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# Translation spreads of the split Cayley hexagon

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**Abstract.** Some results concerning translation spreads of the classical generalized hexagon $H(q)$ are given, motivated by known analogous results for translation ovoids of the generalized quadrangle $Q(4,q)$. In addition, semi-classical spreads are characterized in terms of their kernels. Finally, a new spread of $H(q)$ is described which is also a new 1-system of the non-degenerate parabolic quadric in $PG(6,q)$.

## 1 Introduction

The geometries called generalized polygons were introduced by Tits [15], encapsulating projective planes as generalized triangles and rank 2 polar spaces as generalized quadrangles. Also, it was in that work of Tits that the generalized hexagon $H(q)$ was born. For the sake of brevity, the word 'generalized' will often be omitted in the following.

A spread of a generalized polygon $\Gamma$ is, roughly speaking, a set of lines that are in a sense 'spread' out evenly over $\Gamma$. The dual concept is that of an ovoid. This is made precise in Section 2.1. Translation spreads and ovoids are then essentially those with a high level of symmetry and these were introduced by Bloemen, Thas and Van Maldeghem [1]. In Section 3.1 of that paper, the authors give a range of details about translation spreads of the hexagon $H(q)$, and then in Section 3.2 they discuss translation ovoids of the quadrangle $Q(4,q)$. Although some of the results of the latter section are analogous to results in the former, on the whole they go beyond their partners in the hexagon case. Here we extend on the results pertaining to translation spreads of $H(q)$ to bring them more in line with those of [1, Section 3.2].

In addition to the analogy between them, there is also a more concrete connection between spreads of $H(q)$ and ovoids of $Q(4,q)$ in the form of semi-classical spreads. These were introduced in [1, Section 5], and as a result of their investigations there, the authors discovered new spreads in the case of odd $q$. Here we give an algebraic characterization of semi-classical spreads as those whose kernels are the entire underlying field and subsequently describe a new class of spreads for even $q$.

Most of the contents of this paper are taken from the author's doctoral thesis [6], undertaken under the supervision of Dr C. M. O'Keefe and Dr L. R. A. Casse.

## 2 Preliminaries

**2.1 Generalized polygons, spreads and ovoids.** A *generalized n-gon* of order $(s,t)$, with $s,t>1$, is a geometry $\Gamma=(\mathcal{P},\mathcal{L},\mathrm{I})$ in which each line is incident with exactly $s+1$ points, each point is incident with exactly $t+1$ lines, and whose incidence graph has diameter $n$ and girth $2n$. When $s=t$, we say that $\Gamma$ has order $s$. An *apartment* in $\Gamma$ is a sequence of $n$ points and $n$ lines forming a circuit of the minimum length $2n$ in the incidence graph, which is just an $n$-gon in the ordinary sense. Details can be found in the book [16] by Van Maldeghem and in the chapter [14] by Thas.

Let $\Gamma$ be a generalized $2m$-gon of order $(s,t)$, so $m=2$ for the case of quadrangles and $m=3$ for hexagons. The *distance* $\delta(u,v)$ between two elements $u$ and $v$ of $\Gamma$ is the distance between them in the incidence graph of $\Gamma$. In particular, the value of $\delta(u,v)$ is at most $2m$, the diameter of the incidence graph. When $\delta(u,v)=2m$, the elements $u$ and $v$ are said to be *opposite*. When $\delta(u,v)<2m$, the unique element incident with $u$ and at distance $\delta(u,v)-1$ from $v$ is called the *projection* of $v$ onto $u$.

A *spread* of $\Gamma$ is a set $\mathcal{S}$ of mutually opposite lines such that for each element $u$ of $\Gamma$ there is a line $L\in\mathcal{S}$ such that $\delta(u,L)\leqslant m$. The dual notion is that of an *ovoid*, which is then a similar set of mutually opposite points. Concerning the sizes of these, a set of mutually opposite lines (respectively points) is a spread (respectively ovoid) if and only if: (a) it contains $st+1$ elements in the case that $\Gamma$ is a quadrangle (see [16, 7.2.3(i)]); and (b) it contains $s^3+1$ elements and the order of $\Gamma$ is $s$ $(=t)$ in the case that $\Gamma$ is a hexagon (see [8]).

As a comment on notation used in this paper, whenever two variables are represented by different cases of the same letter, for instance $x$ and $X$, we represent their difference by $\Delta x=x-X$. Also, the operator $\Delta$ is considered as having higher precedence than any arithmetical operation, so an expression like $\Delta x^2$ is to mean $(\Delta x)^2$ and not $\Delta(x^2)$.

**2.2 The quadrangle $\mathbf{Q(4,q)}$.** Let $\mathcal{P}_4$ be the nondegenerate parabolic quadric in $\mathrm{PG}(4,q)$ given by the equation $X_0X_4+X_1X_3+X_2^2=0$. The geometry of points and lines on $\mathcal{P}_4$ with their natural incidence is the classical generalized quadrangle $\mathrm{Q}(4,q)$ of order $q$. Labelling the points and lines as indicated in Table 1, where all variables take values in the field $\mathrm{GF}(q)$, we obtain a coordinatization of $\mathrm{Q}(4,q)$ as described in [16, 3.4.7]. Incidence is then given by the paths

$$[k,b,k']\,\mathrm{I}\,(k,b)\,\mathrm{I}\,[k]\,\mathrm{I}\,(\infty)\,\mathrm{I}\,[\infty]\,\mathrm{I}\,(a)\,\mathrm{I}\,[a,\ell]\,\mathrm{I}\,(a,\ell,a')$$

together with $(a,\ell,a')\,\mathrm{I}\,[k,b,k']$ if and only if

$$\begin{aligned} b&=a'+ak^2+2\ell k\\ k'&=\ell+ak. \end{aligned}\tag{1}$$

Notice that the points of $\mathrm{Q}(4,q)$ opposite $(\infty)$ are those with three coordinates in this coordinatization, and similarly for the lines opposite $[\infty]$. In addition, two points

Table 1. Coordinatization of Q(4, $q$)

| POINTS | |
|---|---|
| Coordinates in Q(4, $q$) | Coordinates in PG(4, $q$) |
| $(\infty)$ | $(1,0,0,0,0)$ |
| $(a)$ | $(a,0,0,1,0)$ |
| $(k,b)$ | $(-b,1,k,-k^2,0)$ |
| $(a,\ell,a')$ | $(-\ell^2+aa',-a,\ell,a',1)$ |
| LINES | |
| Coordinates in Q(4, $q$) | Coordinates in PG(4, $q$) |
| $[\infty]$ | $\langle(1,0,0,0,0),(0,0,0,1,0)\rangle$ |
| $[k]$ | $\langle(1,0,0,0,0),(0,1,k,-k^2,0)\rangle$ |
| $[a,\ell]$ | $\langle(a,0,0,1,0),(-\ell^2,-a,\ell,0,1)\rangle$ |
| $[k,b,k']$ | $\langle(-b,1,k,-k^2,0),(-k'^2,0,k',b-2kk',1)\rangle$ |

$(a,\ell,a')$ and $(A,L,A')$ are opposite if and only if $\Delta\ell^2 \neq \Delta a\Delta a'$ (use the relations in (1) or consider coordinates in $\mathscr{P}_4$ and use the associated bilinear form; alternatively, see [1, Section 3.2]).

