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Rigidity of skew-angled Coxeter groups

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Abstract. A Coxeter system is called skew-angled if its Coxeter matrix contains no entry equal to 2. In this paper we prove rigidity results for skew-angled Coxeter groups. As a consequence of our results we obtain that skew-angled Coxeter groups are rigid up to diagram twisting.

1 Introduction

Given a Coxeter matrix M over a set I, the corresponding Coxeter diagram $\Gamma(M)$ is the graph $(I, E(M))$ where $E(M)$ denotes the set of all 2-element subsets $\{i, j\}$ of I such that $m_{ij} \neq \infty$ and where each edge is labelled by the corresponding m_{ij}. We say that M is indecomposable if $\Gamma(M)$ is connected; we say that M is 1-connected if $\Gamma(M)$ is connected and if $\Gamma(M)$ remains connected if one vertex is removed. We further say that M is edge-connected if M is 1-connected and if $\Gamma(M)$ stays connected if the two vertices of an edge are removed. An edge of $\Gamma(M)$ is called a bridge, if it is not contained in a circuit of $\Gamma(M)$. We say that a Coxeter system is *skew-angled* if the associated Coxeter matrix is skew-angled, i.e. contains no entry equal to 2.

Let (W, S) be a Coxeter system. Following [5] we call a set $S' \subset W$ fundamental if (W, S') is a Coxeter system. In [3] it is defined what it means for two fundamental sets to be *twist equivalent*, see Definition 4.4 of [3] or Section 7 below. It is in particular very easy to decide whether for two given fundamental sets there exist twist equivalent sets that have isomorphic Coxeter graphs. Understanding the equivalence classes of fundamental sets of Coxeter groups would therefore solve the isomorphism problem. Our main result is the following:

Main Theorem. *Suppose that (W, S) is a skew-angled Coxeter system, let $T = S^W$ denote the set of its reflections and suppose that $\overline{S} \subset T$ is a fundamental set. Then the following hold:*

1. *There exists a fundamental set $S' \subset T$ that is twist equivalent to $\overline{S}$ and a bijection $\alpha : S \to S'$ such that α extends to an automorphism of W.*
2. *If $\Gamma(W, S)$ has no bridge, then one can choose $S = S'$ and $\alpha = \mathrm{id}_S$.*

3. *If* S *has at least* 3 *elements and if* $\Gamma(W,S)$ *is edge-connected, then there exists* $w \in W$ *such that* $S^w = \overline{S}$.

Reformulating statement (1) of the main theorem in the language of [3] we get the following corollary. It implies that Conjecture 8.1 in [3] holds in the skew-angled case.

Corollary A. *Skew-angled Coxeter systems are reflection-rigid up to diagram twisting.*

Remark 1. Let (W,S) be a Coxeter system and let $s,t \in S$ be two reflections corresponding to the vertices which are on a bridge of $\Gamma(W,S)$. If there is a non-trivial reflection-preserving outer automorphism α of $\langle s,t\rangle$ (like for instance in the case where st has order 5), then it has an extension to a reflection-preserving automorphism β of W and $\beta(S)$ is not twist equivalent to S because twistings are 'angle-preserving'.

If (W_1,S_1) and (W_2,S_2) are both skew-angled Coxeter systems then any isomorphism $\phi: W_1 \to W_2$ maps reflections onto reflections since the parabolic dihedral subgroups are the maximal finite subgroups and since any automorphism of a dihedral group D_{2n} with $n \geqslant 3$ maps reflections onto reflections. The theorem therefore gives a solution to the isomorphism problem for the class of skew-angled Coxeter groups. This can be rephrased as follows:

Corollary B. *Given two fundamental sets* S,S' *in a Coxeter group* W *such that* (W,S) *and* (W,S') *are skew-angled, then* $\Gamma(W,S)$ *and* $\Gamma(W,S')$ *are twist equivalent.*

If (W,S) is a skew-angled Coxeter system and R is a fundamental set of reflections, then $\Gamma(W,S)$ is twist equivalent to $\Gamma(W,R)$ by Corollary A. It is therefore easy to determine all Coxeter systems (W',S') such that W' is isomorphic to W if we can guarantee that each fundamental set R in W consists of reflections. This motivates the definition of *reflection-independence*. Following [1] we call a Coxeter group *reflection-independent* if $R \subseteq S^W$ for any two fundamental sets S and R of W. Our next result provides an easy criterion to see whether a skew-angled Coxeter group is reflection-independent. We call a vertex in a graph Γ an *end-point* if it is contained in precisely one edge; an edge is called a *spike* if it contains an endpoint.

Theorem (Reflection-independence criterion). *Let* (W,S) *be a skew-angled Coxeter system. Then* W *is reflection-independent if and only if there is no spike whose label is twice an odd number.*

As there are no spikes in an edge-connected graph, Part 3 of the main theorem and the previous theorem have the following consequence:

Corollary C. *Skew-angled Coxeter systems whose diagram has no spike which is labelled by twice an odd number are rigid up to diagram twisting* (*in the sense of* [3]);

they are strongly rigid (*in the sense of* [3]), *if there are at least* 3 *generators and the diagram is edge-connected.*

Remark 2. Complete graphs on at least 3 vertices are edge-connected and hence one recovers a slight generalization of a result of A. Kaul in [14]. We have further learned that F. Haglund [12] has obtained a proof of our main theorem under the additional assumption that the Coxeter graph is a complete graph.

Remark 3. The theorem about reflection-independence follows from Proposition 9.4 below. This proposition can be used to deduce an algorithm to decide for an arbitrary Coxeter system (W, S) whether there is a skew-angled fundamental set R of W.

In Section 2 we fix notation, we recall some definitions concerning the chamber system associated to a Coxeter system (i.e. its Cayley graph) and we deduce a crucial fact concerning roots and finite subgroups (cf. Lemma 2.6).

In Section 3 we consider reflections on thin chamber systems and we introduce *geometric sets of roots* in a thin chamber system. This notion is motivated by a result of Tits in [21]. Much of the content of Section 3 is certainly known to the experts as it is closely related to the results of M. Dyer [8] and V. Deodhar [7] on subgroups generated by reflections in Coxeter groups; the setup which is used here is however more similar to the revision of these results due to J.-Y. Hée [13].

In Section 4 we investigate universal sets of reflections in a Coxeter system, i.e. sets of reflections which constitute a Coxeter system with the subgroup W' they generate. Certain of these universal sets have the property that one can associate a root to each of its reflections such that the intersection of these roots is a fundamental domain for the subgroup W'; these are precisely the geometric sets of reflections.

In Section 5 we recall the definition of strong reflection-rigidity given in [3] and show that this definition is equivalent to a property of Coxeter systems which can be expressed by the notion of a geometric set of reflections.

In Section 6 we show that skew-angled Coxeter systems are strongly reflection-rigid if the underlying diagram is edge-connected by showing that they satisfy the equivalent definition given in Section 5. Here we use a special case of a result of R. Charney and M. Davis on rigidity of Coxeter groups (cf. [5]).

In Section 7 we use techniques introduced by M. Mihalik and S. Tschantz [15] to study how splittings of Coxeter groups over finite special subgroups behave with respect to different fundamental sets.

In Section 8 we give the proof of the main theorem by applying the results of Section 6 and Section 7.

In Section 9 we prove Proposition 9.4 which implies the reflection-independence criterion and which justifies Remark 3.

2 Preliminaries

Graphs. Let X be a set, then $P_2(X)$ denotes the set of all subsets of X having cardinality 2. A *graph* is a pair (V, E) consisting of a set V and a set $E \subseteq P_2(V)$. The elements of V and E are called *vertices* and *edges* respectively.

Let $\Gamma = (V, E)$ be a graph and let W be a subset of V; then Γ_W denotes the graph $(W, P_2(W) \cap E)$.

Let $\Gamma = (V, E)$ be a graph. Let v, w be two vertices of Γ. They are called adjacent if $\{v, w\} \in E$. A *path* from v to w is a sequence $v = v_0, v_1, \ldots, v_k = w$, where v_{i-1} is adjacent to v_i for all $1 \leqslant i \leqslant k$; the number k is the *length* of the path. The *distance* between v and w (denoted by $\delta(v, w)$) is the length of a shortest path joining them; if there is no path joining v and w, we put $\delta(v, w) = \infty$.

A path $v = v_0, v_1, \ldots, v_k = w$ is said to be *closed* if $v = w$; a closed path $v = v_0, \ldots, v_k = v$ is called a *circuit* if $v_1, \ldots, v_k$ are pairwise distinct and if $k \geqslant 2$; a circuit $v = v_0, v_1, \ldots, v_k = v$ is called *chordfree* if $E \cap P_2(\{v_1, \ldots, v_k\}) = \{\{v_0, v_1\}, \{v_1, v_2\}, \ldots, \{v_{k-1}, v_k\}\}$.

The relation $R \subset V \times V$ defined by $R = \{(v, w) \mid \delta(v, w) \neq \infty\}$ is an equivalence relation whose equivalence classes are called the *connected components* of Γ. A graph is said to be *connected* if it has only one connected component.

Let $v \in V$ and let W be the connected component of Γ which contains v. Then v is called a *cut-point* of Γ if $\Gamma_{W \setminus \{v\}}$ is not connected. The graph Γ is called 1-*connected* if it is connected and if there are no cut-points.

Let $e \in E$ be an edge of Γ and let W be the connected component of Γ which contains e. Then e is called a *cut-edge* if $\Gamma_{W \setminus e}$ is not connected. The graph Γ is called *edge-connected* if it is 1-connected and if there are no cut-edges.

We shall need the following facts about 1-connected graphs; the proof is straightforward.

