part 1. Suppose every line on x is a maximal singular subspace. Then in particular, an element $M_x \in \mathscr{M}$ containing x must be a line. By Theorem 17 Γ is either a generalized quadrangle or a product geometry $L \times A$ where L is an $\mathscr{M}$-line and A is a maximal singular subspace. But in the latter case, x lies in a subspace A_x isomorphic to A, and by hypothesis A_x is a line. Thus A is a line, and $\Gamma = L \times A$ is a grid. Thus in either case, Γ is a generalized quadrangle.

Part 2. Now suppose x is a point on exactly two maximal singular subspaces. Now one of these is a maximal singular subspace M_x belonging to the special class $\mathscr{M}$. The other, we shall call A_x, which may or may not belong to $\mathscr{M}$. Thus $x^{\perp} = M_x \cup A_x$.

Now let y be any point of Γ not in $x^{\perp}$. We claim that y is collinear with a unique point y_2 of A_x. By hypothesis (E), y is collinear with a point y_1 of M_x, and $S := \langle\langle x, y\rangle\rangle$ is a symplecton containing x. Then there is a line N of S which contains x but does not lie in $y_1^{\perp}$. Then N must lie in A_x and y is collinear with a unique point y_2 of N. Suppose y_2' were any point of $y^{\perp} \cap A_x$. Now $y_2'^{\perp}$ contains the 2-coclique $\{x, y\}$, and so belongs to S, and in fact $S \cap A_x = N$. Thus $y_2' = y_2$. Thus A_x has the property that every point of $\mathscr{P} - A_x$ is collinear with exactly one point of A_x. By maximality of $\mathscr{M}$ we see that $A_x \in \mathscr{M}$. Thus x lies in two members of $\mathscr{M}$, and so the third conclusion of (C) is impossible. If the first conclusion held, we would have $S = \mathscr{P}$ is a generalized quandrangle with a point x on exactly two lines. In that case Γ is a grid and so is the product of two lines. Thus the second conclusion of (C) must hold in any case.

Part 3. Here, x lies in at least three maximal singular subspaces, one of which is not a line. It follows that Γ cannot be a product geometry nor can it be a generalized quadrangle. Thus the third conclusion of (C) must hold.

12 Proof of Theorem 2

12.1 Uniformity of point residuals.

Lemma 21. *We assume that* Γ *is a gamma space with all singular subspaces projective spaces. Assume that for every point* p*, the point-residual* $Res_{\Gamma}(p) := (\mathscr{L}_p, \Pi_p)$ *is a strong parapolar space of symplectic rank exactly* 2 *satisfying exactly one of the three conclusions of conclusion* (C)*—the choice depending on the particular point* p*.*

Let us define three sets of points:

1. X_1: *the set of all points p which lie in some line L with the property that every maximal singular subspace containing L is a plane.*

2. X_2: *the set of all points p which lie in some line L which lies in exactly two maximal singular subspaces.*

3. X_2: *the set of points p which lie on some line L such that*
   (a) *L lies in at least three maximal singular subspaces,*
   (b) *L lies in a unique member M_L of $\mathcal{M}$, and*
   (c) *M_L has projective rank at least three.*

Then each set X_i is a union of connected components of the point-collinearity graph $\Delta = (\mathcal{P}, \sim)$.

Proof. Suppose $p \in X_1$ and q is another point collinear with p. p lies on a line L with every maximal singular subspace containing L a plane. Now $\mathrm{Res}_\Gamma(p)$ satisfies conclusion (C) with a "point" L having each maximal singular subspace containing it a "line". By Theorem 20, $\mathrm{Res}_\Gamma(p)$ is a generalized quadrangle all of whose "lines" are maximal singular subspaces. Thus the line pq has all the maximal singular subspaces containing it planes. It follows that $q \in X_1$.