**2.3 The hexagon H($q$).** Let $\mathscr{P}_6$ be the nondegenerate parabolic quadric in PG(6, $q$) given by the equation $X_0X_4 + X_1X_5 + X_2X_6 = X_3^2$. This quadric has points, lines and planes lying entirely on it, with the planes being the *generators* of $\mathscr{P}_6$. Following Tits [15], the generalized hexagon H($q$) has a natural embedding in $\mathscr{P}_6$. Specifically, the points of H($q$) are all the points of $\mathscr{P}_6$, and the lines of H($q$) are the lines of this quadric whose Grassmann coordinates satisfy

$$p_{12} = p_{34}, \quad p_{20} = p_{35}, \quad p_{01} = p_{36},$$
$$p_{65} = p_{30}, \quad p_{46} = p_{31}, \quad p_{54} = p_{32}.$$

Incidence in H($q$) is then simply that inherited from $\mathscr{P}_6$. This hexagon H($q$) has order $q$ and is known as the *split Cayley hexagon*.

Labelling the points and lines as indicated in Table 2, where all variables take values in the field GF($q$), we endow H($q$) with a coordinatization as described in [16, 3.5.1].

Incidence is given by the paths

$$\begin{gathered}[k,b,k',b',k''] \,\mathrm{I}\, (k,b,k',b') \,\mathrm{I}\, [k,b,k'] \,\mathrm{I}\, (k,b) \,\mathrm{I}\, [k] \,\mathrm{I}\, (\infty) \,\mathrm{I} \\ [\infty] \,\mathrm{I}\, (a) \,\mathrm{I}\, [a,\ell] \,\mathrm{I}\, (a,\ell,a') \,\mathrm{I}\, [a,\ell,a',\ell'] \,\mathrm{I}\, (a,\ell,a',\ell',a'')\end{gathered} \tag{2}$$

together with $(a,\ell,a',\ell',a'') \,\mathrm{I}\, [k,b,k',b',k'']$ if and only if

Table 2. Coordinatization of $\mathrm{H}(q)$

| POINTS | |
|---|---|
| Coordinates in $\mathrm{H}(q)$ | Coordinates in $\mathrm{PG}(6,q)$ |
| $(\infty)$ | $(1,0,0,0,0,0,0)$ |
| $(a)$ | $(a,0,0,0,0,0,1)$ |
| $(k,b)$ | $(b,0,0,0,0,1,-k)$ |
| $(a,\ell,a')$ | $(-\ell-aa',1,0,-a,0,a^2,-a')$ |
| $(k,b,k',b')$ | $(k'+bb',k,1,b,0,b',b^2-kb')$ |
| $(a,\ell,a',\ell',a'')$ | $(-a\ell'+a'^2+\ell a''+aa'a'',-a'',-a,-a'+aa'',$ $1,\ell+2aa'-a^2a'',-\ell'+a'a'')$ |
| LINES | |
| Coordinates in $\mathrm{H}(q)$ | Coordinates in $\mathrm{PG}(6,q)$ |
| $[\infty]$ | $\langle(1,0,0,0,0,0,0),(0,0,0,0,0,0,1)\rangle$ |
| $[k]$ | $\langle(1,0,0,0,0,0,0),(0,0,0,0,0,1,-k)\rangle$ |
| $[a,\ell]$ | $\langle(a,0,0,0,0,0,1),(-\ell,1,0,-a,0,a^2,0)\rangle$ |
| $[k,b,k']$ | $\langle(b,0,0,0,0,1,-k),(k',k,1,b,0,0,b^2)\rangle$ |
| $[a,\ell,a',\ell']$ | $\langle(-\ell-aa',1,0,-a,0,a^2,-a'),(-a\ell'+a'^2,0,$ $-a,-a',1,\ell+2aa',-\ell')\rangle$ |
| $[k,b,k',b',k'']$ | $\langle(k'+bb',k,1,b,0,b',b^2-kb'),$ $(b'^2+bk'',-b,0,-b',1,k'',-kk''-k'-2bb')\rangle$ |

$$\begin{aligned} \ell &= -a^3k+k''-3a^2b-3ab', \\ a' &= a^2k+b'+2ab, \\ \ell' &= a^3k^2+k'+kk''+3a^2kb+3bb'+3ab^2, \\ a'' &= ak+b. \end{aligned} \tag{3}$$

Notice that the lines opposite $[\infty]$, and similarly the points opposite $(\infty)$, are precisely those that have five coordinates in this coordinatization. Since two points of $\mathrm{H}(q)$ are opposite if and only if they are not collinear in $\mathscr{P}_6$ (see [15, Section 4]), the condition that two points opposite $(\infty)$ should themselves be opposite is readily determined by using the bilinear form associated with the quadric $\mathscr{P}_6$. The result of this computation can be found in the proof of [1, Theorem 23] and in [7, Section 3]. The case of lines opposite $[\infty]$ is treated by the following lemma.

**Lemma 1.** *The two lines* $[k,b,k',b',k'']$ *and* $[K,B,K',B',K'']$ *of* $\mathrm{H}(q)$ *are opposite if and only if*

$$\begin{aligned} &(\Delta b^2-\Delta k\Delta b')(\Delta b'^2+\Delta b\Delta k'') \\ &\quad -(-k''\Delta k-\Delta k'+\Delta b\Delta b'-3b'\Delta b)(K''\Delta k+\Delta k'+\Delta b\Delta b'+3B'\Delta b)\neq 0. \end{aligned}$$

*Proof.* From Table 2, a line $[k, b, k', b', k'']$ is generated by the two points whose coordinates in $\mathrm{PG}(6, q)$ are

$$(k' + bb', k, 1, b, 0, b', b^2 - kb'),$$
$$(b'^2 + bk'', -b, 0, -b', 1, k'', -kk'' - k' - 2bb').$$

Using the bilinear form associated with $\mathscr{P}_6$, the points that lie in the generators containing $[k, b, k', b', k'']$ then satisfy the equations

$$\begin{aligned} b'X_1 \qquad\qquad + (b^2 - kb')X_2 \;\; - 2bX_3 \;\; + (k' + bb')X_4 + kX_5 + X_6 &= 0, \\ X_0 + k''X_1 + (-kk'' - k' - 2bb')X_2 + 2b'X_3 + (b'^2 + bk'')X_4 - bX_5 \qquad &= 0. \end{aligned} \tag{4}$$

Also, the line $[K, B, K', B', K'']$ is given by the system of equations

$$\begin{aligned} X_0 \quad - (K' + BB')X_2 \qquad\qquad - (B'^2 + BK'')X_4 \qquad\qquad &= 0, \\ X_1 \qquad - KX_2 \qquad\qquad\qquad + BX_4 \qquad\qquad &= 0, \\ - BX_2 + X_3 \qquad\qquad + B'X_4 \qquad\qquad &= 0, \\ - B'X_2 \qquad\qquad - K''X_4 + X_5 \qquad &= 0, \\ -(B^2 - KB')X_2 \qquad - (-KK'' - K' - 2BB')X_4 \qquad + X_6 &= 0. \end{aligned} \tag{5}$$

Now two lines of $\mathrm{H}(q)$ are opposite if and only if every generator of $\mathscr{P}_6$ containing one is disjoint from the other (which follows from the fact that two points are opposite precisely when they are not collinear in the quadric; see also [11, 8.2]), and this corresponds to the equations in (4) and (5) having no nonzero solution in common. Taking the equations in the given order to produce a coefficient matrix $A$, what we then require is that $A$ should be nonsingular. Performing the row operations $R_1 := R_1 - b'R_4 + 2bR_5 - kR_6 - R_7$ and $R_2 := R_2 - R_3 - k''R_4 - 2b'R_5 + bR_6$ on $A$, this is equivalent to the $2 \times 2$ determinant

$$\begin{vmatrix} \Delta b^2 - \Delta k \Delta b' & K''\Delta k + \Delta k' + \Delta b \Delta b' + 3B'\Delta b \\ -k''\Delta k - \Delta k' + \Delta b \Delta b' - 3b'\Delta b & \Delta b'^2 + \Delta b \Delta k'' \end{vmatrix}$$

being nonzero. Expanding this determinant gives the desired result.