Lemma 2.1. *Let $\Gamma = (V, E)$ be a* 1-*connected graph with at least three vertices. Then*

i) *every edge is contained in a chordfree circuit*;

ii) *given $v, w, w' \in V$ such that $\{v, w\}, \{v, w'\} \in E$ and such that $w \neq w'$, then there exists a sequence $w = w_0, w_1, \ldots, w_l = w'$ such that $\{v, w_i\} \in E$, such that $w_{i-1} \neq w_i$ and such that w_{i-1}, v, w_i are in a chordfree circuit for $1 \leqslant i \leqslant l$.*

Coxeter matrices, Coxeter diagrams and Coxeter systems. Let I be a finite set. A *Coxeter matrix over I* is a symmetric matrix $M = (m_{ij})_{i, j \in I}$ with entries in $\mathbf{N} \cup \{\infty\}$ such that $m_{ii} = 1$ for all $i \in I$ and $m_{ij} \geqslant 2$ for all $i \neq j \in I$. Given a Coxeter matrix M, we put $E(M) := \{\{i, j\} \subset I \mid 1 \neq m_{ij} \neq \infty\}$. The *Coxeter diagram associated to M* is the graph $(I, E(M))$ whose edges are labelled by the corresponding m_{ij}. The Coxeter matrix (and the associated Coxeter diagram) is called *indecomposable* if the associated diagram is connected.

The *incidence diagram* associated to a Coxeter matrix M is the graph $(I, E'(M))$ where $E'(M) = \{\{i, j\} \subset I \mid m_{ij} \geqslant 3\}$ and where the edges are labelled by the corresponding m_{ij}. The Coxeter matrix (and the Coxeter diagram) is called *irreducible* if the associated incidence diagram is connected.

Let M be a Coxeter matrix over I. A *Coxeter system* of type M is a pair (W, S) consisting of a group W and a set $S = \{s_i \mid i \in I\} \subseteq W$ such that S generates W and such that the relations $((s_i s_j)^{m_{ij}})_{i, j \in I}$ form a presentation of W. Given a Coxeter system

(W, S), an element of W is called a *reflection* if it is conjugate in W to an element of S; the set of all reflections is denoted by T.

Let M be a Coxeter matrix over I. Given a subset J of I, M_J denotes the restriction of M onto J. Let (W, S) be the Coxeter system of type M. We put $W_J = \langle s_j \mid j \in J \rangle$; it is a fact that $(W_J, \{s_j \mid j \in J\})$ is the Coxeter system of type M_J. The groups W_J are called the *special subgroups* of the Coxeter system (W, S); a *parabolic subgroup* is a subgroup which is conjugate to a special subgroup.

A Coxeter matrix (or diagram) is called *spherical* if the corresponding Coxeter group is finite. Given a Coxeter diagram M over I, a subset J of I is called *spherical* if M_J is spherical.

We close this subsection by recalling a well known fact [2].

Proposition 2.2. *Let M be a Coxeter matrix over a set I and let (W, S) be the corresponding Coxeter system. Given a finite subgroup U of W, there exists a spherical subset J of I and an element $w \in W$ such that U^w is a subgroup of W_J.*

The chamber system associated to a Coxeter system. Let (W, S) be a Coxeter system of type M where M is a Coxeter matrix over a set I. The chamber system associated to (W, S) is the graph $\mathbf{C} = (C, P)$, where $C = W$ and $\{c, d\} \in P$ if $c^{-1}d \in S$. The vertices of $\mathbf{C}$ are called *chambers*, the edges are called *panels*. Two chambers c, d are called *i-adjacent* if $c^{-1}d = s_i$. Since the s_i generate W the graph $\mathbf{C}$ is connected. Note that we have a natural mapping *type* : $P \to I$, defined by *type*$(\{c, d\}) = i$ if $c^{-1}d = s_i$. The group W acts from the left (via left translation) on $\mathbf{C}$. This action is regular on C and *type*-preserving on P.

Let $M, I, (W, S)$ and $\mathbf{C} = (C, P)$ be as before, let J be a subset of I and let $c \in C$. The *J-residue* of c is the set $R_J(c) := cW_J$. A *residue* is a subset of C which is a J-residue for some $J \subseteq I$. A residue is called *spherical* if it contains only finitely many chambers. In view of Proposition 2.2 and the regular action of W on C the following holds.

Lemma 2.3. *A subgroup U of W is finite if and only if it stabilizes a spherical residue.*

In view of the previous lemma the following proposition is a consequence of Proposition 5.5 in [6].

Proposition 2.4. *Let (W, S) be a Coxeter system and let $U \leqslant W$ be a finite subgroup of W. Let $\mathbf{S}_U$ denote the set of all spherical residues stabilized by U and consider the graph Γ_U whose set of vertices is $\mathbf{S}_U$ and where two vertices are joined by an edge if one contains the other. Then Γ_U is connected.*

Let M, I and (W, S) be as before. Given a reflection $t \in W$, we put $P(t) := \{p \in P \mid tp = p\}$ and $C(t) := \bigcup_{p \in P(t)} p$ (so $C(t)$ is the collection of all chambers c such that c and tc are adjacent).

Note that for any $w \in W$ and $t \in T$ we have $wP(t) = P(wtw^{-1})$ and $wC(t) = C(wtw^{-1})$.

Lemma 2.5. *Let (W, S) be a Coxeter system and let t be a reflection. Then the graph $\mathbf{C}_t = (C, P \backslash P(t))$ has two connected components.*

Proof. This follows from Proposition 2.6 in [19]. □

The two connected components of $\mathbf{C}_t$ are called the *roots associated to t*. Given a chamber c and a reflection t, $H(t, c)$ (resp. $-H(t, c)$) denotes the root associated to t, which contains c (resp. does not contain c). Given a root α, the reflection to which it is associated is uniquely determined and it is denoted by r_α. Moreover, we denote by $-\alpha$ the root which is associated to r_α and which is not equal to α. The set of roots will be denoted by $\Phi(W, S)$.

Roots and spherical residues. Let $t \in T$ be a reflection and $A \subseteq C$ be a residue. Then t stabilizes A if and only if both roots associated to t have non-trivial intersection with A. If this is not the case, then the unique root associated to t which contains (resp. does not contain) A will be denoted by $H(t, A)$ (resp. $-H(t, A)$).

Lemma 2.6. *Let $U \leqslant W$ be a finite subgroup of W and let $t \in W$ be a reflection such that $\langle t, U \rangle$ is an infinite group. Then there exists a (unique) root associated to t which contains each spherical residue stabilized by U.*

Proof. We define a graph Γ_U as in Proposition 2.4 from which we know that it is connected. By Lemma 2.3 there exists a spherical residue A stabilized by U. If t stabilizes A, then the group $V := \langle t, U \rangle$ stabilizes A and V is finite (again by Lemma 2.3). Therefore t does not stabilize A which is equivalent to the fact that A is contained in a root associated to t. This root will be called α.

We have to show that any residue fixed by U is contained in α. Let B be a spherical residue fixed by U. By the argument above it follows that B is contained in a root associated to t. Suppose now that B is not contained in α, i.e. that B is contained in $-\alpha$. As the graph Γ_U is connected (by Proposition 2.4) we can find a spherical residue B' that is stabilized by U such that $B' \cap \alpha \neq \emptyset \neq B' \cap -\alpha$. Now U and t stabilize the spherical residue B' and therefore $\langle t, U \rangle$ stabilizes B'. It then follows from Lemma 2.3 that $\langle t, U \rangle$ is finite. This yields a contradiction. □

The root of the previous lemma will be denoted by $H(t, U)$. We also consider pairs of reflections t, t' such that tt' has infinite order. For those pairs denote the unique root associated to t which contains all spherical residues fixed by $\langle t' \rangle$ by $H(t, t')$. Thus $H(t, t') = H(t, \langle t' \rangle)$.

The following observation is immediate:

Lemma 2.7. *Let (W, S) be a Coxeter system, let t be a reflection and let U be a finite subgroup of W such that $\langle t, U \rangle$ is infinite. Then each element $w \in W$ maps $H(t, U)$ onto $H(wtw^{-1}, wUw^{-1})$.*

3 Reflections of thin chamber systems

Let $\Gamma = (V, E)$ be a graph and let I be a set. An I-labelling of Γ is a mapping $\tau : E \to I$ whose restriction to the set of edges through any given vertex is a bijection. A *thin chamber system over* I is a pair $\mathscr{C} = (\Gamma, \tau)$ consisting of a connected graph $\Gamma = (V, E)$ and an I-labelling τ of Γ. Throughout this section $\mathscr{C} = (\Gamma, \tau)$ is a thin chamber system over I.

An automorphism of $\mathscr{C}$ is an automorphism of Γ which preserves the labelling; thus an automorphism of $\mathscr{C}$ fixing a vertex is already the identity. A *reflection of* $\mathscr{C}$ is an element $r \in \mathrm{Aut}(\mathscr{C})$ such that the following conditions hold:

(1) $r^2 = \mathrm{id} \neq r$ and

(2) if E_r denotes the set of fixed edges, then the graph $(V, E \backslash E_r)$ has two connected components.

Given a reflection r of $\mathscr{C}$, the two connected components of (2) are called *the roots associated to r* and the set $C(r) := \bigcup_{e \in E_r} e$ is called the *wall* associated to r.

A *root of* $\mathscr{C}$ is a subset ϕ of V such that there exists a reflection to which ϕ is associated as a root; this reflection is uniquely determined by ϕ and it is denoted by r_ϕ. Given a root ϕ of $\mathscr{C}$, then $-\phi := V \backslash \phi$ is also a root. Given a reflection r and a vertex v, let $H(r, v)$ denote the root associated to r which contains v. If v' is another vertex then we call v and v' *r-equivalent* if $H(r, v) = H(r, v')$; in this case we write $v \sim_r v'$.

In the following lemma we summarize some immediate observations:

Lemma 3.1. (1) *Let r be a reflection and $\alpha \in \mathrm{Aut}(\mathscr{C})$. Then $r' = \alpha \circ r \circ \alpha^{-1}$ is a reflection, $\alpha(C(r)) = C(r')$ and $\alpha(\phi)$ is a root associated to r' for each root ϕ associated to r.*

(2) *Each root is convex, i.e. each path of minimal length between two vertices of a root ϕ is contained in ϕ. In particular, intersections of roots are connected.*

Let X be a subgroup of $\mathrm{Aut}(\mathscr{C})$ and let $\mathrm{Refl}(X)$ denote the set of all reflections contained in X. Given $v, v' \in V$ we say that v and v' are *X-equivalent* if $v \sim_r v'$ for all $r \in \mathrm{Refl}(X)$ and we write $v \sim_X v'$ in this case. If $v \in V$ then a reflection r is called an *X-wall* of v if $r \in \mathrm{Refl}(X)$ and if there is a vertex $v' \sim_X v$ which is in $C(r)$. The following proposition is a direct consequence of Proposition 1 in [13]:

Proposition 3.2. *Let X and* $\mathrm{Refl}(X)$ *be as above and let $v_0 \in V$. Let R_0 denote the set of X-walls of v_0, let $W := \langle R_0 \rangle$ and let $l : W \to \mathbf{N}$ be the length function with respect to the generating set R_0. Then the following holds*:

a) $\mathrm{Refl}(X) \subseteq W$ *and (W, R_0) is a Coxeter system*;

b) *for each $r \in R_0$ and each $w \in W$ one has $l(rw) = l(w) + 1$ if and only if $wv_0 \sim_r v_0$ and $l(rw) = l(w) - 1$ if and only if $wv_0 \not\sim_r v_0$*;

c) *the action of W is sharply transitive on the X-equivalence classes of V.*

We shall need also the following observation:

Lemma 3.3. *The situation being as in the previous proposition let* $D := \{v \in V \mid v \sim_X v_0\}$. *Then* $D = \bigcap_{r \in R_0} H(r, v_0)$.

