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# Generalized projective geometries: general theory and equivalence with Jordan structures

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**Abstract.** In this work we introduce *generalized projective geometries* which are a natural generalization of projective geometries over a field or ring $\mathbb{K}$ but also of other important geometries such as Grassmannian, Lagrangian or conformal geometry (see [3]). We also introduce the corresponding *generalized polar geometries* and associate to such a geometry a *symmetric space over* $\mathbb{K}$. In the finite-dimensional case over $\mathbb{K} = \mathbb{R}$, all classical and many exceptional symmetric spaces are obtained in this way. We prove that generalized projective and polar geometries are essentially equivalent to *Jordan algebraic structures*, namely to *Jordan pairs*, respectively to *Jordan triple systems* over $\mathbb{K}$ which are obtained as a linearized tangent version of the geometries in a similar way as a Lie group is linearized by its Lie algebra. In contrast to the case of Lie theory, the construction of the "Jordan functor" works equally well over general base rings and in arbitrary dimension.

**Key words.** projective geometry, polar geometry, symmetric space, Jordan pair, Jordan triple system.

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## 0 Introduction

**0.1 Geometry and algebra.** The aim of this work is to bring together two topics, a geometric one, namely *projective geometry*, and an algebraic one, namely *Jordan algebraic structures*. On the one hand, projective spaces and projective geometry are not only central topics in mathematics but also play a foundational rôle in modern physics, see for example [22]; on the other hand, Jordan algebras have been invented as a concept in the foundation of quantum mechanics, see [13]. Thus it seems not to be without interest to find a geometric concept unifying these two theories; such a concept, called *generalized projective geometries*, is proposed in the present work. In fact, many authors have already remarked that there are important relations between the two topics mentioned (see point 0.6. below); however, most of the literature concerns special cases, and it seems that the problem to establish a general equivalence between the two categories in question has not been raised. As for any equivalence, the problem has two aspects: coming from geometry, we want to find a "linear tangent object" (similar to the Lie algebra of a Lie group) allowing to transform geometric problems into linear algebra. This is achieved in Chapter 9 (Theorems 9.5 and 9.8) in a very general context. In the special case of projective geometry, the tangent object is a trilinear composition of the kind

$$V \times V^* \times V \to V, \quad (x, \phi, y) \mapsto \phi(x)y + \phi(y)x$$

which is well-known to play an important rôle in projective differential geometry. Conversely, coming from algebra, we want to "integrate" our algebraic structure to a global geometric object; in other words, we are looking for a Jordan analogue of Lie's third theorem (this is achieved in Chapter 10, see Theorem 10.1). In the following, we describe our approach in some more detail.

**0.2 Generalized projective geometries.** The essential difference between our approach to projective geometry and the more traditional ones is that we do *not* try to base our theory on "incidence axioms" or other combinatorial structures but on *algebraic laws*. Our guiding model here is the theory of Lie groups (which is based on the group laws) and Loos' theory of symmetric spaces [16], which is based on a set of "non-associative" algebraic identities. Let us briefly describe the main features of the algebraic identities we have in mind: it is convenient to consider a projective space $X = \mathbb{P}(W)$ over a field or ring $\mathbb{K}$ together with its dual space $X' = \mathbb{P}(W^*)$ and to view elements $a \in X'$ as "affinizations" of $X$ and vice versa. In general, a pair of spaces $(X, X')$, each of them parametrizing a family of affinizations on the other, is called an *affine pair geometry* over the base ring $\mathbb{K}$ (Chapter 1). If such a structure is given, then for any fixed scalar $r \in \mathbb{K}$ there is a natural ternary "multiplication map"

$$\mu_r : X \times X' \times X \supset D \to X, \quad (o, a, x) \mapsto \mu_r(o, a, x)$$

associating to an affinization $a$ with affine part $V_a \subset X$ and two points $o, x \in V_a$ the product $rx =: \mu_r(o, a, x)$ in the $\mathbb{K}$-module $V_a$ with zero vector $o$. Exchanging the rôles of $X$ and $X'$, a dual multiplication map $\mu'_r$ is defined. In [3] we give an explicit formula for the multiplication map for the case of projective and Grassmannian geometries and derive its most important properties by elementary linear algebra. As usual, one defines *left*, *right* and *middle multiplications* associated to the ternary map $\mu = \mu_r$ by

$$\mu(x, a, y) = L_{x,a}(y) = M_{x,y}(a) = R_{a,y}(x).$$

*Generalized projective geometries* are now defined by requiring the following *fundamental identities of projective geometry* (Chapter 2):

$$(L_{x,a})^t = L_{a,x}, \quad (R_{a,x})^t = R_{x,a} \tag{PG1}$$

$$(M_{x,y})^t = M_{y,x}, \quad (M_{a,b})^t = M_{b,a}, \tag{PG2}$$

complemented by a property (T) assuring the existence of *translation groups*. Here an identity of the type $g^t = h$ means that the pair $(g, h)$ behaves essentially like a projective map $g$ and its transposed $g^t$ in projective geometry, i.e. that the condition

$$g(\mu_s(x, h(a), y)) = \mu_s(g(x), a, g(y))$$

holds for all $s \in \mathbb{K}$. In [3] we have shown that (PG1) and (PG2) hold for the "classical geometries": projective, Grassmannian, Lagrangian and conformal geometry. For several reasons we believe that these identities are indeed a good starting point for an axiomatic theory: they are simple, highly symmetric and "complete" in the sense that all partial operators obtained by fixing two elements have a good functorial relation with the whole structure—in this sense our equations are the best one could expect and behave nicer than group laws (where left and right translations are not automorphisms) or symmetric space laws (where left translations are automorphisms but right translations are not).

**0.3 Tangent objects and "differential calculus".** Yet in another sense the identities (PG1) and (PG2) behave nicer than group or symmetric space laws: they imply a strong *regularity* of the multiplication maps even in the infinite-dimensional case in the sense that generically everything can be expressed by (quadratic) polynomials combined with inversions in some general linear group—in the finite-dimensional case over a field everything is thus rational over $\mathbb{K}$. The only assumption we need here is that (PG1) and (PG2) hold *in all scalar extensions of* $\mathbb{K}$; we show that the scalar extension of a generalized projective geometry $(X, X')$ over $\mathbb{K}$ by *dual numbers over* $\mathbb{K}$ plays naturally the role of the *tangent bundle* $(TX, TX')$ (Chapter 7; the use of dual numbers in related contexts appears already in [6, Chapter 4], [19] and [20]). Thus we have a sort of differential calculus on $(X, X')$, and in a way similar to the way that one associates a Lie algebra to a Lie group or a Lie triple system to a symmetric space, derivations of the ternary maps $\mu_r$, $\mu_r'$ at a base point $(o, o')$ give rise to a pair of ternary maps of tangent spaces,

$$T : V \times V' \times V \to V, \quad T' : V' \times V \times V' \to V',$$

$V \cong T_o X$, $V' \cong T_{o'} X'$, satisfying the algebraic laws of a (linear) *Jordan pair over* $\mathbb{K}$ (Theorem 9.5). Conversely, to any Jordan pair over $\mathbb{K}$ we can construct a generalized projective geometry $(X, X')$ over $\mathbb{K}$; these constructions are essentially inverse to each other and are functorial (Theorem 10.1). An important rôle in both constructions is played by the Lie algebra of *derivations of* $(X, X')$, known in Jordan theory as the associated *Kantor–Koecher–Tits algebra* (Chapter 9).