A *line regulus* is a set $\mathscr{R}$ of $q + 1$ pairwise opposite lines in $\mathrm{H}(q)$ for which there are two opposite points, say $u$ and $v$, that lie at distance 3 from each of them; that is, such that $\delta(L, u) = \delta(L, v) = 3$ for each $L \in \mathscr{R}$. The name derives from the fact that the lines of $\mathscr{R}$ all lie in a common three dimensional space $\Pi$ that meets $\mathscr{P}_6$ in a hyperbolic quadric (see [1, Section 2.2]), and as such they form a regulus in the usual sense. It follows that $\mathscr{R}$ is uniquely determined by any two of its lines since the space $\Pi$ is uniquely determined by them.

We shall write $[[k, b, b', k'']]$ for the line regulus determined by the lines $[\infty]$ and $[k, b, k', b', k'']$. Noticing that the points $(k, b)$ and $(0, k'', b')$ are at distance 3 from these two lines, another line $[K, B, K', B', K'']$ opposite $[\infty]$ belongs to this line regulus if and only if $(k, b) = (K, B)$ and $(0, k'', b') = (0, K'', B')$. Thus

$$[[k, b, b', k'']] = \{[\infty]\} \cup \{[k, b, x, b', k''] \mid x \in \mathrm{GF}(q)\}.$$

Two line reguli, $\mathscr{R}$ and $\mathscr{R}'$, on a common line $L$ are *compatible* if each line $M \in \mathscr{R} \setminus \{L\}$ is opposite every line of $\mathscr{R}'$. Thus two line reguli on a common line could exist together in a spread only if they are compatible. When the common line is the line $[\infty]$, the following result identifies compatible line reguli.

**Lemma 2.** *Two line reguli* $[[k, b, b', k'']]$ *and* $[[K, B, B', K'']]$ *are compatible if and only if* (i) *for odd* $q$, *the expression*

$$(\Delta b \Delta b' + \Delta k \Delta k'')^2 + 4(\Delta b' \Delta k - \Delta b^2)(\Delta b \Delta k'' + \Delta b'^2)$$

*is a non-square, and* (ii) *for even* $q$,

$$\mathrm{Tr}\left(\frac{\Delta b^2 \Delta b'^2 + \Delta b^3 \Delta k'' + \Delta b'^3 \Delta k}{\Delta b^2 \Delta b'^2 + \Delta k^2 \Delta k''^2}\right) = 1,$$

*where* $\mathrm{Tr}(a) = a + a^2 + a^{2^2} + \cdots + a^{2^{h-1}}$ *is the trace from* $\mathrm{GF}(q)$ *onto its prime field* $\mathrm{GF}(2)$.

*Proof.* By Lemma 1, arbitrary lines $[k, b, k', b', k'']$ and $[K, B, K', B', K'']$ from the line reguli $[[k, b, b', k'']]$ and $[[K, B, B', K'']]$ are opposite if and only if the expression

$$x^2 + (A + B)x + (A - C)(B + C) + D \tag{6}$$

is nonzero, where $x = \Delta k'$ and

$$A = k'' \Delta k + 3b' \Delta b, \quad B = K'' \Delta k + 3B' \Delta b,$$

$$C = \Delta b \Delta b', \quad D = (\Delta b^2 - \Delta k \Delta b')(\Delta b'^2 + \Delta b \Delta k'').$$

Thus the two line reguli are compatible if and only if the quadratic (6) in $x$ is irreducible. For odd $q$, this corresponds to the discriminant $(A + B)^2 - 4(A - C)(B + C) - 4D = (A - B - 2C)^2 - 4D$ being non-square. For even $q$, since $x^2 + (A + B)X + AB$ is reducible, the irreducibility of (6) is equivalent to the irreducibility of $x^2 + (A + B)x + (A + B)C + C^2 + D$, which is in turn equivalent to the given trace condition. For the results used here regarding the irreducibility of quadratics, see for example [3, Section 1.4].

**2.4 Some groups.** Let the generalized $n$-gon $\Gamma$ be either the quadrangle $\mathrm{Q}(4, q)$ or the hexagon $\mathrm{H}(q)$. Let $u$ be an element of $\Gamma$ and let $v \,\mathrm{I}\, u$ be an element incident with it. For the group of collineations that fix both $u$ and $v$ elementwise we write $G^{\{u,v\}}$. Define $G^u = \langle G^{\{u,w\}} \mid w \,\mathrm{I}\, u \rangle$ to be the group generated by all collineations that fix $u$, as well as some element incident with $u$, elementwise.

**Lemma 3.** *The groups defined above have orders $|G^{\{u,v\}}| = q^{n-2}$ and $|G^u| = q^{n-1}$, and the group $G^u$ acts regularly on the set of elements opposite $u$.*

*Proof.* This is a consequence of the fact that each of these groups is a product of so-called *root groups* (see, for example, [10] or [16, 5.2.1]). For the group $G^{\{u,v\}}$, the necessary result can be found in [16, 5.2.3]. In the case of the group $G^u$, it is by the definition of root groups that $G^u$ at least contains the product of $n-1$ adjacent root groups. The reverse containment then follows from a result of Weiss [17, Lemma 2] (see also [16, 5.3.3, Lemma 3]). For the case of $\Gamma = \mathrm{H}(q)$, see also [1, Lemma 4].

When $\Gamma$ is the generalized quadrangle $\mathrm{Q}(4,q)$ and $u$ is the point $(\infty)$, we now have

$$G^{(\infty)} = \{\Psi(a,\ell,a') \mid a,\ell,a' \in \mathrm{GF}(q)\},$$

where $\Psi(a,\ell,a')$ is the unique collineation in $G^{(\infty)}$ that maps the point $(0,0,0)$ to $(a,\ell,a')$.

Similarly, for the hexagon $\mathrm{H}(q)$ we have

$$G^{[\infty]} = \{\Theta[k,b,k',b',k''] \mid k,b,k',b',k'' \in \mathrm{GF}(q)\},$$

where $\Theta[k,b,k',b',k'']$ is the unique collineation in $G^{[\infty]}$ that maps the line $[0,0,0,0,0]$ to the line $[k,b,k',b',k'']$.