Proof. Let $D' := \bigcap_{r \in R_0} H(r, v_0)$. It is clear that $D \subseteq D'$ since $D = \bigcap_{r \in \mathrm{Refl}(X)} H(r, v_0)$. Let $v \in D' \backslash D$. As $v \in D'$ there exists a path $v_0 = x_0, \ldots, x_k = v$ in D' joining v_0 and v by the second part of Lemma 3.1. Let $0 < i \leqslant k$ be minimal for the property that x_i is not in D. It follows that there is a reflection $r \in X$ switching x_{i-1} and x_i. As x_{i-1} is X-equivalent with v_0 and $x_{i-1} \in C(r)$, it follows that $r \in R_0$ which implies $x_i \notin D'$. This contradicts the fact that the path $x_0, \ldots, x_k$ is contained in D'. Hence $D' \backslash D = \emptyset$ and we are done. □

Given any group G acting on a set M, then we call $F \subseteq M$ a *prefundamental domain* if $gF \cap F \neq \emptyset$ implies $g = 1$; we call F a *fundamental domain* if it is a prefundamental domain and if M is the union of the gF where g runs through G.

We obtain the following consequence from Proposition 3.2:

Corollary 3.4. *The situation (and notation) being as in Proposition* 3.2 *set* $D := \{v \in V \mid v \sim_X v_0\}$. *Then* $D = \bigcap_{r \in R_0} H(r, v_0)$ *and* D *is a fundamental domain for the action of* W *on* V. *Moreover, if* $r \in R_0$ *and* $w \in W$, *then* $wD \subseteq H(r, v_0)$ *or* $wD \subseteq -H(r, v_0)$; *in the first case we have* $l(rw) = l(w) + 1$, *in the second we have* $l(rw) = l(w) - 1$. *Finally, if* r *is a reflection in* X, *then there exists a reflection* r_0 *in* R_0 *which is* W-*conjugate to* r.

A *geometric pair of roots in* Γ is a set of two roots $\phi \neq \phi'$ such that $\phi \cap \phi'$ is a fundamental domain for the group $\langle r_\phi, r_{\phi'} \rangle$.

Lemma 3.5. *Let* $\phi_1 \neq \phi_2$ *be a geometric pair of roots and let* $D = \phi_1 \cap \phi_2$. *Put* $r_i := r_{\phi_i}$ *for* $i = 1, 2$ *and* $X := \langle r_1, r_2 \rangle$. *Let* $l : X \to \mathbf{N}$ *denote the length function with respect to the generating set* $\{r_1, r_2\}$. *Then* D *is an* X-*equivalence-class and the following holds for* $i = 1, 2$ *and all* $x \in X$: $xD \subseteq \phi_i$ *or* $xD \subseteq -\phi_i$, $l(r_i x) = l(x) + 1$ *if* $xD \subseteq \phi_i$ *and* $l(r_i x) = l(x) - 1$ *if* $xD \subseteq -\phi_i$.

Proof. Let $v_0 \in D$ and let R_0 be the set of X-walls of v_0. Since the group X is generated by reflections it follows by Proposition 3.2 that (X, R_0) is a Coxeter system and that $R_0 \subseteq \{r_1, r_2\}^X$. We conclude that R_0 has precisely two elements. Put $D' = \bigcap_{r \in R_0} H(r, v_0)$; it follows from Lemma 3.3 that $D' = \{v \in V \mid v \sim_X v_0\}$. Therefore $D' \subseteq D$. On the other hand D' and D are fundamental domains for the action of X on V and therefore equality holds. Hence D is an X-equivalence class. Let $r \in R_0$; as r is an X-wall of v_0 there is an element $v \in C(r) \cap D'$. Thus $v \in C(r) \cap D$ and $r(v)$ is not in $D' = D$; thus there exists an $i \in \{1, 2\}$ such that $r(v) \in -\phi_i$. As $v \in \phi_i$ and $v \in C(r)$ we conclude that $r = r_i$. This shows $R_0 = \{r_1, r_2\}$ and the assertions of the lemma follow now from Corollary 3.4. □

A set of roots Φ is called 2-*geometric*, if each 2-element subset of Φ is a geometric pair of roots; it is called *geometric* if it is 2-geometric and if $\bigcap_{\phi \in \Phi} \phi \neq \emptyset$.

The definition of a geometric set of roots is motivated by the following proposition which is a consequence of Lemme 1 in [21] and the previous Lemma 3.5.

Proposition 3.6. *Let Φ be a geometric set of roots. Let $D = \bigcap_{\phi \in \Phi} \phi$, put $R_0 = \{r_\phi \mid \phi \in \Phi\}$ and $W = \langle R_0 \rangle$. Then (W, R_0) is a Coxeter system and D is a fundamental domain for the action of W on V. If $v_0 \in D$, then R_0 is the set of W-walls of v_0.*

Lemma 3.7. *Let $\phi_1 \neq \phi_2$ be a geometric pair of roots. If $-\phi_1 \neq \phi_2$ is also a geometric pair of roots, then r_{ϕ_1} commutes with r_{ϕ_2}; in this case $\phi_1 \neq -\phi_2$ and $-\phi_1 \neq -\phi_2$ are also geometric pairs. If $D := -\phi_1 \cap -\phi_2 \neq \emptyset$, then r_{ϕ_1} and r_{ϕ_2} generate a finite group and $-\phi_1 \neq -\phi_2$ is a geometric pair as well.*

Proof. Set $r_1 = r_{\phi_1}$, $r_2 = r_{\phi_2}$ and $D = \phi_1 \cap \phi_2$. Let $X = \langle r_1, r_2 \rangle$ and let $l : X \to \mathbf{N}$ denote the length function corresponding to the generating set $\{r_1, r_2\}$. For $i = 1, 2$ and each $x \in X$ we have $l(r_i x) = l(x) - 1$ if and only if $xD \subseteq -\phi_i$ and $l(r_i x) = l(x) + 1$ if and only if $xD \subseteq \phi_i$ by Lemma 3.3.

Suppose that $-\phi_1 \neq \phi_2$ is a geometric pair of roots. Then $\phi_1 \cap \phi_2$, $-\phi_1 \cap \phi_2$, $r_2(\phi_1 \cap \phi_2) = r_2\phi_1 \cap -\phi_2$ and $r_2(-\phi_1 \cap \phi_2) = r_2(-\phi_1) \cap -\phi_2$ are X-equivalence classes which constitute a partition of V and hence X has 4 elements. Hence r_1 commutes with r_2 and $\{r_1, r_2\}$ is the set of reflections of X. Thus $\phi_1 \cap \phi_2$, $\phi_1 \cap -\phi_2$, $-\phi_1 \cap \phi_2$ and $-\phi_1 \cap -\phi_2$ are the four X-equivalence classes of V which shows that the four pairs involved are geometric.

Now suppose that $D' = -\phi_1 \cap -\phi_2 \neq \emptyset$ and choose $v \in D'$. There exists $x \in X$ such that $v \in xD$. It follows that $xD \subseteq D'$ and therefore $l(r_1 x) = l(r_2 x) = l(x) - 1$ which implies that X is finite. Suppose that there is $v' \in D'$ and let $x' \in X$ be the unique element such that $v' \in x'D$. Then $l(r_1 x') = l(r_2 x') = l(x') - 1$ and therefore $x = x'$ beause the longest element in X is unique. This shows that $D' = xD$ is a fundamental domain for X. Hence $-\phi_1 \neq -\phi_2$ is a geometric pair. $\square$

4 Universal and geometric sets of reflections

Throughout this section (W_0, S_0) is a Coxeter system, T denotes the set of its reflections, $\mathbf{C} = (C, P)$ is the associated chamber system and Φ denotes the set of roots. Note that the pair $\mathscr{C} = (\mathbf{C}, type)$ is a thin chamber system over I in the sense of the previous section. Moreover, the elements of T are reflections of the thin chamber system $\mathscr{C}$ in the sense of the previous section.

Let R be a subset of T. Then we put $M(R) = (o(rr'))_{r, r' \in R}$ where $o(rr')$ denotes the order of rr'. The set R is called *universal* if $(\langle R \rangle, R)$ is a Coxeter system.

Let Ψ be a set of roots. We put $R(\Psi) := \{r_\psi \mid \psi \in \Psi\}$ and $M(\Psi) := (o(r_\psi r_{\psi'}))_{\psi, \psi' \in \Psi}$.

A set R of reflections will be called *geometric* (resp. 2-*geometric*) if there exists a geometric (resp. 2-geometric) set Ψ of roots such that $R = R(\Psi)$; it will be called *sharp-angled* if each 2-element-subset of R is geometric. We note that 'sharp-angled' is weaker than '2-geometric'.

The following observation is immediate from the definitions:

Lemma 4.1. *Let R (resp. Ψ) be a geometric set of reflections (resp. roots) and let $R' \subseteq R$ (resp. $\Psi' \subseteq \Psi$). Then R' (resp. Ψ') is a geometric set of reflections (resp. roots).*

Proposition 4.2. *Let $R \subseteq T$ be a set of reflections and let $X = \langle R \rangle$. Then there exists a geometric set or reflections R_0 such that $\langle R_0 \rangle = X$ and $X \cap T = R_0^X$.*

Proof. The group X is generated by $\mathrm{Refl}(X)$ and therefore we have $W = X$ in Proposition 3.2 and Corollary 3.4. Let v_0, D and R_0 be as in Corollary 3.4 and put $\Psi := \{H(r_0, v_0) \mid r_0 \in R_0\}$. We claim that R_0 is geometric.