**0.4 Polar and null geometries.** Besides Jordan pairs, there are two other important algebraic categories in Jordan theory, namely *Jordan triple systems* and *Jordan algebras*. The former are the same as *Jordan pairs with involution*, see [17, I.1]. Geometrically, they correspond to *generalized polar geometries*. Polarities can be defined in our general situation in the same way as in ordinary projective geometry: they are anti-automorphisms $(p : X \to X', p' : X' \to X)$ of order 2 which are not a *null-system*, i.e., not all $x \in X$ are *isotropic* with respect to $p$ (Chapter 3). The *quadric* of a polarity $p$ is the set of isotropic points of $p$; it can be described by an algebraic condition in terms of the associated Jordan triple system (see Section 11.1).

The geometric object corresponding to *unital Jordan algebras* is closely related to *null systems* and to *inner polarities*; we will come back to this point in subsequent work (cf. Section 11.3).

**0.5 Symmetric spaces and Jordan–Lie functor.** It is well-known that any Jordan triple system $T$ gives rise to a Lie triple system $R = R_T$ defined by

$$R(X, Y)Z = -(T(X, Y, Z) - T(Y, X, Z)); \tag{0.1}$$

the correspondence $T \mapsto R_T$ is called the *Jordan–Lie functor* (see [5]). In Chapter 4 we construct the corresponding functor on a geometric level: for any polar geometry $(X, X', p)$ the complement $M^{(p)}$ of the associated quadric is a symmetric space over $\mathbb{K}$ in an appropriate sense (Theorem 4.1). In the finite-dimensional case over $\mathbb{K} = \mathbb{R}$ we get a new and more conceptual construction of the *geometric Jordan–Lie functor* from [5], and for a general field $\mathbb{K}$ we obtain a class of symmetric spaces which is *algebraic over* $\mathbb{K}$ (cf. Section 11.2). In the real case, it is known by classification (work of E. Neher; cf. [5]) that all classical and about half of the exceptional simple symmetric spaces are obtained by our construction; therefore one may conjecture that also in the general case our construction yields an important part of the finite- and even infinite-dimensional symmetric spaces over a general base field or ring. The Jordan algebraic description is a very effective and powerful tool in the study of such spaces. It seems to be a rather deep problem to understand the Jordan–Lie functor in a conceptual way: which intrinsic property of a symmetric space makes it associated to one or several generalized polar geometries? We hope that the approach presented here will help to solve this problem.

**0.6 Related work.** As already mentioned, the relation between geometry and Jordan structures has attracted the attention of many authors. A quite extensive bibliography on the geometry of *exceptional* Jordan structures, going back to the work of Freudenthal, Springer and others, can be found in [12]. Very closely related to our approach are the papers [18], [19] and [21] by O. Loos and the papers [6] and [8] by J. Faulkner. Although the latter papers are placed in an incidence geometric context, our formalism is surprisingly close to the one developed there. Our presentation of the Kantor–Koecher–Tits algebra (Section 9) is motivated by Section 4 of [6]. Comparing with the above mentioned papers by O. Loos, the reader will find that our identity (PG1) for invertible scalars is in fact implicitly contained in [19] (and also in our approach [5] to the real case), whereas the identity (PG2) seems to be completely new—in fact, the discovery of the identity (PG2) was a big surprise to us; since right multiplications in symmetric spaces have no known functorial interpretation, we did not expect the situation in Jordan theory to be that much better. Correspondingly, the central part in the proof of the existence theorem for generalized projective geometries (Theorem

10.1) is the verification of (PG2). It is precisely the identity (PG2) that allows one to get rid of regularity assumptions and thus to develop the theory in full generality, including the infinite-dimensional case.

**Organization of the paper.** The contents is as follows:



The main results can be found in Chapters 9 and 10. In Chapter 11 we mention some further topics and problems which we hope to investigate in subsequent work. Examples and motivation for the axiomatic approach presented here are given in our paper [3]. I am grateful to John R. Faulkner and to Ottmar Loos for helpful comments during the 2000 Oberwolfach conference on Jordan algebras which lead to the general approach, including the case of base *rings*, which is presented here.

**Notation.** $\mathbb{K}$ denotes a commutative ring with unit 1 and $1 + 1 \in \mathbb{K}^*$.

## 1 Affine pair geometries

**1.1.** A *pair geometry* is given by two sets $X$, $X'$ and a subset $M \subset X \times X'$; if $(x, a) \in M$ we say that *x and a are remote* or *in general position*. If $(x, a) \notin M$, we say that $(x, a)$ are *neighboring*. For $o' \in X'$, respectively $o \in X$, we denote by

$$V_{o'} := \{x \in X \mid (x, o') \in M\}, \quad V'_o := \{x' \in X' \mid (o, x') \in M\} \tag{1.1}$$

the sets of objects remote from $o'$, respectively from $o$. We assume that the sets $V_{o'}$, $o' \in X'$ (resp. $V'_o$, $o \in X$) cover $X$ (resp. $X'$) or, equivalently, that all $V_{o'}$ and all $V'_o$ are non-empty. The case that $X$ is covered already by one of the $V_{o'}$'s is not excluded.

**1.2.** *Homomorphisms* of pair geometries are remoteness-preserving pairs of maps, that is, pairs $g : X \to Y$, $g' : X' \to Y'$ such that $g(V_{o'}) \subset V_{g'(o')}$ and $g'(V'_o) \subset V'_{g(o)}$

for all $o \in X$, $o' \in X'$. *Local homomorphisms* are defined by the same property; they are required to be defined at least on one pair of sets $(V_a, V'_x)$ with $(x, a) \in M$.

**1.3.** If $(X, X', M)$ is a pair geometry, then so is $(X', X, M^d)$ with $M^d = \{(q, p) \mid (p, q) \in M\}$; we call it the *dual pair geometry*. All axioms we are going to add will appear together with their dual version, thus assuring the existence of dual objects.