The explicit forms of the collineations $\Psi(a,\ell,a')$ and $\Theta[k,b,k',b',k'']$ appear in [1]. In particular, the action of $\Theta[K,B,K',B',K'']$ is given by

$$\begin{aligned}(a,\ell,a',\ell',a'') \mapsto{} & (a, \ell + K'' - a^3K - 3a^2B - 3aB', a' + B' + a^2K + 2aB, \\ & \ell' + K' + KK'' + \ell K + a^3K^2 + 3BB' + 3aa'K + 3aB^2 \\ & + 3a'B + 3a^2KB, a'' + B + aK), \\ [k,b,k',b',k''] \mapsto{} & [k+K, b+B, k'+K' - kK'' - 3bB', b'+B', k''+K''],\end{aligned}$$

where the action on all other elements of $\mathrm{H}(q)$ follows from (2) together with the fact that the elements $(\infty)$ and $[\infty]$ are fixed.

**2.5 Translation spreads and ovoids.** In this section, we introduce translation spreads and ovoids of generalized $2m$-gons following the paper [1], although here we only describe translation spreads explicitly, leaving it to the reader to dualize for the case of translation ovoids.

Let $\Gamma$ be a generalized $2m$-gon of order $(s,t)$, where $m = 2$ or $3$. Let $\mathscr{S}$ be a spread of $\Gamma$ (so if $m = 3$ then necessarily $s = t$) and let $L$ be a line of $\mathscr{S}$. We will write $\mathscr{S}^+ = \mathscr{S} \backslash \{L\}$.

First we define what it means for $\mathcal{S}$ to be a translation spread with respect to a flag $\{L, u\}$, where $u$ is a point incident with $L$. For each line $M \neq L$ incident with $u$, let $V_M$ be the set of lines of $\mathcal{S}$ that are at distance $2m - 2$ from $M$. Then the $t$ sets $V_M$ contain $s^{m-1}$ lines each and they partition $\mathcal{S}^+$. We say that the spread $\mathcal{S}$ is a *translation spread with respect to the flag* $\{L, u\}$ if there exists a group $G_{\{L,u\}}$ of collineations of $\Gamma$ that fixes the spread $\mathcal{S}$, fixes $L$ pointwise and $u$ linewise, and in addition, acts transitively on each of the sets $V_M$. The group $G_{\{L,u\}}$ is called the group *associated* with $\mathcal{S}$ with respect to the flag $\{L, u\}$. By [16, 4.4.2], the group $G_{\{L,u\}}$ is uniquely determined and it is sufficient to require only that it acts transitively on one of the sets $V_M$ rather than all of them.

We say that $\mathcal{S}$ is a *translation spread with respect to the line* $L$ if it is a translation spread with respect to the flag $\{L, u\}$ for every point $u$ incident with $L$. The group *associated* with $\mathcal{S}$ with respect to $L$ is then the group $G_L = \langle G_{\{L,u\}} \mid u \, \mathrm{I} \, L \rangle$ generated by the associated groups with respect to each flag.

Suppose now that $\Gamma$ is either $\mathrm{Q}(4, q)$ or $\mathrm{H}(q)$. Then $G_{\{L,u\}} \leqslant G^{\{L,u\}}$ and $G_L \leqslant G^L$. In addition, since $|\mathcal{S}^+| = q^m$ and the group $G^L$ acts regularly on the lines opposite $L$, it follows that $|G_L| \leqslant q^m$ with equality if and only if $G_L$ acts transitively on $\mathcal{S}^+$.

It should be noted that we are only interested in the ovoid case for $\mathrm{Q}(4, q)$ as it is known that this generalized quadrangle only admits spreads when it is self-dual anyway (see [12]). Similarly, in $\mathrm{H}(q)$ it is sufficient to only consider the case of translation spreads as this generalized hexagon only admits translation ovoids when it is self-dual (see [7]).

## 3 Translation ovoids of Q(4, q)

In this section, we overview some results from [1] about translation ovoids of $\mathrm{Q}(4, q)$ for comparison with the results for translations spreads of $\mathrm{H}(q)$ that appear in the next section.

Let $\mathcal{O}$ be an ovoid of $\mathrm{Q}(4, q)$. We may suppose without loss of generality that $\mathcal{O}$ contains the points $(\infty)$ and $(0, 0, 0)$ so that it then takes the form

$$\mathcal{O} = \{(\infty)\} \cup \{(x, y, f(x, y)) \mid x, y \in \mathrm{GF}(q)\}, \tag{7}$$

where $f(0, 0) = 0$ (see [1, Section 3.2]).

It is shown in [1] that an ovoid $\mathcal{O}$ is a translation ovoid with respect to the point $u$ if and only if the stabilizer $G^u_{\mathcal{O}}$ of $\mathcal{O}$ in the group $G^u$ has order $q^2$, and that the associated group $G_u$ is then equal to this stabilizer (cf. Theorem 5). As a consequence, an ovoid $\mathcal{O}$ as represented in (7) is a translation ovoid with respect to the point $(\infty)$ if and only if the function $f$ takes the form

$$f(x, y) = \sum_{i=0}^{h-1} (f_{1i} x^{p^i} + f_{2i} y^{p^i}), \tag{8}$$

where $f_{ki} \in \mathrm{GF}(q)$ and $p$ is the prime such that $q = p^h$ (cf. Theorem 6).

Suppose that $\mathcal{O}$, represented as in (7), is a translation ovoid with respect to the point $(\infty)$ so that the function $f$ has the form shown in (8). The *kernel* of $\mathcal{O}$ is defined to be the set

$$\ker \mathcal{O} = \{a \in \mathrm{GF}(q) \mid f(ax, ay) = af(x, y) \text{ for all } x, y \in \mathrm{GF}(q)\},$$

which is then a subfield of $\mathrm{GF}(q)$.

It is proved in [1] that if $\ker \mathcal{O} = \mathrm{GF}(q)$ then $\mathcal{O}$ is a *classical ovoid* (cf. Corollary 8). These ovoids form one isomorphism class and those containing the points $(\infty)$ and $(0, 0, 0)$ are

$$\mathcal{O}_E(\mu, v) = \{(\infty)\} \cup \{(x, y, -vx + \mu y) \mid x, y \in \mathrm{GF}(q)\}, \tag{9}$$

where $r(x) = x^2 - \mu x + v$ is an irreducible quadratic. From this representation, it is readily seen that $O_E(\mu, v)$ is indeed a translation ovoid with respect to $(\infty)$ and that $\ker \mathcal{O} = \mathrm{GF}(q)$.

## 4 Translation spreads of H(*q*)

Let $\mathcal{S}$ be a spread of $\mathrm{H}(q)$. Without loss of generality, we may suppose that $\mathcal{S}$ contains the lines $[\infty]$ and $[0, 0, 0, 0, 0]$. Then $\mathcal{S}$ takes the form

$$\mathcal{S} = \{[\infty]\} \cup \{[x, y, z, f(x, y, z), g(x, y, z)] \mid x, y, z \in \mathrm{GF}(q)\},$$

where $f(0, 0, 0) = g(0, 0, 0) = 0$ (see [1, Lemma 3]). The spread $\mathcal{S}$ is said to be *locally hermitian* in $[\infty]$ if it is a union of line reguli on the common line $[\infty]$. The functions $f$ and $g$ are then independent of $z$ and the spread $\mathcal{S}$ can be expressed in the form

$$\mathcal{S} = \bigcup_{x, y \in \mathrm{GF}(q)} [[x, y, f(x, y), g(x, y)]], \tag{10}$$

where $f(0, 0) = g(0, 0) = 0$.