Let $r_0 \neq r_0' \in R_0$. We have to show that $\phi := H(r_0, v_0)$ and $\phi' := H(r_0', v_0)$ constitute a geometric pair. Obviously $\phi \neq \phi'$. Let Y be the group generated by r_0 and r_0' and let $D' = \phi \cap \phi'$. Suppose that $y(D') \cap D' \neq \emptyset$ for some $y \in Y$. If $l : W \to \mathbf{N}$ denotes the length function with respect to the Coxeter sytem (W, R_0) it follows that $l(r_0 y) = l(r_0' y) = l(y) + 1$. The length function on Y with respect to $\{r_0, r_0'\}$ is obtained by restricting l to Y because (X, R_0) is a Coxeter system by 3.2. Therefore it follows that y is the identity. This shows that D' is a prefundamental domain for the action of Y on C. As D' contains a Y-equivalence class of C and as each Y-equivalence class is a fundamental domain by Corollary 3.4 (applied to Y), it follows that D' is a fundamental domain for Y. $\square$

In the following three lemmas we summarize basic observations on subgroups generated by 2 reflections; the first two of them are immediate consequences of Lemma 3.7.

Lemma 4.3. *Let $t \neq t'$ be two reflections of the Coxeter system (W, S), and suppose $tt' = t't$. If α is a root associated to t and if α' is a root associated to t', then $\alpha \cap \alpha'$ is a fundamental domain for $\langle t, t' \rangle$. In particular the set $\{t, t'\}$ is geometric.*

Lemma 4.4. *Let $t \neq t'$ be two reflections of the Coxeter system (W, S) and suppose that $\{t, t'\}$ is geometric and that tt' has finite order strictly greater than 2. If α is a root associated to t, then there is a unique root α' associated to t' such that $\alpha \cap \alpha'$ is a fundamental domain for the group $\langle t, t' \rangle$; in this case the set $-\alpha \cap -\alpha'$ is a fundamental domain for $\langle t, t' \rangle$ as well.*

Lemma 4.5. *Let $t \neq t'$ be two reflections and suppose that tt' has infinite order. Then $H(t, t') \cap H(t', t)$ is a fundamental domain for the group $\langle t, t' \rangle$. In particular, the set $\{t, t'\}$ is geometric. Moreover we have $-H(t, t') \subseteq H(t', t)$, $-H(t', t) \subseteq H(t, t')$ and $-H(t, t') \cap -H(t', t) = \emptyset$; in particular, $H(t, t') \neq H(t', t)$ is the only geometric pair of roots associated to $\{t, t'\}$.*

Proof. By Proposition 4.2 there exists a geometric set of reflections R_0 such that $X := \langle t, t' \rangle = \langle R_0 \rangle$. As X is an infinite dihedral group, it follows that R_0 is X-conjugate to $\{t, t'\}$. Hence $\{t, t'\}$ is a geometric set of reflections. Hence there is a geometric pair of roots $\phi \neq \phi'$ such that $r_\phi = t$ and $r_{\phi'} = t'$; as tt' has infinite order it follows from Lemma 3.7 that $-\phi \cap -\phi' = \emptyset$. As $H(t,t') \cap H(t',t) \neq \emptyset \neq H(t,t') \cap -H(t',t)$ and $H(t',t) \cap -H(t,t') \neq \emptyset$ we conclude that $\phi = H(t,t')$ and $\phi' = H(t',t)$. The remaining assertions in the statement are now immediate. □

Lemma 4.6. *Let t, t', t'' be three pairwise distinct reflections of a Coxeter system (W,S) and suppose that tt', tt'' have infinite order and that $H(t,t') \neq H(t,t'')$. Then $t't''$ has infinite order. Moreover, if $t' = s_0, s_1, \ldots, s_k = t''$ is a sequence of reflections with the property that $s_{i-1}s_i$ has finite order for $1 \leqslant i \leqslant k$, then ts_l has finite order for some $l \in \{1, \ldots, k-1\}$.*

Proof. Suppose $t't''$ has finite order. Then there is a spherical residue A stabilized by $\langle t', t'' \rangle$; as A is stabilized by t' and by t'' it follows that A is contained in $H(t,t') \cap H(t,t'')$. As $H(t,t') = -H(t,t'')$, this intersection is empty and we have a contradiction. The second assertion is an immediate consequence of the first. □

Lemma 4.7. *Let $R \subseteq T$ be a finite sharp-angled set of reflections, suppose that $M(R)$ is irreducible and let Ψ, Ψ' be 2-geometric sets of roots such that $R = R(\Psi) = R(\Psi')$.*

a) *If $\Psi \cap \Psi' \neq \emptyset$, then $\Psi = \Psi'$.*

b) *If $\Psi \cap \Psi' = \emptyset$, then $\Psi' = \{-\psi \mid \psi \in \Psi\}$ and rr' has finite order for all $r, r' \in R$.*

Proof. This is a consequence of Lemmas 4.4 and 4.5. □

Lemma 4.8. *Let $R \subseteq T$ be a geometric set of reflections, suppose that $M(R)$ is irreducible and let $\Psi \neq \Psi'$ be geometric sets of roots such that $R = R(\Psi) = R(\Psi')$. Then $M(R)$ is spherical.*

Proof. By Part b) of the previous lemma we have $\Psi' = -\Psi$ and as $D = \bigcap_{\psi \in \Psi} \psi \neq \emptyset \neq \bigcap_{\psi \in \Psi} -\psi = D'$ we can find a chamber c in D'. As D is a fundamental domain for $W = \langle R \rangle$, it follows that there exists $w \in W$ such that $c \in w(D)$. By Proposition 3.6 R is the set of W-walls of any element in D; from Corollary 3.4 it follows that $l(r_\psi w) = l(w) - 1$ for all $\psi \in \Psi$ where l denotes the length function for the Coxeter system (W,R). This shows that W is finite and the claim follows. □

Proposition 4.9. *Let R be a geometric subset of T. If $M(R)$ is non-spherical and irreducible then there is a unique geometric set of roots Ψ such that $R(\Psi) = R$.*

Proof. This is an immediate consequence of Lemma 4.8. □

In the situation of the previous proposition the unique geometric set of roots will be denoted by $\Psi(R)$; for each element $r \in R$, the unique root associated to r which is in $\Psi(R)$ will be denoted by $H(r,R)$.

Lemma 4.10. *Let $\Psi = \Psi' \cup \{\pi\}$ be a geometric set of roots such that $M(\Psi)$ is not spherical and irreducible and that $M(\Psi')$ is spherical. Set $t = r_\pi$ and*

$U = \langle r_{\psi'} \mid \psi' \in \Psi' \rangle$. *Then* $\pi = H(t, U)$. *In particular, if* A *is a spherical residue stabilized by the group* U, *then* A *is contained in* π.

Proof. Let A be a spherical residue stabilized by U. If there is an infinity in $M(\Psi)$ then there is a root $\psi' \in \Psi'$ such that $r_\pi r_{\psi'}$ has infinite order. As $\pi \neq \psi'$ is a geometric pair of roots, it follows that $\pi = H(r_\pi, r_{\psi'}) = H(r_\pi, A)$ by Lemma 4.6. Thus we can assume that all entries in $M(\Psi)$ are finite.

As Ψ' is a geometric set of roots and as the group U stabilizes A it follows that $\bigcap_{\psi' \in \Psi'} \psi' \cap A$ is a fundamental domain for the action of U on A; in particular, it is not empty. It follows therefore that $(\bigcap_{\psi' \in \Psi'} -\psi') \cap A$ is not empty.

Suppose now that $\pi = -H(t, U)$. Then $A \subseteq -\pi$ and by the considerations above it follows that $\bigcap_{\psi' \in \Psi'} -\psi' \cap -\pi = \bigcap_{\psi \in \Psi} -\psi$ is not empty. On the other hand $\{-\psi \mid \psi \in \Psi\}$ is 2-geometric because Ψ is geometric; hence $\{-\psi \mid \psi \in \Psi\}$ is geometric. As Ψ is geometric and $M(\Psi)$ is not spherical we obtain a contradiction to Lemma 4.8. Thus $\pi = H(t, U)$ and A is contained in $H(t, U)$. □

5 Strong reflection-rigidity

We recall the definition of a strongly reflection-rigid Coxeter system as it is given in [3]: A Coxeter system (W, S) is called *strongly reflection-rigid* if the following holds for each Coxeter system (W', S') (whose set of reflections is denoted by T'): Given an isomorphism $\alpha : W \to W'$ with $\alpha(S) \subseteq T'$, then $\alpha(S)$ is W'-conjugate to S'. We call a Coxeter diagram strongly reflection-rigid if the associated Coxeter system is strongly reflection-rigid.

Lemma 5.1. *Let* (W, S) *be a Coxeter system, let* T *denote the set of reflections, let* $R \subseteq T$ *be universal and suppose that* $M(R)$ *is strongly reflection-rigid. Then* R *is geometric.*

Proof. Let W' be the subgroup generated by R and let $T' = T \cap W'$ denote the set of reflections in W'. By Proposition 4.2 there is a geometric set of reflections R' such that W' is the group generated by R' and such that T' is the set of reflections of the Coxeter system (W', R'). Now the identity on W' is an isomorphism mapping R onto a subset of T'. As $M(R)$ is strongly reflection-rigid it follows that we can find $w' \in W'$ such that $R' = R^{w'}$. This shows that R is geometric. □

We say that a Coxeter diagram M satisfies Condition (F) if the following is satisfied:

(F) Each universal set R of reflections in a Coxeter system (W, S) with $M(R) = M$ is geometric.

Proposition 5.2. *A Coxeter diagram is strongly reflection-rigid if and only if it satisfies Condition* (F).