**1.4.** An *affine pair geometry (over* $\mathbb{K}$*)* is a pair geometry $(X, X', M)$ such that for every element $o' \in X'$ (resp. $o \in X$) a structure of an affine space over $\mathbb{K}$ is defined on $V_{o'}$ (resp. on $V'_o$). In other words, for each pair $(o, o') \in M$ there is a structure of a $\mathbb{K}$-module with zero vector $o$ on $V_{o'}$ and a $\mathbb{K}$-module structure with zero vector $o'$ on $V'_o$. The affine parts $V_{o'} \subset X$, resp. $V'_o \subset X'$ are called *affine cells* of $X$, resp. of $X'$, and we say that $X'$ (resp. $M$) is the *space of affinizations* (resp. *space of vectorializations*) of $X$, and vice versa.

**1.5 The multiplication maps.** Given an affine pair geometry, we denote for any $(o, o') \in M$ and $r \in \mathbb{K}$ by $r_{o,o'} : V_{o'} \to V_{o'}$ the multiplication by the scalar $r$ in the $\mathbb{K}$-module $V_{o'}$ with zero vector $o$, and define dually $r_{o',o} : V'_o \to V'_o$ as multiplication by $r$ in the $\mathbb{K}$-module $V'_o$ with zero vector $o'$. Putting for a fixed $r \in \mathbb{K}$ all these together, we define the *multiplication maps associated to an affine pair geometry* by

$$\mu_r : X \times X' \times X \supset D \to X, \quad (o, o', x) \mapsto \mu_r(o, o', x) := r_{o,o'}(x) \tag{1.2}$$

and dually

$$\mu'_r : X' \times X \times X' \supset D' \to X', \quad (o', o, x') \mapsto \mu'_r(o', o, x') := r_{o',o}(x') \tag{1.3}$$

where

$$D = \{(o, o', x) \in X \times X' \times X \mid (o, o') \in M, (x, o') \in M\}, \tag{1.4}$$

and dually for $D'$. Thus, if we fix the middle element $o'$, the partial maps

$$\mu_r(\cdot\,, o', \cdot) : V_{o'} \times V_{o'} \to V_{o'}, \quad \mu'_r(\cdot\,, o, \cdot) : V'_o \times V'_o \to V'_o \tag{1.5}$$

define on $V_{o'}$ (resp. $V'_o$) the structure of an affine space over $\mathbb{K}$ in the sense of Chapter 1 of [3], and the concept of an affine pair geometry can be expressed by several identities for the multiplication maps which are denoted by (Af1)–(Af5) in [3] and from which one can recover the whole of affine geometry over $\mathbb{K}$ ([3, Theorem 1.1]). For instance, the identity (Af3) reads in the notation used here

$$\mu_r(p, a, q) = \mu_{1-r}(q, a, p), \quad \mu'_r(a, p, b) = \mu'_{1-r}(b, p, a). \tag{1.6}$$

We will not need the explicit form of the other identities in the sequel; it suffices here to remark that affine maps are precisely the algebraic homomorphisms of the multiplication maps and that translations can be recovered from the multiplication maps via Formula (2.4) given below.

**1.6.** *Homomorphisms* of affine pair geometries $(X, X')$, $(Y, Y')$ are remoteness-preserving pairs $(g, g')$ of maps $g : X \to Y$, $g' : X' \to Y'$ such that the multiplication maps are respected:

$$\begin{aligned} g(\mu_r(o, o', p)) &= \mu_r(g(o), g'(o'), g(o)), \\ g'(\mu_r'(o', o, p')) &= \mu_r'(g'(o'), g(o), g'(p')). \end{aligned} \tag{1.7}$$

Equivalently, for all $o' \in X'$ the restrictions

$$g : X \supset V_{o'} \to V_{g'(o')} \subset Y, \quad g' : X' \supset V_o' \to V_{g(o)}' \subset Y'$$

are affine maps. *Local homomorphisms* are local homomorphisms of pair geometries (in the sense of 1.2) respecting the multiplication maps. (Note that we do not require $g'$ to be determined uniquely by $g$ or vice versa; however, in the "non-degenerate case" this property holds, see 2.8.) *Isomorphisms* are homomorphisms $(g, g')$ for which $g$ and $g'$ are bijections. The composition of homomorphisms is again a homomorphism (with $(gh)' = g'h'$); thus affine pair geometries form a category. In particular, we may speak of the *automorphism group* $\mathrm{Aut}(X, X')$ of $(X, X')$.

**1.7.** A *base point* in $(X, X')$ is a fixed element $(o, o') \in M$; a *homomorphism of affine pair geometries with base point* is a base point preserving homomorphism. The corresponding automorphism group $\mathrm{Aut}(X, X'; o, o') = \mathrm{Aut}(X, X')_{o,o'}$ is called the *structure group of* $(X, X'; o, o')$; by definition of a homomorphism it acts linearly on $V_{o'} \times V_o'$.

**1.8.** An *adjoint pair of morphisms* of affine pair geometries $(X, X')$, $(Y, Y')$ is a pair of maps $g : X \supset U \to Y$, $h : Y' \supset U' \to X'$ preserving remoteness in the sense that

$$g(V_{h(p)} \cap U) \subset V_p, \quad h(V_{g(q)}' \cap U') \subset V_q'$$

(where the subsets $U \subset X$, $U' \subset X'$ shall contain at least one affine cell) and satisfying the relations

$$\begin{aligned} g(\mu_r(o, h(o'), p) &= \mu_r(g(o), o', g(p)), \\ h(\mu_r'(o', g(o), p')) &= \mu_r(h(o'), o, h(p')). \end{aligned} \tag{1.8}$$

These relations can be rephrased by saying that $g|_{V_{h(o')} \cap U}$ has an extension by an affine map $V_{h(o')} \to V_{o'}$, and dually. We write $h = g^t$ if $(g, h)$ is an adjoint pair; this means just that $(g, h) = (g, g^t)$ satisfies the relations (1.8) and shall not be interpreted in the sense that $g^t$ be uniquely determined by $g$ (although in the non-degenerate case it actually is, see 2.8 and 2.9). The conditions $h = g^t$ for $(X, X')$, $(Y, Y')$ and $g^t = h$ for the dual pairs $(Y', Y)$, $(X'X)$ are equivalent. Clearly, if $(g, g^t)$ and $(f, f^t)$ are adjoint pairs such that $g$, $f$ and $f^t$, $g^t$ are composable, then $(g \circ f, f^t \circ g^t)$ is an adjoint pair (this may be expressed by writing $(gf)^t = f^t g^t$). In particular, the affine pair geometries with base point form a category with respect to base-point preserving adjoint pairs of morphisms.

**1.9.** The preceding discussion shows that affine pair geometries can be turned into a category in two essentially different ways—in general, homomorphisms do not give rise to adjoint pairs of morphisms or vice versa. (This is very well known from projective geometry: some authors require homomorphisms of projective spaces to be induced by *injective* linear maps, whereas others allow morphisms to be possibly defined only on some affine part, see e.g. [1]; algebraically, this corresponds exactly to the distinction of two categories made here. Since the Jordan pair of ordinary projective geometry is *simple*, homomorphisms have to be injective or trivial.) However, if $(g, g')$ is an isomorphism, then clearly $(g, g^t)$ with $g^t = (g')^{-1}$ is an adjoint pair. In general, categorial notions will refer to the category defined in 1.6 and not to the category defined in 1.8.