Examples of locally hermitian spreads are the *hermitian spreads*. These form one isomorphism class and those containing the lines $[\infty]$ and $[0, 0, 0, 0, 0]$ are

$$\mathcal{S}_H(\mu, v) = \bigcup_{x, y \in \mathrm{GF}(q)} [[x, y, -vx + \mu y, \mu vx - (\mu^2 - v)y]], \tag{11}$$

where $r(x) = x^2 - \mu x + v$ is an irreducible quadratic.

In analogy with the situation for translation ovoids of $\mathrm{Q}(4, q)$, one might hope that a spread $\mathcal{S}$ of $\mathrm{H}(q)$ is a translation spread with respect to a line $L$ if and only if the stabilizer $G^L_{\mathcal{S}}$ of $\mathcal{S}$ in the group $G^L$ has order $q^3$, corresponding to its action on $\mathcal{S} \backslash \{L\}$ being regular. From [1, Theorem 6], this is indeed so when $q \not\equiv 2 \pmod 3$, and in this case, the stabilizer is the associated group $G_L$. In addition, the desired result in the 'if' direction holds for general $q$ by [1, Lemma 5]. Here now we complete the task for all $q$ and as a consequence obtain a result similar to the one for quadrangles involving (8).

**Theorem 4.** *Let $\mathscr{S}$ be a spread of* $\mathrm{H}(q)$, $q = p^h$, *and let $L$ be a line of $\mathscr{S}$. If*

(i) $q \not\equiv 2 \pmod 3$ *and $\mathscr{S}$ is a translation spread with respect to two distinct flags on $L$;*

(ii) $q \equiv 2 \pmod 3$, *$q$ odd, and $\mathscr{S}$ is a translation spread with respect to* $2 + \frac{2(q-1)}{p-1}$ *distinct flags on $L$; or*

(iii) $q = 2^{2e+1}$ *and $\mathscr{S}$ is a translation spread with respect to* $2(q+1)/3$ *distinct flags on $L$;*

*then $\mathscr{S}$ is a translation spread with respect to $L$. Moreover, the stabilizer $G^L_{\mathscr{S}}$ of $\mathscr{S}$ in $G^L$ acts transitively on $\mathscr{S}\backslash\{L\}$ so $|G^L_{\mathscr{S}}| = q^3$, and either $G_L = G^L_{\mathscr{S}}$ or $q = 2^{2e+1}$ and $[G^L_{\mathscr{S}} : G_L] = 2$.*

*Proof.* Since $\mathscr{S}$ is a translation spread with respect to at least two distinct flags on $L$, it is locally hermitian in $L$ (see [1, Theorem 6]), and so we may choose coordinates such that $\mathscr{S}$ is represented as in (10) with $L$ being the line $[\infty]$ and such that $\{[\infty], (\infty)\}$ is one of the flags with respect to which $\mathscr{S}$ is a translation spread. By its transitive action on the set $V_{[0]}$ (see Section 2.5), the associated group for the flag $\{[\infty], (\infty)\}$ is then

$$G_{\{[\infty],(\infty)\}} = \{\Theta[0, y, z, f(0, y), g(0, y)] \mid y, z \in \mathrm{GF}(q)\}.$$

What we aim to show is that $G_{\{[\infty],(\infty)\}}$ acts transitively on the set of lines not equal to $[\infty]$ that are incident with some point $(a)$ on $[\infty]$, although we drop a little short of this goal when $q = 2^{2e+1}$. Then if $\mathscr{S}$ is also a translation spread with respect to the flag $\{[\infty], (a)\}$, the group $G = \langle G_{\{[\infty],(\infty)\}}, G_{\{[\infty],(a)\}}\rangle$ acts transitively on the set $\mathscr{S}\backslash\{[\infty]\}$, so $|G| = q^3$ and $\mathscr{S}$ is a translation spread with respect to the line $[\infty]$ by [1, Lemma 5]. Furthermore, since $G \leqslant G_{[\infty]} \leqslant G^{[\infty]}_{\mathscr{S}}$ and $|G^{[\infty]}_{\mathscr{S}}| \leqslant q^3$, we will also have $G_{[\infty]} = G^{[\infty]}_{\mathscr{S}} = G$.

Applying the collineation $\Theta[0, Y, Z, f(0, Y), g(0, Y)]$ in $G_{\{[\infty],(\infty)\}}$ to the line regulus $[[x, y, f(x, y), g(x, y)]]$ of $\mathscr{S}$ gives $[[x, y + Y, f(x, y) + f(0, Y), g(x, y) + g(0, Y)]]$, which must also be a line regulus of $\mathscr{S}$. Thus $f(x, y) = f_1(x) + f_2(y)$ and $g(x, y) = g_1(x) + g_2(y)$, where $f_2(y + Y) = f_2(y) + f_2(Y)$ and $g_2(y + Y) = g_2(y) + g_2(Y)$.

The projection of the line $[0, y, z, f_2(y), g_2(y)]$ of $\mathscr{S}$ onto a point $(a)$ is the line $[a, h_a(y)]$, where

$$h_a(y) = -3a^2 y - 3af_2(y) + g_2(y)$$

as determined from the incidence equations in (3). For each $a \in \mathrm{GF}(q)$, let

$$K_a = \ker h_a = \{y \in \mathrm{GF}(q) \mid h_a(y) = 0\}.$$

Now as $G_{\{[\infty],(\infty)\}}$ acts transitively on the lines $[0, y, z, f_2(y), g_2(y)]$, it acts transitively on the lines $[a, h_a(y)]$ for each $a$. So what we want to show is that for some $a$, the function $h_a(y)$ is a bijection, or equivalently, that $K_a = \{0\}$.

First consider when $q = 3^h$ so then $h_a(y) = g_2(y)$. Applying Lemma 2 to the line reguli $[[0, y, f_2(y), g_2(y)]]$ and $[[0, 0, 0, 0]]$, we have $-y^3 g_2(y)$ is non-square whenever $y \neq 0$, so in particular, $h_a(y) \neq 0$ whenever $y \neq 0$. Thus for all $a$ we have $K_a = \{0\}$ and hence the result follows.

Suppose now that $3 \nmid q$. Treating $h_a(y) = 0$ with $y \neq 0$ as a quadratic in $a$, for odd $q$, its discriminant is

$$9f_2(y)^2 + 12yg_2(y) = -3(-3f_2(y)^2 - 4yg_2(y)), \tag{12}$$

and for even $q$, the $S$-invariant is

$$\frac{yg_2(y)}{f_2(y)^2}. \tag{13}$$

Since $\mathscr{S}$ is a spread, the line reguli $[[0, y, f_2(y), g_2(y)]]$ and $[[0, 0, 0, 0]]$ are compatible, so Lemma 2 says that for odd $q$

$$y^2 f_2(y)^2 - 4y^2(yg_2(y) + f_2(y)^2) = y^2(-3f_2(y)^2 - 4yg_2(y))$$

is a non-square, and for even $q$, $f_2(y) \neq 0$ and

$$\mathrm{Tr}\left(\frac{y^2 f_2(y)^2 + y^3 g_2(y)}{y^2 f_2(y)^2}\right) = \mathrm{Tr}(1) + \mathrm{Tr}\left(\frac{yg_2(y)}{f_2(y)^2}\right) = 1.$$

Thus when $q \equiv 1 \pmod 3$, if $q$ is odd then $-3$ is a square so the discriminant in (12) is non-square, and if $q$ is even then $\mathrm{Tr}(1) = 0$ so the $S$-invariant in (13) has trace one. Either way, $h_a(y) = 0$ has no solutions in $a$ for $y \neq 0$. Thus $K_a = \{0\}$ for all $a \in \mathrm{GF}(q)$ and the result follows for $q \equiv 1 \pmod 3$.