Proof. Suppose that M satisfies condition (F), let (W, S) be a Coxeter system of type M, let (W', S') be an arbitrary Coxeter system whose set of reflections is T' and let $\alpha : W \to W'$ be an isomorphism such that $\alpha(S) \subseteq T'$. Then $\alpha(S)$ is a universal subset of reflections in W' and as M satisfies Condition (F), it follows that $\alpha(S)$ is a geometric subset of T'. Let $\Psi' \subseteq \Phi(W', S')$ be a geometric set of roots such that $\alpha(S) = R(\Psi')$. According to Proposition 3.6 $D' = \bigcap_{\psi' \in \Psi'} \psi'$ is a fundamental domain for the action of W' on its chamber system. This means that D' consists of one chamber and hence $\alpha(S)$ is W'-conjugate to S'. This shows one direction; the other direction is provided by the previous lemma. □

Let (W, S) be a Coxeter system and let T be the set of reflections of (W, S); we call a subset R of T a *chordfree circuit* if the Coxeter diagram associated to $M(R)$ is a chordfree circuit.

Lemma 5.3. *Let (W, S) be a Coxeter system, let T denote the set of its reflections and let $R \subseteq T$ be a universal set of reflections which is a chordfree circuit and which generates an infinite group. Then R is geometric.*

Proof. Since R is a cordfree circuit and $\langle R \rangle$ is infinite it follows that $\langle R \rangle$ is a cocompact reflection group of the hyperbolic plane $\mathbb{H}^2$ or the Euclidean plane $\mathbb{E}^2$, see for example [23]. In particular $\langle R \rangle$ acts effectively, properly and cocompactly on a contractible manifold. The main result of [5] then implies that the Coxeter system $(\langle R \rangle, R)$ is strongly reflection-rigid. Thus $M(R)$ is strongly reflection-rigid and R is geometric by Proposition 5.2. □

6 The edge-connected case

Throughout this section we have the following setup: (W, S) is a Coxeter system, T is the set of its reflections and $R \subseteq T$ is a universal set of reflections such that $|R| \geqslant 3$, such that $M(R)$ is skew-angled and such that the Coxeter diagram associated to $M(R)$ is edge-connected.

The goal of this section is to prove the following

Theorem 6.1. *The set R is geometric.*

In view of Proposition 5.2 the previous theorem has following consequence.

Corollary 6.2. *A skew-angled, edge-connected Coxeter diagram of rank at least 3 is strongly reflection-rigid.*

Lemma 6.3. *Given three pairwise distinct elements $r, s, t \in R$, the product $rsrt$ has infinite order.*

Proof. If the order of all products rs, rt, st is 3, then the three reflections generate the affine Coxeter group $\tilde{A}_2$ and the claim can be proved by considering the action of this group on the Euclidean plane. In the remaining cases one uses the solution of the word problem in Coxeter groups (cf. [20]). □

Since in the skew-angled case any chordfree circuit of a universal set generates an infinite group we have the following consequence of Lemma 5.3:

Lemma 6.4. *Each chordfree circuit $X \subseteq R$ is geometric.*

It is further clear that for any chordfree circuit $X \subseteq R$ the Coxeter matrix $M(X)$ is irreducible. It follows that there exists a unique geometric set of roots $\Psi(X)$ such that $X = R(\Psi(X))$ (by Proposition 4.9). We recall that for each reflection $r \in X$ the root which is contained in $\Psi(X)$ and associated to r is denoted by $H(r, X)$.

Lemma 6.5. *The set R is sharp-angled.*

Proof. Let r, s be two distinct reflections in R. If rs has infinite order, then there is nothing to prove (by Lemma 4.5); if rs has finite order, then, by Lemma 2.1 i), we can find a chordfree circuit $X \subseteq R$ containing s and r as $M(R)$ is edge-connected. By the previous lemma we know that X is geometric therefore $\{r, s\} \subseteq X$ is geometric by Lemma 4.1. □

Proposition 6.6. *Let $r, s \in R$ be two distinct reflections such that rs has finite order and let C, C' be two chordfree circuits of R which contain r and s. Then $H(r, C) = H(r, C')$.*

Proof. Let $C = \{r = t_0, t_1, \dots, t_k = s\}$ and $C' = \{r = t_0', t_1', \dots, t_l' = s\}$.

Suppose first that $t_1 = t_1'$. Then the group $\langle r, s, t_1 \rangle$ is infinite and the set $X = \{r, s, t_1\}$ is geometric, because it is a subset of the geometric set C (cf. Lemmas 4.1 and 6.4). This shows $H(r, C) = H(r, X) = H(r, C')$.

Suppose now that $t_1 \neq t_1'$. As $M(R)$ is edge-connected we can find a sequence $t_1 = s_0, s_1, \dots, s_m = t_1'$ such that $s_i \notin \{r, s\}$ and such that $s_{i-1}s_i$ has finite order for any $1 \leqslant i \leqslant m$.

Assume now that $H(r, C) = -H(r, C')$. As $\{H(r, C), H(s, C)\}$ and $\{H(r, C'), H(s, C')\}$ are both geometric pairs of roots, it follows from Lemma 4.4 that $H(s, C) = -H(s, C')$. Applying Lemma 4.10 twice it follows that $H(s, \langle r, t_1 \rangle) = H(s, C) = -H(s, C') = -H(s, \langle r, t_1' \rangle)$. Now we apply Lemma 2.7 with $w := r$ to obtain $H(rsr, \langle r, t_1 \rangle) = -H(rsr, \langle r, t_1' \rangle)$. By Lemma 6.3 we know that $rsrt_1$ and $rsrt_1'$ have both infinite order and therefore we obtain $H(rsr, t_1) = -H(rsr, t_1')$. By the second part of Lemma 4.6 there is an index j such that $rsrs_j$ has finite order which yields a contradiction to Lemma 6.3.

Hence $H(r, C) = H(r, C')$ and we are done. □

Corollary 6.7. *Given $r \in R$ and two chordfree circuits $C, C' \subseteq R$ which contain r, we have $H(r, C) = H(r, C')$.*

Proof. By Proposition 6.6 the assertion is true if there is a reflection s different from r which is contained in C and C'. If there is no such reflection we choose $s \in C$, $s' \in C'$

such that sr and rs' have finite order. Considering a sequence $s = s_0, \dots, s_l = s'$ as in Lemma 2.1 ii), the claim follows by an obvious induction. $\square$

Given a reflection $r \in R$, we define the root ψ_r by choosing a chordfree circuit $C \subseteq R$ which contains r and setting $\psi_r = H(r, C)$, and we put $\Psi = \{\psi_r \mid r \in R\}$; the previous corollary ensures that the roots $(\psi_r)_{r \in R}$ are well-defined, and as each edge of the diagram $M(R)$ is contained in a chordfree circuit (by Lemma 2.1 i)) we have the following.

Lemma 6.8. *If $r \neq s \in R$ are such that rs has finite order, then $\{\psi_r, \psi_s\}$ is a geometric pair of roots.*

We will now prove the same result for two reflections in R whose product has infinite order:

Lemma 6.9. *Let $r, s, t, \tau \in R$ be pairwise distinct reflections such that $tr, ts, t\tau$ have finite order and such that sr has infinite order and suppose that there are two chordfree circuits $X, X' \subseteq R$ containing $\{r, t, \tau\}$ and $\{s, t, \tau\}$ respectively. Then $H(r, s) = \psi_r$.*

Proof. Let $X' = \{t, \tau = s'_0, s'_1, \dots, s'_l = s\}$. By Lemma 6.3 it follows that $st\tau t$ and $rt\tau t$ have infinite order. As X is a chordfree circuit it follows from Lemma 4.10 that $\psi_r = H(r, X) = H(r, \langle t, \tau \rangle) = H(r, t\tau t)$.

Suppose that $\psi_r = -H(r, s)$. Then $H(r, t\tau t) = -H(r, s)$. Setting $\sigma_j = ts'_j t$ for $0 \leqslant j \leqslant l$ and $\sigma_{l+1} = s$ we have $H(r, \sigma_0) = H(r, t\tau t) = -H(r, s) = H(r, \sigma_{l+1})$ and $\sigma_{i-1}\sigma_i$ is of finite order for $1 \leqslant i \leqslant l+1$. It follows by Lemma 4.6 that $r\sigma_k$ has finite order for some $k \in \{1, \dots, l\}$. This contradicts Lemma 6.3 because $r\sigma_k = rts'_k t$. $\square$

Lemma 6.10. *Let $r, t, s \in R$ be such that rt and st have finite order and such that rs has infinite order. Then $\psi_r = H(r, s)$.*

Proof. By Lemma 2.1 there exists a sequence $r = s_0, \dots, s_l = s$ such that ts_i has finite order, such that $s_{i-1} \neq s_i$ and such that $\{s_{i-1}, t, s_i\}$ is contained in some chordfree circuit X_i for $1 \leqslant i \leqslant l$. We choose such a sequence with l minimal.

If $l = 1$ we have $\psi_r = H(r, X_1) = H(r, s)$ because $s \in X_1$ (cf. Lemma 4.10). If $l = 2$ the assertion follows by the previous lemma.

Suppose $l > 2$. Then rs_j has infinite order for $2 \leqslant j \leqslant l$ by the minimality of l and $\psi_r = H(r, s_2)$ by the previous lemma. Suppose now that $H(r, s) = -\psi_r$. Then there exists $j \in \{3, \dots, l\}$ such that $H(r, s_{j-1}) = -H(r, s_j)$. Let $X_j = \{t, s_{j-1} = \sigma_0, \dots, \sigma_k = s_j\}$; by Lemma 4.6 there exists $i \in \{1, \dots, k\}$ such that $r\sigma_i$ has finite order. Choosing $i \in \{1, \dots, k\}$ maximal for this property, we obtain a chordfree circuit $\{t, r, \sigma_i, \sigma_{i+1}, \dots, \sigma_k = s_j\}$ contradicting the minimality of l. $\square$

Proposition 6.11. *Let $s, r \in R$ be such that sr has infinite order. Then $\psi_r = H(r, s)$ and $\psi_r \cap \psi_s$ is a fundamental domain for $\langle r, s \rangle$. In particular, $\{\psi_r, \psi_s\}$ is geometric.*

Proof. As the Coxeter diagram associated to $M(R)$ is connected, we have a sequence $r = s_0, \dots, s_k = s$ in R such that $s_{i-1}s_i$ has finite order for $1 \leqslant i \leqslant k$; we choose a sequence with k minimal. The minimality of k implies that rs_i has infinite order for $2 \leqslant i \leqslant k$. The previous lemma yields $\psi_r = H(r, s_2)$. As $H(r, s_{i-1}) = H(r, \langle s_{i-1}, s_i \rangle) = H(r, s_i)$ for $3 \leqslant i \leqslant k$ (by Lemma 4.10) an obvious induction shows $H(r, s) = \psi_r$.