**1.10 Scalar extensions.** By a *scalar extension* of $\mathbb{K}$ we mean a unital commutative and associative $\mathbb{K}$-algebra $R$. Let us denote by $\phi : \mathbb{K} \to R$, $r \mapsto r1_R$ the natural homomorphism (it need not be injective). A corresponding *scalar extension of the affine pair geometry* $(X, X')$ over $\mathbb{K}$ is an affine pair geometry $(X, X')_R := (X_R, X'_R)$ over $R$ together with a homomorphism of affine pair geometries over $\mathbb{K}$

$$(\Phi, \Phi') : (X, X') \to (X, X')_R$$

such that, for all $(x, a) \in M$, the $R$-module $(V_{\Phi'(a)}, \Phi(x))$ (zero vector $\Phi(x)$) is the usual scalar extension of the $\mathbb{K}$-module $(V_a, x)$, i.e.

$$(V_{\Phi'(a)}, \Phi(x)) \cong (V_a, x) \otimes_{\mathbb{K}} R$$

as an $R$-module; more precisely, the following diagram shall commute:

$$\begin{array}{ccc} (V_a, x) & = & (V_a, x) \\ \downarrow & & \downarrow \\ (V_a, x) \otimes_{\mathbb{K}} R & \cong & (V_{\Phi'(a)}, \Phi(x)), \end{array} \tag{1.9}$$

where the first column is the natural map $v \mapsto v \otimes_{\mathbb{K}} 1_R$ and the second column is given by restriction of $\Phi$. Dually, for $(V'_x, a)$ a similar condition is required to hold. Moreover, we require $(X, X')_R$ to be *minimal* in the following sense: for every homomorphism $(g, g')$ of $(X, X')$ into a geometry $(Y, Y')$ defined over $R$, considered as a geometry over $\mathbb{K}$, there exists a unique extension $(g, g')_R := (g_R, g'_R)$ to a homomorphism $(X, X')_R \to (Y, Y')$ defined over $R$. It is clear that this property determines $(X, X')_R$ up to isomorphisms. (One might be tempted to define scalar extensions only by this universal property; however, it seems not to be possible to deduce from it the isomorphism (1.9). Note that we do not make here any claims of existence of scalar extensions for general affine pair geometries.)

## 2 Generalized projective geometries

**2.1.** Assume $(X, X')$ is an affine pair geometry over $\mathbb{K}$ with multiplication maps $\mu_r$, $\mu_r'$. As for any ternary map, we define *right*, *left* and *middle multiplication* associated to the ternary maps $\mu_r$ and $\mu_r'$ by

$$\begin{aligned} L_{x,a}(y) &:= R_{a,y}(x) := M_{x,y}(a) := \mu_r(x,a,y) \\ L_{a,x}(b) &:= R_{x,b}(a) := M_{a,b}(x) := \mu_r'(a,x,b) \end{aligned} \tag{2.1}$$

(where we add the superscript $(r)$ if the dependence on $r$ shall be indicated). Then (1.6) says that $R_{a,y}^{(r)} = L_{y,a}^{(1-r)}$, and thus we can transform left multiplications into right multiplications and vice versa. However, they cannot be transformed into middle multiplications since the latter exchange the partners $X$ and $X'$ whereas the former preserve them. We call operators of the type of left, right or middle multiplications altogether *interior operators of the geometry*.

**2.2 The fundamental identities.** We say that an affine pair geometry satisfies the *fundamental identities* if, for all $r \in \mathbb{K}$, the following holds:

$$(L_{x,a})^t = L_{a,x}, \quad (R_{a,x})^t = R_{x,a}, \tag{PG1}$$

$$(M_{x,y})^t = M_{y,x}, \quad (M_{a,b})^t = M_{b,a}, \tag{PG2}$$

where $(x,a) \in M$, and $(x,y) \in X \times X$, $(a,b) \in X' \times X'$ are such that the middle multiplications are defined at at least one point. Since $L_{x,a}^{(r)} = R_{a,x}^{(1-r)}$, the first and the second condition in (PG1) are equivalent. Since $g^t = h$ and $h^t = g$ are equivalent, another equivalent formulation is $(L_{a,x})^t = L_{x,a}$. However, the two conditions in (PG2) are not equivalent: the first one says that $(M_{x,y}, M_{y,x})$ is an adjoint pair for $(X', X)$, $(X, X')$ and the second that $(M_{a,b}, M_{b,a})$ is an adjoint pair for $(X, X')$, $(X', X)$; both are self-dual. Using (1.8), the conditions (PG1) and (PG2) can be written more explicitly

$$\begin{aligned} L_{x,a}(\mu_s(y, L_{a,x}(b), z)) &= \mu_s(L_{x,a}(y), b, L_{x,a}(z)), \\ L_{a,x}(\mu_s'(b, L_{x,a}(y), c)) &= \mu_s'(L_{a,x}(b), y, L_{a,x}(c)), \\ M_{x,y}(\mu_s'(a, M_{y,x}b, c)) &= \mu_s(M_{x,y}(a), b, M_{x,y}(c)), \\ M_{a,b}(\mu_s'(x, M_{b,a}y, z)) &= \mu_s(M_{a,b}(x), y, M_{a,b}(z)). \end{aligned}$$

for all $r, s \in \mathbb{K}$ and $x, y, z \in X$, $a, b, c \in X'$ where the expressions are defined. With all variables included, the preceding formulas read

$$\begin{aligned}
\mu_r(x, a, \mu_s(y, \mu_r'(a, x, b), z)) &= \mu_s(\mu_r(x, a, y), b, \mu_r(x, a, z)),\\
\mu_r'(a, x, \mu_s'(b, \mu_r(x, a, y), c)) &= \mu_s'(\mu_r'(a, x, b), y, \mu_r'(a, x, c)),\\
\mu_r(x, \mu_s'(a, \mu_r(y, bx), c), y) &= \mu_s(\mu_r(x, a, y), b, \mu_r(x, c, y)),\\
\mu_r'(a, \mu_s'(x, \mu_r(b, y, a), z), b) &= \mu_s(\mu_r(a, x, b), y, \mu_r(a, z, b)).
\end{aligned}$$

We also require that for all $r \in \mathbb{K}^*$ the left multiplications $L_{x,y}^{(r)} = r_{x,y}$ extend to bijections $X \to X$ such that all identities introduced so far still hold whenever all expressions are defined. This implies that, if both $r$ and $1 - r$ belong to $\mathbb{K}^*$, the multiplication map $\mu_r$ is defined on the *extended domain*

$$D^e := \{(x, a, y) \in X \times X' \times X \mid (x, a) \in M \text{ or } (y, a) \in M\}. \tag{2.2}$$

By elementary properties of affine geometry we have (cf. the identity (Af1) from [3]),

$$r_{x,a} \circ s_{x,a} = (rs)_{x,a}, \quad 1_{x,a} = \mathrm{id}_{V_a}.$$

Thus for $r \in \mathbb{K}^*$ the condition (PG1) can be rephrased by saying that

$$(r_{x,a}, r_{a,x}^{-1}) \in \mathrm{Aut}(X, X'). \tag{2.3}$$

The automorphisms $(r_{x,a}, r_{a,x}^{-1})$ $((x, a) \in M, r \in \mathbb{K}^*)$ will also be called *inner automorphisms* or *major dilatations*, and the subgroup $\mathrm{Int}(X, X')$ of $\mathrm{Aut}(X, X')$ generated by them is called the *inner automorphism group*.