Now suppose $q \equiv 2 \pmod 3$. Then the discriminant in (12) is a nonzero square and the $S$-invariant in (13) has trace zero. Therefore, for each $y \neq 0$ there are exactly two values, $a$ and $b$, such that $y \in K_a$ and $y \in K_b$. Also, for each $a$, the function $h_a$ is a linear operator of $\mathrm{GF}(q)$ over $\mathrm{GF}(p)$ so $K_a$ is a vector space over $\mathrm{GF}(p)$ and $|K_a|$ is a power of $p$. Let $N_i$ be the number of values of $a$ for which $|K_a| = p^i$, where $i = 0, 1, \ldots, h$. Counting the nonzero elements in the sets $K_a$, we then have

$$(p-1)N_1 + (p^2-1)N_2 + \cdots + (p^h-1)N_h = 2(q-1). \tag{14}$$

Writing $N_{\geqslant i} = N_i + N_{i+1} + \cdots + N_h$, this gives us $(p-1)N_{\geqslant 1} \leqslant 2(q-1)$. Hence, if in addition to the flag $\{[\infty], (\infty)\}$ we have at least $\frac{2(q-1)}{p-1} + 1$ other flags on $[\infty]$ with respect to which $\mathscr{S}$ is a translation spread, then for at least one of these, say $\{[\infty], (a)\}$, we will have $K_a = \{0\}$. The result now follows for odd $q \equiv 2 \pmod 3$.

Finally, suppose $q = 2^{2e+1}$. From (14) we have $3N_{\geqslant 2} \leqslant 2(q-1)$, which we can tighten to $N_{\geqslant 2} \leqslant 2(q-2)/3$ since $N_{\geqslant 2}$ is an integer. Thus if, including the flag $\{[\infty], (\infty)\}$, the spread $\mathscr{S}$ is a translation spread with respect to at least $\frac{2(q-2)}{3} + 2 = 2(q+1)/3$ flags on $[\infty]$, then for at least one of these, say $\{[\infty], (a)\}$, we will have $|K_a| = 1$ or 2. If the group $H = \langle G_{\{[\infty],(\infty)\}}, G_{\{[\infty],(a)\}}\rangle$ has order $q^3$, as certainly happens when $|K_a| = 1$, then the result follows as in the previous cases, so we suppose now that $|K_a| = 2$ in which case $|H| = q^3/2$. The orbit of $[0,0,0,0,0]$ under $H$ is then

$$\mathscr{S}' = \{[x, y, z, f(x, y), g(x, y)] \mid x \in A \text{ and } y, z \in \mathrm{GF}(q)\}$$

for some subset $A \subseteq \mathrm{GF}(q)$ of order $q/2$, and the group $H$ is

$$H = \{\Theta[x, y, z, f(x, y), g(x, y)] \mid x \in A \text{ and } y, z \in \mathrm{GF}(q)\}.$$

The collineation $\Theta[X, 0, 0, f_1(X), g_1(X)] \in H$ sends the line regulus $[[x, 0, f_1(x), g_1(x)]]$ of $\mathscr{S}$ to $[[x+X, 0, f_1(x)+f_1(X), g_1(x)+g_1(X)]]$, which is then another line regulus of $\mathscr{S}$. Thus $f_1(x+X) = f_1(x) + f_1(X)$ for all $x \in \mathrm{GF}(q)$ and $X \in A$. Also, since the set $\mathscr{S}'$ is fixed by $H$, we have that for $x \in A$ and $X \in A$, the sum $x + X$ belongs to $A$ as well, so $A$ is a subgroup of $(\mathrm{GF}(q), +)$ of index 2. Consequently, if $x \notin A$ and $X \notin A$, then $x + X \in A$ so $f_1(x) = f_1((x+X)+X) = f_1(x+X) + f_1(X)$ and therefore $f_1(x+X) = f_1(x) + f_1(X)$. Similarly, $g_1(x+X) = g_1(x) + g_1(X)$. It now follows that the set $\{\Theta[x, y, z, f(x, y), g(x, y)] \mid x, y, z \in \mathrm{GF}(q)\}$ of collineations forms a subgroup of $G^{[\infty]}$ fixing $\mathscr{S}$. Since its order is $q^3$, this subgroup is the stabilizer $G^{[\infty]}_{\mathscr{S}}$ of $\mathscr{S}$ in $G^{[\infty]}$, and the spread $\mathscr{S}$ is a translation spread with respect to the line $[\infty]$ by [1, Lemma 5]. Also, we have $H \leqslant G_{[\infty]} \leqslant G^{[\infty]}_{\mathscr{S}}$ so either $G_{[\infty]} = G^{[\infty]}_{\mathscr{S}}$ or $G_{[\infty]} = H$, in which case $[G^{[\infty]}_{\mathscr{S}} : G_{[\infty]}] = 2$.

**Remark.** The exceptional case that $G_L \neq G^L_{\mathscr{S}}$ when $q = 2^{2e+1}$ does indeed occur. For example, in the hermitian spread $\mathscr{S}_H(1,1)$ in H[2], we have $G_{[\infty]} = G_{\{[\infty],(0)\}} = G_{\{[\infty],(1)\}} = G_{\{[\infty],(\infty)\}}$.

**Theorem 5.** *A spread $\mathscr{S}$ of* H$(q)$ *is a translation spread with respect to a line L if and only if the stabilizer $G^L_{\mathscr{S}}$ of $\mathscr{S}$ in $G^L$ has order $q^3$.*

*Proof.* This follows from Theorem 4 and [1, Lemma 5].

**Theorem 6.** *Let $\mathscr{S}$ be a spread of* H$(q)$, $q = p^h$, *that is represented as in* (10). *Then $\mathscr{S}$ is a translation spread with respect to the line $[\infty]$ if and only if the functions of the representation have the forms*

$$f(x, y) = \sum_{i=0}^{h-1}(f_{1i}x^{p^i} + f_{2i}y^{p^i}) \quad \textit{and} \quad g(x, y) = \sum_{i=0}^{h-1}(g_{1i}x^{p^i} + g_{2i}y^{p^i}),$$

*with the coefficients $f_{ni}, g_{ni} \in \mathrm{GF}(q)$.*

*Proof.* Suppose $\mathscr{S}$ is a translation spread with respect to $[\infty]$. Using Theorem 5, the stabilizer of $\mathscr{S}$ in $G^{[\infty]}$ is $G_{\mathscr{S}}^{[\infty]} = \{\Theta[x, y, z, f(x, y), g(x, y)] \mid x, y, z \in \mathrm{GF}(q)\}$. Since a collineation $\Theta[X, Y, Z, f(X, Y), g(X, Y)]$ from $G_{\mathscr{S}}^{[\infty]}$ maps the line regulus $[[x, y, f(x, y), g(x, y)]]$ of $\mathscr{S}$ to the line regulus $[[x + X, y + Y, f(x, y) + f(X, Y), g(x, y) + g(X, Y)]]$ which must also belong to $\mathscr{S}$, we see that $f(x + X, y + Y) = f(x, y) + f(X, Y)$ and $g(x + X, y + Y) = g(x, y) + g(X, Y)$. It now follows that the functions $f$ and $g$ have the claimed forms (see [9, Theorem 9.4.4]).