In view of Lemma 4.5 the second assertion is an immediate consequence of the first. □

Corollary 6.12. *Let $r \neq s \in R$ be such that rs has finite order, let A be a spherical residue stabilized by $\langle r, s \rangle$ and let $t \in R \backslash \{r, s\}$. Then $A \subseteq \psi_t$ and $\bigcap_{\psi \in \Psi} \psi \neq \emptyset$.*

Proof. If rt and st have finite order, then $\{r, s, t\}$ is a chordfree circuit. Therefore it follows from Lemma 4.10 that $\psi_t = H(t, \langle r, s \rangle)$ and hence $A \subseteq \psi_t$.

If rt has infinite order, then $\psi_t = H(t, r)$ by the previous proposition and as r stabilizes A it follows that $A \subseteq \psi_t$. If st has infinite order the same argument applies and the first assertion is proved.

As A is a spherical residue stabilized by r and s it follows that $Y = A \cap \psi_r \cap \psi_s \neq \emptyset$ and as $A \subseteq \psi_t$ for all $t \in R \backslash \{r, s\}$ it follows that $\emptyset \neq Y \subseteq \bigcap_{\psi \in \Psi} \psi$. □

Proof of Theorem 6.1. It follows from Lemma 6.8 and Proposition 6.11 that the set $\Psi = \{\psi_r \mid r \in R\}$ is 2-geometric. Moreover, by Corollary 6.12 we have $\bigcap_{r \in R} \psi_r \neq \emptyset$ and therefore Ψ is geometric; as $R = R(\Psi)$ it follows that R is geometric.

7 Visual decompositions and diagram twisting

We study decompositions of Coxeter groups as fundamental groups of graphs of groups. We therefore apply the ideas of M. Mihalik and S. Tschantz [15]. Suppose that (W, S) is a Coxeter system. Following [15] we call a splitting of W as a fundamental group of a graph of groups $\mathbb{A}$ *visual* (*with respect to* S) if every edge and vertex group is *special*, i.e. is generated by a subset of S. It is clear that $\mathbb{A}$ must be a tree of groups since W is generated by elements of finite order and therefore admits no non-trivial homomorphism to $\mathbb{Z}$; W can therefore not be an HNN-extension. We use the following facts from [15], the second is actually a corollary of the first. Note that the definition of T_S in Proposition 7.1 below makes sense since any $s \in S$ is of finite order and therefore acts with a fixed point. The fact that W acts without fixed point guarantees its uniqueness.

Proposition 7.1 (Mihalik, Tschantz). *Suppose that (W, S) is a Coxeter group that acts simplicially without inversion on a simplicial tree T such that W fixes no vertex of T. Then*

$$W = \pi_1(\mathbb{A})$$

where $\mathbb{A} = (T_S, \{G_e \mid e \in ET_S\}, \{G_v \mid v \in VT_S\}, \{\phi_e \mid e \in ET_S\})$ is the graph of groups where the objects are defined as follows: *$T_S \subset T$ is the unique minimal tree such that* *for any $s \in S$ there exists $x \in T_S$ such that $sx = x$. The vertex and edge groups are defined as $G_e = \langle s \in S \mid se = e \rangle$ for each edge e of T_S and $G_v = \langle s \in S \mid sv = v \rangle$ for each vertex v of T_S. All boundary monomorphisms are simply the inclusion maps.*

Suppose now that a Coxeter group splits as a proper amalgamated product $W = A *_C B$. Then W acts on the associated Bass–Serre tree T. Since the amalgamated product is proper it follows that W acts without a fixed point. Proposition 7.1 therefore guarantees that W splits visually over a subgroup that fixes an edge of T. Since any edge stabilizer is conjugate to C we have the following:

Theorem 7.2 (Mihalik, Tschantz). *Let (W, S) be a Coxeter group and suppose that W splits as a proper amalgamated product $W = A *_C B$. Then there exists a proper decomposition $W = A' *_{C'} B'$ that is visual with respect to S such that C' is conjugate to a subgroup of C.*

We recall the notion of *diagram twisting* as defined in [3]. Note that the operations we describe here are only a subset of the operations defined in [3], but that they coincide when we restrict our attention to skew-angled Coxeter groups.

Suppose that a Coxeter group (W, S) splits visually as an amalgamated product $W = A *_C B$ where C is a finite subgroup. This means that we have sets $S_1, S_2 \subset S$ such that $A = \langle S_1 \rangle$, $B = \langle S_2 \rangle$ and $C = \langle S_1 \cap S_2 \rangle$. Let w be the longest element of the Coxeter group $(C, S_1 \cap S_2)$. This implies that $w(S_1 \cap S_2)w^{-1} = S_1 \cap S_2$. It follows that

$$W = A *_C B = A *_{wCw^{-1}} wBw^{-1} = A *_C wBw^{-1}$$

where $W = A *_C wBw^{-1}$ is visual with respect to the set $S_1 \cup wS_2w^{-1}$ which is fundamental. This is clear since wS_2w^{-1} is obviously fundamental for wBw^{-1} and since $S_1 \cap S_2 = S_1 \cap wS_2w^{-1}$ is fundamental for $C = A \cap B = A \cap wBw^{-1}$. We say that the fundamental sets S and $\overline{S} = S_1 \cup wS_2w^{-1}$ are *elementarily equivalent*. We further say that fundamental sets S and $\overline{S}$ are *twist equivalent* if there exists a finite sequence of fundamental sets $S = S^1, S^2, \dots, S^{k-1}, \overline{S} = S^k$ such that S^i and S^{i+1} are elementarily equivalent for $1 \leqslant i \leqslant k$. Note that we do not require the amalgamated product to be proper, i.e. possibly $A = C$. This implies that conjugate fundamental sets are equivalent.

The names *diagram twisting* and *twist equivalent* stem from the fact that the diagrams $\Gamma(W, S)$ and $\Gamma(W, \overline{S})$ are related by a twisting operation. Namely both $\Gamma(W, S)$ and $\Gamma(W, \overline{S})$ are obtained from $\Gamma(A, S_1)$ and $\Gamma(B, S_2)$ by identifying the subdiagrams $\Gamma(C, S_1 \cap S_2)$. In the first case the identification is the identity, in the second by the automorphism induced by conjugation with the longest element of C.

Remark. Suppose that (W, S), $W = A *_C B$ and S_1 and S_2 are as above. Instead of replacing $S = S_1 \cup S_2$ by $S_1 \cup wS_2w^{-1}$ we can replace it by $w^{-1}S_1w \cup S_2$. The resulting diagram is clearly isomorphic to the first one since the two sets are conjugate. Since any finite special subgroup is generated by either a subset of S_1 or of S_2 this implies that any twisting operation on the diagram level can be realized such that any finite given spherical subset of S is preserved.

This implies that the twist equivalences preserve angles, i.e. that if a fundamental set $\overline{S}$ is obtained from a fundamental set S of W then any generating pair $\{s_1, s_2\} \subset S$ of a finite dihedral group gets replaced with a pair $\{\bar{s}_1, \bar{s}_2\} \subset \overline{S}$ such that $\{s_1, s_2\}$ and $\{\bar{s}_1, \bar{s}_2\}$ are conjugate in W.

The proof of the main theorem is by induction on $\#S$, the cardinality of S. We can assume that $\Gamma(W, S)$ is not edge-connected otherwise the results follows from Section 6. In the case that $\Gamma(W, S)$ is not edge-connected W decomposes visually as an amalgamated product $A *_C B$ where A, B and C are special subgroups of (W, S) and C is either trivial or of order 2 or dihedral. In particular the Coxeter generating sets of A and B are of smaller cardinality than S, i.e. we can assume that the respective results hold for each factor. We therefore need to study how a given visual splitting behaves with respect to another fundamental set.

Crucial to our arguments later is the following lemma which is a consequence of the work of V. V. Deodhar [6].

Lemma 7.3. *Let (W, S) be a Coxeter system and suppose that the decomposition $W = G_1 *_{C_1} G_2 *_{C_2} G_3$ is visual with respect to S. Suppose further that C_1 is finite and that $g_2 C_2 g_2^{-1} = C_1$ for some $g_2 \in G_2$ (possibly $C_1 = C_2$ and $g_2 \neq 1$). Then there exists a fundamental set S' such that S is twist equivalent to S' and such that the splitting $W = G_1 *_{C_1} G_2 *_{g_2 C_2 g_2^{-1}} g_2 G_3 g_2^{-1} = G_1 *_{C_1} G_2 *_{C_1} g_2 G_3 g_2^{-1}$ is visual with respect to S'.*

Proof. By assumption C_1 and C_2 are special subgroups of the Coxeter group (G_2, S') where $S' = S \cap G_2$. By Proposition 5.5 of [6] there exist sequences $C_1 = U_1, U_2, \dots, U_{k-1}, U_k = C_2$ and $W_1, \dots, W_{k-1}$ of finite special subgroups of (G_2, S') such that $U_i, U_{i+1} \subset W_i$, such that $U_i = w_i U_{i+1} w_i^{-1}$ where w_i is the longest element of W_i and such that $g_2 = w_1 \cdot \ldots \cdot w_{k-1}$. This clearly implies that the $k-1$ diagram twists give the assertion of the lemma. □

Lemma 7.4. *Let (W, S) be Coxeter system and C be a finite special subgroup such that W does not split over a proper subgroup of C. Then there exists a finite decomposition $W = \underset{i \in I}{*_C} G_i$ such that the following hold:*

1. *G_i does not split over a subgroup of C for $i \in I$.*
2. *$W = \underset{i \in I}{*_C} G_i$ is visual with respect to a fundamental set S' that is twist equivalent to S.*

Proof. We start with the trivial splitting, i.e. we put $G_1 = W$. In particular we have a fundamental set $S' = S$ that is twist equivalent to S and a splitting $W = \underset{i \in I}{*_C} G_i$ that is visual with respect to this fundamental set.