**2.3 Translations.** Since by assumption 2 is invertible in $\mathbb{K}$, we can express *translations* via major dilatations: for $(o, o') \in M$,

$$\tau_v := \tau_v^{(o,o')} := 2_{o,o'} 2_{v,o'}^{-1} \tag{2.4}$$

is the translation by $v$ in the $\mathbb{K}$-module $(V_{o'}, o)$. If the fundamental identity (PG1) in its version (2.3) holds, then the pair

$$(\tau_v, \tilde{\tau}_v) := (2_{o,o'} 2_{v,o'}^{-1}, 2_{o',o}^{-1} 2_{o',v}) \tag{2.5}$$

belongs to $\mathrm{Aut}(X, X')$; this can also be written

$$(\tilde{\tau}_v)^t = \tau_{-v}. \tag{2.6}$$

The identities of affine geometry imply that $\tau_v \tau_w = \tau_{v+w}$ (sum in $(V_{o'}, o)$). If the transpose is unique (as in ordinary projective geometry), then we obtain by transposing:

$$\tilde{\tau}_v \tilde{\tau}_w = \tilde{\tau}_{v+w} \tag{T}$$

which means that

$$\tau_{V_{o'}} := \{(\tau_v, \tilde{\tau}_v) \mid v \in V_{o'}\} \tag{2.7}$$

is a group, called the *translation group with respect to the affinization* $o'$; as a group, it is isomorphic to $(V_{o'}, o, +)$. We say that our geometry *satisfies the translation property* if (T) holds together with its dual, for all $(o, o') \in M$. The dual of (T) implies that

$$\tilde{\tau}_{V'_o} := \{(\tilde{\tau}_w, \tau_w) \mid w \in V'_o\} \tag{2.8}$$

with

$$\tilde{\tau}_w := \tilde{\tau}_w^{(o,o')} := 2_{o,o'}^{-1} 2_{o,w} \tag{2.9}$$

is an abelian group, isomorphic to $(V'_o, o', +)$. The maps $\tilde{\tau}_w : X \to X$ are called *dual translations*. Our notational convention is such that groups denoted by $\tau_{V_a}$ $(a \in X')$ act by usual translations on the *first* factor of the geometry $(X, X')$ and groups denoted by $\tilde{\tau}_{V'_x}$ $(x \in X)$ act by usual translations on the *second* factor. Thus, for instance, the group denoted by $\tau_{V'_x}$ $(x \in X)$ acts by usual translations on the first factor of the dual geometry $(X', X)$; as a group, it is of course isomorphic to $\tilde{\tau}_{V'_x}$. (Note that (T) is a consequence of the preceding identities if the transpose is unique; we do not know whether this is true also in the general case. It may be conjectured that this is indeed so since for Jordan pairs a *duality principle* holds, see [17, Proposition 2.9], which is used in the proof of the addition formula [17, Theorem 3.7] that corresponds to our formula (T).)

**2.4.** A *generalized projective geometry over* $\mathbb{K}$ is an affine pair geometry $(X, X')$ over $\mathbb{K}$ in which the fundamental identities (PG1), (PG2) and the translation property (T) hold in all scalar extensions of $\mathbb{K}$—this means that, if $\phi : \mathbb{K} \to R$ is any scalar extension, then there exists a scalar extension $(X, X')_R$ of $(X, X')$ in the sense of 1.10 satisfying (PG1) (also in its extended version, if the scalar is invertible), (PG2) and (T) (together with its dual) over $R$.

**2.5.** *Homomorphisms* and *adjoint pairs of morphisms* of generalized projective geometries are those of the underlying affine pair geometry. Therefore, generalized projective geometries can be turned into a category in two essentially different ways.

**2.6.** For a definition of generalized projective geometries if $2 \notin \mathbb{K}^*$, it will be necessary to add axiomatically the structure given by the following maps of *four* arguments:

$$\begin{aligned} &X \times X' \times X \times X' \supset W \to X, \quad (o, o', v, w) \mapsto \tilde{\tau}_w^{(o,o')}(v), \\ &X \times X' \times X \times X \supset W' \to X, \quad (o, o', v, w) \mapsto \tau_w^{(o,o')}(v) \end{aligned} \tag{2.10}$$

where $W$, $W'$ are defined by conditions similar to (2.2). This idea will be taken up in subsequent work.

**2.7 Categorial constructions.** Since the category of generalized projective geometries is essentially defined by algebraic laws, it behaves fairly well with respect to some standard categorial constructions:

(1) (*Duality.*) The rôle played by the spaces $X$ and $X'$ in our axioms is completely symmetric; thus $(X', X)$ with the space $M_d = \{(a, x) \mid (x, a) \in M\}$ of vectorializations is again a generalized projective geometry, called the *dual pair* or *dual geometry* of $(X, X')$.

(2) (*Direct products.*) The *direct product* of $(X, X')$, $(Y, Y')$ is $(X \times Y, X' \times Y')$ with remoteness given by $V_{(o', p')} = V_{o'} \times V_{p'}$, and dually, which we require to carry the direct product structure $V_{o'} \times V_{p'}$ of affine spaces. It is then easily verified that $(X \times Y, X' \times Y')$ is again a generalized projective geometry; the multiplication map is just the direct product of the ones of $(X, X')$ and $(Y, Y')$. In particular, we can define the direct product $(X, X') \times (X', X) = (X \times X', X' \times X)$ which will play an important role later on.

(3) (*Subspaces.*) These are subsets $Y \subset X$, $Y' \subset X'$ such that $(Y, Y')$ is closed under all multiplications maps. Thus in the affinizations $y' \in Y'$, $Y$ is an affine subspace, and vice versa.

(4) (*Inner ideals.*) An *inner ideal* of $X$ as a subset $Y \subset X$ which is linear w.r.t. *all* possible affinizations, i.e. satisfying $\mu_r(Y, a, Y) \subset Y$ for all $a \in X'$; inner ideals of $X'$ are defined dually.