Suppose $p \in X_2$ and q is another point collinear with p. p lies on a line L lying on exactly two maximal singular subspaces. Since $\mathrm{Res}_\Gamma(p)$ satisfies conclusion (C) with a "point" L in just two maximal singular spaces, by Theorem 20, $\mathrm{Res}_\Gamma(p)$ is a product geometry, all of whose "points" lie in just two maximal singular subspaces. In particular, the line pq is in exactly two maximal singular subspaces, and so $q \in X_2$.

Suppose $p \in X_3$ and q is another point collinear with p. p lies on a line L lying in a unique singular subspace M_L of $\mathcal{M}$, with all other singular subspaces on L, of which there are at least two, being planes. By Theorem 20, every line on p has this property—in particular line pq does. Thus $q \in X_3$.

Thus for all i, if $p \in X_i$ and q is a point collinear with p, then $q \in X_i$. The proof is complete.

Theorem 22. *Suppose Γ is a connected parapolar space satisfying condition* (D). *Then one of the following three conclusions holds*:

1. *Γ is a non-degenerate polar space of rank exactly three.*

2. *Γ is either* (1) *the Grassmannian $A_d(V)$ of d-subspaces of a* (*possibly infinite-dimensional*) *vector space V, where d is a positive integer or* (2) *is the quotient geometry $A_{2n-1,n}(D)/\langle\sigma\rangle$ of the Grassmannian of n spaces of a 2n-dimensional vector space by a polarity σ of Witt index at most $n-5$.*

3. *In the geometry Γ, every line L lies in a unique member M_L of $\mathcal{M}$ which is of finite projective dimension $d > 2$ and in at least two other maximal singular subspaces—all of these maximal singular subspaces being planes. Moreover, each point lies in a proper rank three symplecton which is not oriflamme. For each plane π that is not contained in a member of $\mathcal{M}$ and for each point p in π, there exists a member of $\mathcal{M}$ meeting π exactly at point p.*

Proof. Suppose Γ is a connected parapolar space satisfying condition (D). Then every point residual $\mathrm{Res}_\Gamma(p)$ satisfies condition (E), and so, by Theorem 1, such a point-residual must satisfy one of the three conclusions of (C). This means $\mathscr{P} = X_1 + X_2 + X_3$, where the X_i are defined as in Lemma 21. By assumption, Γ has a connected point-collinearity graph, so we have three cases emanating from $\mathscr{P} = X_i$, $i = 1, 2, 3$.

Case 1. $\mathscr{P} = X_1$. Here, every point-residue $\mathrm{Res}_\Gamma(p)$ is a generalized quadrangle. It follows from the main result of [13] that Γ is a polar space of rank three.

Case 2. $\mathscr{P} = X_2$. Here, every point-residue $\mathrm{Res}_\Gamma(p)$ is a product geometry $A_p \times B_p$, where A_p and B_p are two distinct maximal singular subspaces of Γ which intersect at a line on p. Conclusion 2 now follows from Corollary 6.

Case 3. $\mathscr{P} = X_3$. Every point-residue $\mathrm{Res}_\Gamma(p)$ has its "points" $\mathscr{L}_p$ partitioned by the elements of $\mathscr{M}$ on p. It also contains a proper symplecton which is not a grid, each of its "lines" (elements of Π_p) is either contained in an element of $\mathscr{M}$, or is a maximal singular subspace. Now suppose a plane π on p were contained in no member of $\mathscr{M}$. Then π corresponds to a "line" of $\mathrm{Res}_\Gamma(p)$. If this "line" met every member of $\mathscr{M}_p$ and a "point", then it could be adjoined to the collection $\mathscr{M}_p$ without changing the hypothesis (E), and Theorem 17 would then force $\mathrm{Res}_\Gamma(p)$ to be a product geometry of the form $L \times M_p$ where L is a "line" and M_p is a singular space. But in that case, every symplecton of $\mathrm{Res}_\Gamma(p)$ would be a grid contrary to the fact that it contains a proper symplecton which is not a grid. Thus there must be a member M of $\mathscr{M}_p$ which intersects the plane π exactly at point p.

Now Theorem 2 is an immediate corollary of the above Theorem 22.

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