If none of the G_i splits over a subgroup of C there is nothing to show. If some G_i splits over a subgroup of C we show how to replace S' by a twist equivalent set again denoted by S' and how to refine the splitting $W = \underset{i \in I}{*_C} G_i$ by replacing G_i by two new factors such that the obtained splitting is visual with respect to the new fundamental set. Since S is finite this process terminates and yields a decomposition and a set S' with the desired properties.

Suppose that there exists $i \in I$ such that G_i splits over a subgroup of C. If G_i splits over a proper subgroup C' of C then W also splits over C' since both C and C' are finite which contradicts our assumption. It follows that G_i splits over C. Theorem 7.2 implies that G_i splits visually over a subgroup C' that is conjugate to C. Lemma 7.3 now guarantees that we can replace S' with a equivalent set such that $C' = C$. This means that we can refine the visual splitting by replacing G_i with two factors. □

Lemma 7.5. *Let* (W, S) *and* C *be as in Lemma* 7.4. *Choose a set* S' *and a decomposition* $W = \ast_{C\, i \in I} G_i$ *as in the conclusion of Lemma* 7.4. *Let* $\overline{S} \subset T$ *be a fundamental set of generators of* W. *Then there exists a fundamental set* $\overline{S}'$ *that is equivalent to* $\overline{S}$ *such that the splitting* $W = \ast_{C\, i \in I} G_i$ *is visual with respect to* $\overline{S}'$.

Proof. We consider the amalgamated product $W = \ast_{C\, i \in I} G_i$ as the graph of groups whose underlying graph has vertex set $\{x\} \cup I$, edge set $\{[x, i] \mid i \in I\}$, vertex groups G_i for $i \in I$ and $G_x = C$ and all edge groups equal to C. The boundary monomorphisms are the inclusion maps. We consider the action of W on the Bass–Serre tree T with respect to this splitting.

We assume that among all sets that are equivalent to $\overline{S}$ the set $\overline{S}$ is the one such that the tree $T_{\overline{S}}$ (of Proposition 7.1) has the smallest complexity, i.e. the least number of edges. We choose the associated graph of groups $\mathbb{A} = (T_{\overline{S}}, \{G_e \mid e \in ET_{\overline{S}}\}, \{G_v \mid v \in VT_{\overline{S}}\}, \{\phi_e \mid e \in ET_{\overline{S}}\})$ as in Proposition 7.1.

Claim 1. $G_e = \mathrm{Stab}_W(e)$ for every edge $e \in ET_{\overline{S}}$.

Proof. Suppose that G_e is a proper subgroup of $\mathrm{Stab}_W(e)$ for some $e \in ET_{\overline{S}}$. This implies that W splits over a proper subgroup of $\mathrm{Stab}_W(e)$. Since the stabilizer of any edge is conjugate to C this implies that W splits over a proper subgroup of C which contradicts our assumption. This proves the claim.

Claim 2. If $v \in VT_{\overline{S}}$, $e_1, e_2 \in ET_{\overline{S}}$ and $e_1 \neq e_2$ then e_1 and e_2 are not G_v-equivalent.

Proof. Suppose that e_1 and e_2 are G_v equivalent, i.e. that there exists a $g_v \in G_v$ such that $g_v e_1 = e_2$. Since our ambient space is a tree we can clearly assume that both e_1 and e_2 are incident with v. Since G_{e_1} and G_{e_2} are the full edge stabilizers of e_1 and e_2 we have $G_{e_1} = g_v^{-1} G_{e_2} g_v$. After collapsing all edges in the graph of groups except e_1 and e_2 we see that W splits as an amalgamated product $W = W_1 \ast_{G_{e_1}} W_2 \ast_{G_{e_2}} W_3$. Because of Lemma 7.3 we can pass to an equivalent set $\overline{S}'$ such that the splitting $W = W_1 \ast_{G_{e_1}} W_2 \ast_{G_{e_2}} \ast W_3 = W_1 \ast_{G_{e_1}} W_2 \ast_{g_v^{-1} G_{e_2} g_v} g_v^{-1} W_3 g_v$ is visual with respect to $\overline{S}'$. It is clear that the tree $T_{\overline{S}'}$ is contained in the tree $T_1 \cap g_v^{-1} T_2$ where T_1 is the component of $T_{\overline{S}} - e_2$ that contains v and T_2 is the component that does not contain v. (Note that $T_1 \cap g_v^{-1} T_2$ is connected since both sets contain the terminal vertex of e_1

different from v). The tree $T_1 \cap g_v^{-1} T_2$ however has smaller complexity than $T_{\bar{S}}$ which contradicts the minimality assumption. Thus the claim is proven.

The two claims imply that W is controlled by the tuple $(T_{\bar{S}}, T_{\bar{S}}, \{G_v \mid v \in VT\bar{S}\}, \emptyset)$ in the sense of [9]. It therefore follows from Proposition 5 of [9] that the splitting associated to $T_{\bar{S}}$ is the induced splitting of W, i.e. the splitting that W inherits from the action on the tree. Since the group action stems from a decomposition of W itself the induced splitting must recover the original splitting of W. It follows that after conjugation the claim of the lemma holds. □

8 The proof of the main theorem

In this section we give the proof of the main theorem. If $\Gamma(W, S)$ is edge-connected then the three conclusions of the main theorem follow directly from Theorem 6.1.

We first show that the first two conclusions hold if $\Gamma(W, S)$ is 1-connected, i.e. that any fundamental set of reflections is twist equivalent to S. Since twist equivalence is an equivalence relation we can clearly also modify S within its equivalence class. The proof is by induction on $\#S$, the cardinality of S.

If $\Gamma(W, S)$ is edge connected there is nothing to show, i.e. we can assume that $\Gamma(W, S)$ contains a cut edge. It follows that W splits visually (with respect to S) as an amalgamated product $A *_{D_{2n}} B$. Now W does not split over a proper subgroup of D_{2n} since then W would split visually over the trivial group or a group of order 2 by Theorem 7.2 which contradicts our assumption that $\Gamma(W, S)$ is 1-connected.

Let now $W = \mathop{*_{D_{2n}}}\limits_{i \in I} G_i$ be a maximal decomposition as in the conclusion of Lemma 7.4. After replacing S and $\bar{S}$ by equivalent sets we know that this decomposition is visual with respect to S and $\bar{S}$ because of Lemma 7.5. After joining factors we have a new decomposition $A *_{D_{2n}} B$ that is visual with respect to both S and $\bar{S}$. This means there are sets $S_A, S_B, S_C, \bar{S}_A, \bar{S}_B$ and $\bar{S}_C$ such that $S = S_A \cup S_B$, $S_C = S_A \cap S_B$, $\bar{S} = \bar{S}_A \cup \bar{S}_B$, $\bar{S}_C = \bar{S}_A \cap \bar{S}_B$, that $A = \langle S_A \rangle = \langle \bar{S}_A \rangle$, $B = \langle S_B \rangle = \langle \bar{S}_B \rangle$ and $C = D_{2n} = \langle S_C \rangle = \langle \bar{S}_C \rangle$. Clearly $\Gamma(A, S_A)$ and $\Gamma(B, S_B)$ are also 1-connected. By induction we know that S_A and $\bar{S}_A$ are equivalent. Since twist equivalence preserves angles (use the remark of Section 7 after Theorem 7.2) this implies that $S_C = \{s_1, s_2\}$ and $\bar{S}_C = \{\bar{s}_1, \bar{s}_2\}$ are conjugate in C, i.e. that $s_1 s_2$ and $\bar{s}_1 \bar{s}_2$ are rotations about the same angle. It follows that after conjugating $\bar{S}$ with an element of C, i.e. after some twisting operations, we can assume that $S_C = \bar{S}_C$.

By induction we know that S_A is equivalent to $\bar{S}_A$ and that S_B is equivalent to $\bar{S}_B$, i.e. $\Gamma(A, S_A)$ can be obtained from $\Gamma(A, \bar{S}_A)$ by a finite number of twisting operations and $\Gamma(B, S_B)$ can be obtained from $\Gamma(B, \bar{S}_B)$ by a finite number of twisting operations. Since C is a dihedral group, the subdiagrams of $\Gamma(A, \bar{S}_A)$ and $\Gamma(B, \bar{S}_B)$ corresponding to C are preserved by the twists.

After these operations we have sets $\bar{S}_A$ and $\bar{S}_B$ that contain S_C and are conjugate to S_A and S_B, respectively. Since the subgroup S_C is dihedral and since in skew-angled Coxeter groups edges are in 1-to-1 correspondence with conjugacy classes of parabolic dihedral groups and parabolic dihedral groups are self-normalizing it follows that the conjugacy factors must lie in $C = \langle S_C \rangle$. As this conjugation must further preserve S_C, it must either be the longest element in C or be trivial. This implies that possibly after twisting along S_C we have $S = S_A \cup S_B = \bar{S}_A \cup \bar{S}_B$.

We proceed with the case where $\Gamma(W, S)$ is connected but not 1-connected and contains no bridge. Again we have to show that any fundamental set of reflections if twist equivalent to S. The proof is again by induction on the complexity of the Coxeter system.

Clearly W splits visually over a subgroup of order 2. As before we find a decomposition $W = A *_C B$ that is visual with respect to S and $\bar{S}$ after replacing them by twist equivalent sets. We choose $S_A, S_B, S_C, \bar{S}_A, \bar{S}_B$ and $\bar{S}_C$ as before. Since C is of order 2 we clearly have $S_C = \bar{S}_C = \{c\}$ for some reflection $c \in S$.

It is clear that neither $\Gamma(A, S_A)$ nor $\Gamma(B, S_B)$ contains a bridge, i.e. the induction hypothesis holds. As before we see that we can replace $\bar{S}_A$ and $\bar{S}_B$ by equivalent sets again denoted by $\bar{S}_A$ and $\bar{S}_B$ such that $\{c\} \subset \bar{S}_A$, $\{c\} \subset \bar{S}_B$ and that S_A is conjugate to $\bar{S}_A$ and S_B is conjugate to $\bar{S}_B$. We give the argument for S_A, the case of S_B is analogous. Suppose that the conjugacy factor is a, i.e. $aS_Aa^{-1} = \bar{S}_A$. Since C is special with respect to $\bar{S}_A$ we have $c \in \bar{S}_B$, since $c \in S_A$ we have that $aca^{-1} \in \bar{S}_A$. We now see as in the proof of Lemma 7.3 that we can perform the conjugation with a by a sequence of twisting operations along dihedral groups. Since at every step c is one of the generators of the relevant dihedral groups, c is preserved during this process and we obtain that S is equivalent to $\bar{S}$.