(5) (*Congruences and quotient spaces.*) A *congruence* is a subspace $(R, R') \subset (X \times X, X' \times X')$ which is an equivalence relation—similar to the case of symmetric spaces ([16, Chapter III]) one shows that $(X, X')/(R, R')$ is again a generalized projective geometry.

(6) (*Tangent bundle.*) One can construct a tangent bundle $(TX, TX')$ of $(X, X')$ which is essentially scalar extension by dual numbers over $\mathbb{K}$—see Chapter 7.

(7) (*Flat geometries.*) The category of pairs of affine spaces over $\mathbb{K}$ is imbedded in the category of generalized projective geometries as follows: let $V, W$ be affine spaces in the usual sense; let $X := V$, $X' := W$, $M = V \times W$ and let $r_{o,o'} = r_o$ (i.e. all affine charts of $X$ yield the same structure of affine space on $V$), and dually $r_{o',o} = r_{o'}$. The axioms are easily verified.

**2.8 Faithful and non-degenerate geometries.**

(i) We say that the generalized projective geometry is *non-degenerate* if the map assigning to $a \in X'$ the set $V_a \subset X$ is injective, and dually.

(ii) We say that the generalized projective geometry is *faithful* if the map assigning to $a \in X'$ the affine structure $(V_a, A_a := \mu_\cdot(\cdot, a, \cdot))$ is injective, and dually.

It is clear that a non-degenerate geometry is faithful; the converse is not true. If $(g, g')$ is an automorphism of a faithful geometry, then $g$ determines $g'$ uniquely by the condition $A_{g'(a)} = g_*(A_a)$, where $g_*$ is the push-forward of the affine structure $A_a$ by $g$, and conversely $g$ is determined by $g'$.

**2.9 Conformal group.** The *projective* or *conformal group of X* is the group

$$\mathrm{Co}(X) := \{g : X \to X \mid \exists g' : X' \to X' : (g, g') \in \mathrm{Aut}(X, X')\}. \tag{2.11}$$

The *inner conformal group* is the subgroup of $\mathrm{Co}(X)$ generated by the dilatations $r_{x,a}$, $r \in \mathbb{K}^*$, $(x, a) \in M$, and dually we define $\mathrm{Co}(X')$ and its inner conformal group. If $(X, X')$ is faithful, then the surjective homomorphisms

$$\mathrm{Aut}(X, X') \to \mathrm{Co}(X), \quad (g, g') \mapsto g, \quad \mathrm{Aut}(X, X') \to \mathrm{Co}(X'), \quad (g, g') \mapsto g'$$

are injective, hence $\mathrm{Co}(X)$ and $\mathrm{Co}(X')$ are isomorphic to $\mathrm{Aut}(X, X')$, and we have an isomorphism

$$\mathrm{Co}(X) \to \mathrm{Co}(X'), \quad g \mapsto g'$$

and a canonical anti-isomorphism

$$\mathrm{Co}(X) \to \mathrm{Co}(X'), \quad g \mapsto g^t = (g')^{-1}.$$

For instance, this is the case in ordinary projective geometry (which is non-degenerate).

## 3 Generalized polar geometries

**3.1.** An *antiautomorphism* of a generalized projective geometry $(X, X')$ is an isomorphism

$$(p, p') : (X, X') \to (X', X)$$

onto the dual pair. A *correlation* of $(X, X')$ is an antiautomorphism $(p, p')$ such that $p' = p^{-1}$, or, equivalently, such that $p^t = p$. This in turn means that the identity

$$p(\mu_r(x, p(y), z)) = \mu'_r(p(x), y, p(z)) \tag{3.1}$$

and its dual hold.

**3.2.** With respect to a fixed correlation $p$, a point $x \in X$ is called *non-isotropic* if $x$ and $p(x)$ are remote (i.e. $(x, p(x)) \in M$) and *isotropic* if $x$ and $p(x)$ are neighboring. A correlation is called a *null-system* if all points $x \in X$ are isotropic and a *polarity* if there exist non-isotropic points. A *generalized polar geometry* is a generalized projective geometry $(X, X')$ together with a polarity $p$; *homomorphisms of generalized polar geometries* are homomorphisms $(g, g')$ of generalized projective geometries commuting with the respective polarities; in particular, the *automorphism group* $\mathrm{Aut}(X, X', p)$ is the group of all elements $(g, g') \in \mathrm{Aut}(X, X')$ such that $g' \circ p = p \circ g$.

**3.3.** For a fixed polarity $p$, the set

$$Q^{(p)} := \{x \in X \mid (x, p(x)) \notin M\} \tag{3.2}$$

of isotropic points is called the *associated quadric*, and its complement is denoted by

$$M^{(p)} = \{x \in X \mid (x, p(x)) \in M\}. \tag{3.3}$$

By the very definition of a polarity the set $M^{(p)}$ is not empty. A polarity is called *elliptic* if the quadric is empty.

**3.4.** If a correlation $p : X \to X'$ is fixed we will use it frequently to identify $X$ with $X'$. Thus the product maps $\mu_r$ and $\mu_r'$ are both represented by a ternary map

$$\tilde{\mu}_r : X \times X \times X \supset D \to X, \quad (x, y, z) \mapsto \mu_r(x, p(y), z)$$

satisfying identities which are obtained from (PG1) and (PG2) by simply forgetting the distinction between $\mu_r$ and $\mu_r'$. Polar geometries are then characterized by the fact that some element of the diagonal in $X \times X$ belongs to $M$, whereas for null geometries this is not the case. One may use these properties for an axiomatic definition of a *generalized polar geometry* (which is thus a set $X$ together with a subset of $X \times X$ containing some element of the diagonal and a family of ternary maps $\mu_r$ defined on a subset of $X \times X \times X$ and satisfying certain identities). Homomorphisms are then precisely the maps which are compatible with the multiplication maps $\tilde{\mu}_r$.

**3.5.** Not every generalized projective geometry does admit a polarity—take e.g. the flat case given a by a pair of non-isomorphic vector spaces. It is all the more important that one can associate to any generalized projective geometry a polar geometry in a canonical way:

**Proposition 3.6.** *For any generalized projective geometry $(X, X')$, the generalized projective geometry $(X \times X', X' \times X)$ admits a canonical polarity, given by the exchange map $p(x, x') = (x', x)$. The corresponding space $M^{(p)} \subset X \times X'$ is equal to the space $M$ of vectorializations of $X$.*

*Proof.* It is clear that $(x, x')$ and $p(x, x') = (x', x)$ are remote if and only if $x$ and $x'$ are remote, whence the last claim. The first claim is proved by the following calculation:

$$\begin{aligned} p\mu_r((x, x'), p(y, y'), (z, z')) &= p\mu_r((x, x'), (y', y), (z, z')) = p(\mu_r(x, y', z), \mu_r'(x', y, z')) \\ &= (\mu_r'(x', y, z'), \mu_r(x, y', z)) = \mu_r'((x', x), (y, y'), (z', z)) \\ &= \mu_r(p(x, x'), (y, y'), p(z, z')). \quad \square \end{aligned}$$

**3.7.** The preceding proposition yields an imbedding of the category of generalized projective geometries into the category of generalized polar geometries (if to a homomorphism $(g, g')$ we associate the pair $(g \times g', g' \times g))$. In Jordan theory the corresponding construction is given by imbedding the category of Jordan pairs into the category of Jordan triple systems by constructing the *associated polarized Jordan triple system* ([17, p. 10]). In a somewhat related context, similar constructions have already been used by Rozenfeld, see [9, p. 172].