Suppose now that $f$ and $g$ have the given forms. Then the collineations $\Theta[x, y, z, f(x, y), g(x, y)]$ form a group of order $q^3$ fixing $\mathscr{S}$ and the result then follows from Theorem 5.

**Remark.** In addition to the representation in (10), a locally hermitian spread $\mathscr{S}$ may also be representable as

$$\bigcup_{x, y \in \mathrm{GF}(q)} [[x, f(x, y), g(x, y), y)]], \tag{15}$$

with $f(0, 0) = g(0, 0) = 0$. For example, using Lemma 2 it can be shown that this is always so when $q \not\equiv 2 \pmod 3$. If this representation is used rather than that in (10) then Theorem 6 holds without change.

Let $\mathscr{S}$ be a translation spread of $\mathrm{H}(q)$ with respect to the line $[\infty]$ represented as in (10). The *kernel* of $\mathscr{S}$ is defined to be the set

$$\begin{aligned} \ker \mathscr{S} = \{a \in \mathrm{GF}(q) \mid & f(ax, ay) = af(x, y) \text{ and} \\ & g(ax, ay) = ag(x, y) \text{ for all } x, y \in \mathrm{GF}(q)\}, \end{aligned}$$

which is then a subfield of $\mathrm{GF}(q)$. It can be shown that if $\theta$ is any collineation of $\mathrm{H}(q)$ that fixes the line $[\infty]$ and such that $\mathscr{S}^\theta$ contains the line $[0, 0, 0, 0, 0]$ so that $\mathscr{S}^\theta$ also has a representation as given in (10), then $\ker \mathscr{S}^\theta = \ker \mathscr{S}$. In addition, $\ker \mathscr{S}$ is independent of whether $\mathscr{S}$ is represented as in (10) or as in (15), when the latter of these applies.

## 5 Semi-classical spreads

Let $\mathscr{S}$ be a spread of $\mathrm{H}(q)$ that is locally hermitian in the line $L$. For each point $x$ on $L$, the spread $\mathscr{S}$ gives rise to an ovoid of $\mathrm{Q}(4, q)$ by a process called *projection along reguli* (see [1]). Specifically, for each of the $q^2$ line reguli on $L$ that comprise $\mathscr{S}$, there is a unique transversal through $x$, and these together with the line $L$ determine $q^2 + 1$ points of $\mathrm{Q}(4, q)$ that are then the points of an ovoid. We shall denote the resulting ovoid by $\mathcal{O}(x)$. In the event that the ovoids $\mathcal{O}(x)$ with $x \,\mathrm{I}\, L$ are all classical, the spread $\mathscr{S}$ is called *semi-classical*. Here now we give the following characterization of semi-classical spreads.

**Theorem 7.** *Let $\mathscr{S}$ be a locally hermitian spread of* $\mathrm{H}(q)$ *in the line $L$. Then the following are equivalent*:

(i) *$\mathscr{S}$ is semi-classical*;

(ii) *Two of the ovoids $\mathcal{O}(u)$ and $\mathcal{O}(v)$, with $u \neq v$, are classical*;

(iii) *$\mathscr{S}$ is a translation spread with respect to the line $L$ and* $\ker \mathscr{S} = \mathrm{GF}(q)$.

*Proof.* Without loss of generality, we suppose that coordinates are chosen such that $L$ is the line $[\infty]$ and such that $\mathscr{S}$ contains the line $[0,0,0,0,0]$, so $\mathscr{S}$ is represented as in (10). We begin by determining $\mathcal{O}(\infty)$.

Let $\Pi_4$ be the four dimensional space given by $X_0 = X_4 = 0$. Choose coordinates in $\Pi_4$ such that the point whose coordinates are $(0, x_1, x_2, x_3, 0, x_5, x_6)$ in $\mathrm{PG}(6, q)$ has coordinates $(-x_6, -x_1, x_3, x_5, x_2)$ in $\Pi_4$. Then the equation in $\Pi_4$ of the quadric $\mathscr{P}_4 = \mathscr{P}_6 \cap \Pi_4$ is as in Section 2.2, so we use Table 1 for the coordinatization of the quadrangle $\mathrm{Q}(4, q)$.

We must identify the point in $\mathrm{Q}(4, q)$ that is determined by a line regulus $[[x, y, f(x, y), g(x, y)]]$ of $\mathscr{S}$. The transversal $M$ of this line regulus through the point $(\infty)$ also passes through the point $(x, y, 0, f(x, y))$ of $\mathrm{H}(q)$. Using Table 2, the line $M$ is then determined by the two points whose coordinates in $\mathrm{PG}(6, q)$ are $(1, 0, 0, 0, 0, 0, 0)$ and $(yf(x, y), x, 1, y, 0, f(x, y), y^2 - xf(x, y))$, so $M$ meets $\Pi_4$ in the point whose coordinates in $\Pi_4$ are $(-y^2 + xf(x, y), -x, y, f(x, y), 1)$. Using Table 1, this is seen to be the point $(x, y, f(x, y))$ of the quadrangle $\mathrm{Q}(4, q)$. Similarly, the line $[\infty]$ of $\mathrm{H}(q)$ meets $\Pi_4$ in the point that is the point $(\infty)$ of $\mathrm{Q}(4, q)$. Thus the ovoid $\mathcal{O}(\infty)$ is given by

$$\mathcal{O}(\infty) = \{(\infty)\} \cup \{(x, y, f(x, y)) \mid x, y \in \mathrm{GF}(q)\}. \tag{16}$$

Next we take $\Pi'_4$ to be the subspace given by $X_2 = X_6 = 0$ and we choose coordinates in $\Pi'_4$ such that the point whose coordinates are $(x_0, x_1, 0, x_3, x_4, x_5, 0)$ in $\mathrm{PG}(6, q)$ has coordinates $(-x_0, x_1, -x_3, -x_5, x_4)$ in $\Pi'_4$. Then the equation in $\Pi'_4$ of the quadric $\mathscr{P}'_4 = \mathscr{P}_6 \cap \Pi'_4$ is as in Section 2.2, so again we use Table 1 for the correspondence with coordinates in the quadrange $\mathrm{Q}(4, q)$. Proceeding as in the previous paragraph, we then find that the ovoid $\mathcal{O}(0)$ is given by

$$\mathcal{O}(0) = \{(\infty)\} \cup \{(y, f(x, y), -g(x, y)) \mid x, y \in \mathrm{GF}(q)\}. \tag{17}$$

Suppose that (ii) holds. Without loss of generality, we may suppose that coordinates have been chosen such that the two points $u$ and $v$ are $(\infty)$ and $(0)$. Considering the ovoid $\mathcal{O}(\infty)$ in (16) and the representation of classical ovoids in (9), we see that the function $f(x, y)$ is linear in $x$ and $y$. Similarly from (17), the function $g(x, y)$ is linear in $f(x, y)$ and $y$, and therefore, in $x$ and $y$. It now follows from Theorem 6 that $\mathscr{S}$ is a translation spread with respect to the line $[\infty]$ and, by the linearity of the functions $f$ and $g$, its kernel is all of $\mathrm{GF}(q)$.