It remains to show that if we admit bridges we still get that after equivalence of S and $\bar{S}$ we have that $\Gamma(W, S)$ and $\Gamma(W, \bar{S})$ are isomorphic. Again the proof is by induction on $\#S$. We can clearly assume that $\Gamma(W, S)$ is not 1-connected since we have shown the stronger result for this case. It follows that W splits visually over $C \cong \mathbb{Z}_2$ and we obtain as in the case before that we can assume that $W = A *_C B$ is visual with respect to S and $\bar{S}$. Choose $S_A, S_B, S_C, \bar{S}_A, \bar{S}_B$ and $\bar{S}_C$ as before. By induction we know that after some twisting operations that preserve S_C we have that $\Gamma(A, S_A)$ is isomorphic to $\Gamma(A, \bar{S}_A)$ and that $\Gamma(B, S_B)$ is isomorphic to $\Gamma(B, \bar{S}_B)$.

It follows that $\Gamma(W, S)$ and $\Gamma(W, \bar{S})$ are both obtained from $\Gamma(A, S_A)$ and $\Gamma(B, S_B)$ by identifying a vertex v_1, v_1', respectively, of $\Gamma(A, S_A)$ with a vertex v_2, v_2' respectively, of $\Gamma(B, S_B)$. Since the twists preserve conjugacy classes of reflections by hypothesis it follows that v_i and v_i' must be connected by a path with odd labels only for $i = 1, 2$. It follows that we obtain $\Gamma(W, S)$ from $\Gamma(W, \bar{S})$ by a finite number of twists.

If $\Gamma(W, S)$ is not connected then we have a free product and the result follows immediately from the above and the work of Fouxe-Rabinovitch on automorphisms of free products [10], [11].

9 Reflection-independence

In this section we will use the following observation which is an immediate consequence of [4]:

Lemma 9.1. *Let (W, S) be a Coxeter system, let $s \in S$ and suppose that S is the disjoint union of S_1 and S_2 where S_1 is the set of all elements in S which commute with* *s and where S_2 is the set of all elements $s' \in S$ such that ss' has infinite order. Then $C_W(s) := \{w \in W \mid sw = ws\}$ is the special subgroup generated by S_1. In particular, $(C_W(s), S \cap C_W(s))$ is a Coxeter system.*

For the rest of this section we assume that (W, S) is a skew-angled Coxeter system and T is its set of reflections.

For all $s \neq s'$ such that ss' has finite order, we have a longest element $r(s, s')$ in the corresponding dihedral group, which is an involution. If the order of ss' is odd, then $r(s, s')$ is a reflection. Using the geometric representation of W it is easily verified that this is not the case if the order of ss' is even. A *rotation* is an element in W which is conjugate to $r(s, s')$ such that ss' has even order; the set of rotations in W (which depends of course on S) is denoted by $\mathrm{Rot}(W, S)$. As a consequence of a result of Richardson [18] the set of involutions in W is the disjoint union of the set of reflections and the set of rotations.

Lemma 9.2. *Let $s \neq s' \in S$ be such that ss' has finite and even order and put $\alpha = r(s, s')$. Then $X = \langle s, s' \rangle$ is the only spherical residue in the chamber system $\mathbf{C} = (C, P)$ associated with (W, S) which is stablized by α.*

Proof. Suppose that there exists a spherical residue $Y \neq X$ stabilized by α. By Proposition 2.4 there is a sequence $X = R_0, R_1, \dots, R_k = Y$ such that $R_i \neq R_{i+1}$ and $R_i \subseteq R_{i+1}$ or $R_{i+1} \subseteq R_i$ for $i = 1 \dots k$ and such that each R_i is stabilized by α. As $X \neq Y$ we can assume that $Y = R_1$. As (W, S) is assumed to be skew-angled the group $\langle s, s' \rangle$ is maximal finite and hence X is a maximal spherical residue of $\mathbf{C}$. It follows that Y is properly contained in X. As X has rank 2 the residue Y has rank one or it is a chamber; hence α is a reflection or $\alpha = \mathrm{id}_C$. This contradicts $\alpha \in \mathrm{Rot}(W, S)$. □

Lemma 9.3. *Let s, s', α and X be as in the previous lemma. Then $X = \{w \in W \mid \alpha w = w\alpha\}$ and if $r \in W$ is any involution such that αr has finite order, then $r \in X$.*

Proof. If $w \in W$ centralizes α then α fixes the spherical residue $w(X)$; hence $w(X) = X$ by the previous lemma; this means $w \in X$.

If $r \in W$ is an involution such that $r\alpha$ has finite order, then $\langle r, \alpha \rangle$ is a finite group, which stabilizes a spherical residue Y. By the previous lemma $Y = X$ and as α is in the center of X the claim follows. □

Proposition 9.4. *Let $R \subseteq W$ be fundamental and suppose that $\alpha \in R$ is in $\mathrm{Rot}(W, S)$. Then the following holds:*

(1) *$X := \{w \in W \mid \alpha w = w\alpha\}$ is a dihedral group whose order is $4m$ for some odd $m > 2$ and X is a maximal finite subgroup of W.*

(2) *There are precisely 3 elements in $R' := R \cap X$ and (X, R') is a Coxeter system; moreover, the set $R' \setminus \{\alpha\}$ consists of two reflections $t_1 \neq t_2$ which generate a dihedral group Y of order $2m$.*

(3) *Let $\beta \in X \backslash Y$ be an involution. If ω is an involution in W such that $\beta\omega$ has finite order, then $\omega \in X$.*

(4) *There exists a spike labelled by $2m$ in $\Gamma(W,S)$.*

(5) *The edge $\{t_1, t_2\}$ is a bridge of Γ' where Γ' is the graph obtained from the diagram $\Gamma(W,R)$ by removing the vertex α.*

Proof. As α is a rotation it follows from Lemma 9.3 that X is a finite parabolic subgroup of rank 2 with respect to the skew-angled system (W,S). Thus X is a dihedral group and X is a maximal finite subgroup of W. Again by Lemma 9.3 it follows that an involution ω in W with the property that $\alpha\omega$ has finite order already commutes with α; this is in particular true for each involution in R and hence we can apply Lemma 9.1. We conclude that (X, R') is a Coxeter system where $R' = R \cap X$. As X is a finite dihedral group and there is a central involution in R' which has a complement in X we conclude that the order of X is divisible by 4 but not by 8. As X is a parabolic subgroup for the skew-angled system (W,S) it cannot have order 4. Moreover, α is the only rotation in X and therefore $R' \backslash \{\alpha\}$ consists of two reflections $t_1 \neq t_2$ which generate a dihedral subgroup Y of order $2m$. This completes the proof of Assertions (1) and (2).

As X is the centralizer of α it follows from Lemma 9.3 that $r\alpha$ has infinite order for all $r \in R'' := R \backslash R'$. Let Y be the subgroup of W generated by t_1 and t_2. As there are no relations between α and the elements of R'' it follows that $W = X *_Y Z$ where $Z = \langle R \backslash \{\alpha\} \rangle$. Let β be an involution in $X \backslash Y$; as Y has a central complement in X of order 2 it follows that β is not conjugate in X to an involution in Y. Let ω be an involution in W such that $\beta\omega$ has finite order. Then $U := \langle \beta, \omega \rangle$ is a finite group. The group W acts on the Bass–Serre tree T corresponding to the amalgamated product $W = X *_Y Z$ and therefore U stabilizes a vertex of T. But the only vertex stabilized by β is the vertex v with $\operatorname{Stab} v = X$ because β is not conjugate in X to an element of Y; it follows that ω fixes v, i.e. that ω is an element of X. This completes (3).

As X is a maximal finite group, it is a parabolic subgroup of rank 2 with respect to the system (W,S). Therefore there exists an element $w \in W$ such that $X' := wXw^{-1}$ is a special subgroup of rank 2 with respect to (W,S); put $Y' = wYw^{-1}$. It follows that $S \cap X' = \{s, s'\}$ for some $s, s' \in S$. Without loss of generality we can assume that $s \notin Y'$ as $S \cap X'$ generates X' and Y' is a proper subgroup of X'. Let now $s'' \in S$ be such that ss'' has finite order. Then $w^{-1}ss''w = w^{-1}sww^{-1}s''w$ has finite order as well and, as s is not in Y', $\beta := w^{-1}sw$ is not in Y. As $\omega := w^{-1}s''w$ is an involution, Part (3) yields $\omega \in X$ and therefore $s'' \in X \cap S = \{s, s'\}$. This proves that s' is the only element in S with which s is connected in the graph $\Gamma(W,S)$. As X has order $4m$ and as $X' = \langle s, s' \rangle$ is conjugate to X Part (4) follows.

By (4) the group W can be written as a visual amalgamated product $W = \langle S \backslash \{s\} \rangle *_{\langle s' \rangle} X'$ with respect to S. As in the proof of the main theorem we see that this splitting is also visual with respect to $\bar{R}$ for some fundamental set $\bar{R}$ that is twist equivalent to R. As $R' = \{\alpha, t_1, t_2\} \subset R$ is spherical it is conjugate to a spherical subset $\bar{R}' = w\{\alpha, t_1, t_2\}w^{-1}$ of $\bar{R}$. In particular $\bar{R}'$ generates a conjugate of X'. As no other spherical subset of $\bar{R}$ generates a subgroup conjugate of X' and as the above splitting is visual with respect to $\bar{R}$ it follows that $X' = \langle \bar{R}' \rangle$. As $w\alpha w^{-1}$ is not a reflection of (W, S) it follows that either $s' = wt_1w^{-1}$ or that $s' = wt_2w^{-1}$. Removing the vertex $w\alpha w^{-1}$ form the graph $\Gamma(R, \bar{R})$ clearly yields a graph such that the edge $\{wr_1w^{-1}, wr_2w^{-1}\}$ is a spike and therefore a bridge. As twist equivalence preserves the property that an edge is a bridge this implies (5). $\square$

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