## 4 Associated symmetric spaces over $\mathbb{K}$

**4.1.** A *symmetric space* (in the sense of O. Loos [16]) is a real smooth manifold $M$ with a smooth binary map $\mu : M \times M \to M$, $(x, y) \mapsto \mu(x, y) = \sigma_x(y)$ satisfying

(M1) $\sigma_x(x) = x$

(M2) $\sigma_x \circ \sigma_x = \mathrm{id}_M$

(M3) $\sigma_x \in \mathrm{Aut}(\mu)$, i.e. $\sigma_x(\mu(y, z)) = \mu(\sigma_x(y), \sigma_x(z))$

(M4) the fixed point $x$ of $\sigma_x$ is isolated.

The automorphism $\sigma_x$ is called the *symmetry w.r.t. x*, and the *transvection group* $G(M)$ of a symmetric space is the group generated by all $\sigma_x\sigma_y$ with $x, y \in M$. A *connected* symmetric space is homogeneous under the group $G(M)$ and is of the form $G/H$ where $G = G(M)$ is a Lie group and $H$ an open subgroup of the group of fixed points of a non-trivial involution of $G$; such spaces will be called *homogeneous symmetric spaces*. There exists a theory of *symmetric k-varieties* (see [11]) over general base fields, but not of general symmetric spaces in the sense of Loos (possibly infinite-dimensional and defined over rings)—one reason for this is certainly that symmetric spaces (in the sense of Loos) over a general base field or ring will be "less homogeneous" than the real or complex ones (see examples in Chapter 4 of [3]). We do not try to define here formally what a "symmetric space over $\mathbb{K}$" should be, but the following theorem shows that any generalized polar geometry over $\mathbb{K}$ defines a structure which certainly is one:

**Theorem 4.2.** *Assume* $(X, X', p)$ *is a generalized polar geometry over* $\mathbb{K}$. *Then the complement* $M^{(p)}$ *of the associated quadric is stable under the binary map*

$$\mu(x, y) := \mu_{-1}(x, p(x), y)$$

*which satisfies the properties* (M1)–(M3), *and the symmetry with respect to a base point* $o \in M^{(p)}$ *is* $\sigma_o = (-1)_{o,p(o)}$ *(the negative of the identity of the* $\mathbb{K}$*-module* $V_{p(o)}$*).*

*Proof.* If $(g, g')$ is an automorphism of the polar geometry $(X, X', p)$, then $g$ preserves the set $M^{(p)}$ and is compatible with $\mu$; in fact,

$$\begin{aligned} g\mu(x, y) &= g\mu_{-1}(x, p(x), y) \\ &= \mu_{-1}(gx, g'p(x), gy) \\ &= \mu_{-1}(gx, p(gx), gy) \\ &= \mu(gx, gy). \end{aligned}$$

Applying this to $(g, g') = ((-1)_{o,p(o)}, (-1)_{p(o),o})$, we get the identity (M3). (Note that in general $(g, g') = (r_{x,p(x)}, r^{-1}_{p(x),x})$ is an automorphism of the polar geometry if and only if $r = r^{-1}$; thus the construction works precisely for the scalars $r$ such that $r^2 = 1$.) The properties (M1) and (M2) are clear, and the last statement holds by the very definition of $\mu$. □

It is indeed reasonable to say that the fixed point $o$ of $\sigma_o = (-1)_{o,p(o)}$ is isolated: since 2 is invertible in $\mathbb{K}$, we have, in every $\mathbb{K}$-module, $-x = x$ iff $x = 0$.

**4.3 The Jordan–Lie functor.** We say that $(M^{(p)}, \mu)$ is the *symmetric space associated to the generalized polar geometry* $(X, X', p)$. *Homomorphisms of symmetric spaces* are maps $\phi : M \to N$ commuting with multiplication maps $\mu$ of $M$ and $N$. The arguments given in the preceding proof show that then homomorphisms of polar geometries induce homomorphisms of the associated symmetric spaces. Thus we have defined a covariant functor from generalized polar geometries over $\mathbb{K}$ into spaces having the properties from Theorem 4.2. This functor is called the *geometric Jordan–Lie functor*; as mentioned in the introduction, it generalizes the geometric Jordan–Lie functor from the real finite-dimensional case considered in [5].

**Corollary 4.4.** *If $(X, X')$ is a generalized projective geometry, then the space $M \subset X \times X'$ with the multiplication map*

$$\mu((o, o'), (x, x')) = (\mu_{-1}(o, o', x), \mu_{-1}(o', o, x'))$$

*is a symmetric space over $\mathbb{K}$ in the sense of* Theorem 4.2.

*Proof.* Apply Theorem 4.2 to the polarity of $(X \times X', X' \times X)$ from Proposition 3.6. □

**4.5.** The symmetric space $M$ from the preceding corollary has as additional structure a *double fibration* over $X$ and over $X'$ such that the fibers are affine spaces:

$$M = \bigcup_{o \in X} \{o\} \times V'_o = \bigcup_{o' \in X'} \{o'\} \times V_{o'}. \tag{4.1}$$

This can be seen as a sort of "polarization" on $M$; in fact these spaces generalize the *para-Hermitian symmetric spaces* introduced by Kozai and Kaneyuki ([14]). If $(g, g')$ is an automorphism of $(X, X')$, then, by the proof of Theorem 4.2, $g \times g'$ preserves $M$ and induces an automorphism of the symmetric space $M$ which preserves the double fibration (4.1). Similarly, it can be verified that, if $(g, g')$ is an antiautomorphism of the generalized projective geometry $(X, X')$, then

$$\tilde{g} : X \times X' \to X \times X', \quad (x, a) \mapsto (g'(a), g(x))$$

preserves $M$ and induces an automorphism of the symmetric space $M$ which exchanges the fibers of the double fibration (4.1). Automorphisms of the first type can be considered as "para-holomorphic" whereas the ones of the second type are sort of "anti para-holomorphic". The automorphism $\tilde{g}$ is involutive iff $g' = g^{-1}$, that is, iff $(g, g')$ is a correlation, and it has a fixed point in $M$ iff it is a polarity. Thus the polarities correspond precisely to the anti-paraholomorphic involutions having a fixed point in $M$. The whole fixed point set of a correlation $p$ is

$$(X \times X)^{\tilde{p}} = \{(x, p(x)) \mid x \in X\} \subset X \times X';$$

if we identify $X$ and $X'$ via $p$ then this is just the diagonal in $X \times X$. The intersection of this set with $M$ is the fixed point set of $\tilde{p}$ in $M$; it is non-empty iff $p$ is a polarity, and then

$$M^{(p)} \to M^{\tilde{p}}, \quad x \mapsto (x, p(x))$$

is an isomorphism of symmetric spaces. Thus $M^{(p)}$ is imbedded in $M$ as a sort of "para-real form".