Consider now the case that (iii) holds. By Theorem 6 together with the analogous result for ovoids of $\mathrm{Q}(4, q)$, the ovoid $\mathcal{O}(\infty)$ is a translation ovoid with respect to the point $(\infty)$. Furthermore, in general we have $\ker \mathscr{S} \subseteq \ker \mathcal{O}(\infty)$. Since here $\ker \mathscr{S} = \mathrm{GF}(q)$, it follows that $\ker \mathcal{O}(\infty) = \mathrm{GF}(q)$, so the ovoid $\mathcal{O}(\infty)$ is a classical ovoid. Since coordinates may be chosen such that any particular point on $[\infty]$ is the point $(\infty)$, it follows that $\mathcal{O}(u)$ is a classical ovoid for every point $u$ on $[\infty]$ and we conclude that $\mathcal{S}$ is semi-classical.

Finally, notice that (ii) is implied by (i) simply by the definition of semi-classical, and this completes the cycle of implications.

**Corollary 8.** *Let $\mathcal{S}$ be a spread of* $\mathrm{H}(q)$, *with $q$ odd, that is a translation spread with respect to the line $[\infty]$. If* $\ker \mathcal{S} = \mathrm{GF}(q)$ *then $\mathcal{S}$ is either hermitian or isomorphic to the spread $\mathcal{S}_{[9]}$ of* [1].

*Proof.* In view of Theorem 7, this follows from the classification in [1, Theorems 30–32] of semi-classical spreads for odd $q$.

## 6 A new class of spreads

Just as Bloemen *et al.* discovered the spreads $\mathcal{S}_{[9]}$ while considering the semi-classical spreads of $\mathrm{H}(q)$ for odd $q$, so too has another new class of spreads arisen while investigating semi-classical spreads of $\mathrm{H}(q)$ for even $q$. These spreads, which are described by the following theorem, were discovered immediately after submission of [6] and have now already been known in the community for some time. Since they are non-hermitian, these spreads are not contained in a hyperplane of $\mathrm{PG}(6,q)$ (see [13]), and thus they provide us with a new class of 1-systems of the quadric $\mathcal{P}_6$ (see [11]). It has been remarked by Luyckx in [4] that these also yield new 1-systems of the polar space $W_5(2^{2e})$ by projection from the nucleus, and hence new semi-partial geometries.

**Theorem 9.** *Let $q = 2^{2e}$ and let $\delta$ be some fixed element of* $\mathrm{GF}(q)$ *with* $\mathrm{Tr}(\delta) = 1$. *Then the set*

$$\mathcal{S}_\delta = \bigcup_{x,y \in \mathrm{GF}(q)} \left[\left[x, y, \frac{\delta^3}{(\delta+1)^2}x + \frac{\delta}{\delta+1}y, \frac{\delta^3}{(\delta+1)^2}x + \frac{\delta^2}{\delta+1}y\right]\right] \tag{18}$$

*is a non-hermitian semi-classical spread of* $\mathrm{H}(q)$.

*Proof.* Notice that if $\mathcal{S}_\delta$ is a spread then its kernel is $\ker \mathcal{S}_\delta = \mathrm{GF}(q)$ and so it is semi-classical by Theorem 7. Also, $\mathcal{S}_\delta$ is not a hermitian spread as otherwise we would have from comparison with (11) that $\delta/(\delta+1) = 1$, which is absurd.

For each nonzero pair $(x,y) \in \mathrm{GF}(q)^2$, we apply Lemma 2 to the corresponding line regulus of $\mathcal{S}_\delta$ and $[[0,0,0,0]]$. Let $T(x,y)$ be the resulting expression in the trace in the statement of that lemma. By the linearity of the terms in (18), to show that $\mathcal{S}_\delta$ is a spread we have only to show that $\mathrm{Tr}(T(x,y)) = 1$ for all pairs $(x,y) \neq (0,0)$.

Applying Lemma 2 to the hermitian spread $\mathcal{S}_H(1,\delta)$ shows that

$$S(x,y) = \frac{A(x,y)}{B(x,y)^2} \tag{19}$$

has trace equal to 1 for all $(x,y) \neq (0,0)$, where

$$A(x, y) = \delta^3 x^4 + \delta^2 x^3 y + (\delta^2 + \delta) x^2 y^2 + (\delta + 1) x y^3 + \delta y^4,$$
$$B(x, y) = \delta x^2 + xy + y^2.$$

In terms of $A(x, y)$ and $B(x, y)$, the function $T(x, y)$ is found to be

$$T\left(\frac{x}{\delta}, \frac{y}{\delta + 1}\right) = \frac{A(x, y) + xy^3}{(B(x, y) + x^2)^2}. \tag{20}$$

In order to compare $T(x, y)$ more readily with $S(x, y)$, we make a change of variables so that the denominator in (20) is identical to that which appears in (19). Since $q = 2^{2e}$, there is an element $\theta \in \mathrm{GF}(q)$ such that $\theta^2 = \theta + 1$. Then $B(x, \theta x + y) + x^2 = B(x, y)$, so we let $T'(x, y) = T(x/\delta, (\theta x + y)/(\delta + 1))$. Now to show that $T(x, y)$ has trace equal to 1 for all nonzero pairs $(x, y)$, we have only to show the same for $T'(x, y)$, and since we know that this is so for $S(x, y)$, we can instead endeavour to show that $\mathrm{Tr}(S(x, y) + T'(x, y)) = 0$ for all nonzero pairs $(x, y)$. Now

$$S(x, y) + T'(x, y) = \frac{C(x, y)}{B(x, y)^2}$$

where

$$\begin{aligned} C(x, y) &= A(x, y) + A(x, \theta x + y) + x(\theta x + y)^3 \\ &= \delta^2 x^4 + (\delta + \delta\theta) x^3 y + \delta\theta x^2 y^2 + xy^3. \end{aligned}$$

It can be checked by direct substitution that $X = \delta\theta x^2 + xy$ is a solution of the quadratic equation $X^2 + B(x, y)X + C(x, y) = 0$ for all nonzero pairs $(x, y)$, thus this quadratic is always reducible. Consequently, we have $\mathrm{Tr}(C(x, y)/B(x, y)^2) = \mathrm{Tr}(S(x, y) + T'(x, y)) = 0$. It now follows that $\mathscr{S}_\delta$ is a spread of $\mathrm{H}(q)$.

**Additional note.** During the refereeing process of this paper, Cardinali *et al.* [2] announced a new construction of non-hermitian semi-classical spreads $S_\ell$ for $3 \nmid q$. As noted there, for odd $q$ the spreads $S_\ell$ are isomorphic to the spreads $\mathscr{S}_{[9]}$ of [1]. For even $q$, the spreads $S_\ell$ are isomorphic to the spreads $\mathscr{S}_\delta$ presented here, as is remarked in [5]. Thus they have provided $\mathscr{S}_\delta$ and $\mathscr{S}_{[9]}$ with a common description. It should be noted further that in [5], Luyckx and Thas go on to classify, for even $q$, the locally hermitian semi-classical 1-systems of the quadric $\mathscr{P}_6$ that do not lie in a hyperplane.

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