## 5 Associated group actions

**5.1.** We have already defined the groups $\mathrm{Aut}(X, X')$ and $\mathrm{Int}(X, X')$ associated to a generalized projective geometry $(X, X')$ (Sections 1.6 and 2.2). If $(o, o') \in M$ is a base point, we write $V := V_{o'}$, $V' := V'_o$ and consider the following stabilizer groups:

$$\begin{aligned} P &:= \{(g, g') \in \mathrm{Aut}(X, X') \mid g.o = o\}, \\ P' &:= \{(g, g') \in \mathrm{Aut}(X, X') \mid g'.o' = o'\}, \\ \mathrm{Str}(V, V') &:= P \cap P' = \{(g, g') \in \mathrm{Aut}(X, X') \mid g.o = o, g'.o' = o'\}. \end{aligned} \tag{5.1}$$

The group $\mathrm{Str}(V, V')$ is called the *structure group of* $(V, V')$; by the very definition of an automorphism, it acts as a subgroup of $\mathrm{Gl}(V) \times \mathrm{Gl}(V')$. Note that, with $\tilde{\tau}_w$ defined by (2.9),

$$\tilde{\tau}_w(o) = 2_{o,o'}^{-1} 2_{o,w}(o) = o, \tag{5.2}$$

and thus $\tilde{\tau}_{V'} \subset P$ and, dually, $\tau_V \subset P'$.

**Lemma 5.2.** (i) $P = \mathrm{Str}(V, V') \cdot \tilde{\tau}_{V'}$ (*semidirect product*)
(ii) $P' = \mathrm{Str}(V, V') \cdot \tau_V$ (*semidirect product*)

*Proof.* (i) Since automorphisms are remoteness preserving we have $g.o \in V$ iff $(g.o, o') \in M$ iff $(g')^{-1}.o' \in V'$. Therefore, if $g.o = o$, then $w := (g')^{-1}.o' \in V'$, and we let

$$(h, h') := (g, g') \circ (\tilde{\tau}_w, \tau_w) = (g \circ \tilde{\tau}_w, g' \circ \tau_w).$$

It follows that

$$h.o = g(\tilde{\tau}_w.o) = g.o = o, \quad h'.o' = g'\tau_w.o' = g'(w) = o',$$

and therefore $(h, h') \in \mathrm{Str}(V, V')$. This proves existence of the decomposition. Let us prove uniqueness: if $(g, g') = (h, h') \circ (\tilde{\tau}_v, \tau_v) \in P$, then $(g')^{-1}.o' = (\tau_v)^{-1}.o' = -v$; thus $v$ and hence also $(h, h')$ are uniquely determined. Part (ii) is the dual version of (i). □

For a fixed base point $(o, o') \in M$, let $\Omega \subset \mathrm{Aut}(X, X')$ be the subset

$$\Omega := \{(g, g') \in \mathrm{Aut}(X, X') \mid g.o \in V\}. \tag{5.3}$$

Then, as remarked in the preceding proof,

$$\Omega^{-1} = \{(g, g') \in \mathrm{Aut}(X, X') \mid g'.o' \in V'\}. \tag{5.4}$$

It is clear that $P \subset \Omega$ and $P' \subset \Omega^{-1}$.

**Lemma 5.3.** $\Omega = \tau_V \cdot P = \tau_V \cdot \mathrm{Str}(V, V') \cdot \tilde{\tau}_{V'}$ (*unique decomposition*)

*Proof.* Let $(g, g') \in \Omega$ and put $v := g.o$. Then $(\tau_{-v}, \tilde{\tau}_{-v}) \circ (g, g') \in P$, and we can apply the preceding lemma. Conversely, if $(g, g') = (\tau_w, \tilde{\tau}_w) \circ (p, p')$ with $w \in V$, $(p, p') \in P$, then $g.o = w \in V$, and moreover $w$ and thus $(p, p')$ is uniquely determined. □

**5.4.** The preceding decompositions do not extend to the whole group $\mathrm{Aut}(X, X')$. However, in certain special cases we get the *Harish–Chandra decomposition* known from the theory of Hermitian symmetric spaces: let us say that a generalized projective geometry $(X, X')$ is *stable* if the intersection $V_a \cap V_b$ is non-empty for all $a, b \in X'$, and dually (cf. [21, Proposition 3.2] for this terminology). As a consequence, $X$ is then already covered by the $V_a$ with $a \in V'$, and dually:

$$X = \bigcup_{a \in V'} V_a, \quad X' = \bigcup_{x \in V} V'_x.$$

**Proposition 5.5** ("Harish–Chandra decomposition"). *If* $(X, X')$ *is stable, then*

$$\mathrm{Aut}(X, X') = \Omega \cdot \tau_V = \tau_V \cdot \mathrm{Str}(V, V') \cdot \tilde{\tau}_{V'} \cdot \tau_V$$

(*non-unique decomposition*).

*Proof.* Let $(g, g') \in \mathrm{Aut}(X, X')$; pick $x \in g^{-1}(V) \cap V = V_{g'.o'} \cap V_{o'}$; then $v := g(x) \in V$. Thus $(g, g') \circ (\tau_x, \tilde{\tau}_x) \in \Omega$, and we can apply the preceding lemma. □

It can be shown that the stability condition is fulfilled for instance in the finite-dimensional case over a field (cf. [19], [21]) and for some infinite-dimensional geometries modelled on complex or real Banach spaces.

**5.6 Connectedness.** We say that $(x,a)$ and $(y,b) \in M$ are *connected* if there is a sequence $(p_1,q_1),\dots,(p_n,q_n) \in M$ such that $(p_1,q_1)=(x,a)$, $(p_n,q_n)=(y,b)$ and

$$(p_{j+1},q_{j+1}) \in (V_{q_j}, V'_{p_j}), \quad j=1,\dots,n-1.$$

It is clear that connectedness is an equivalence relation; thus we get a partition of $M=\bigcup_{i\in I} M_i$ into connected components, and it is easily verified that all $(X_i,X_i',M_i)$ with $X_i := \mathrm{pr}_1(M_i)$, $X_i' = \mathrm{pr}_2(M_i)$ are subspaces of $(X,X',M)$, called *connected components of* $(X,X')$. (A natural example of a non-connected geometry is the Grassmannian $(X,X)$ of *all* subspaces of a given $\mathbb{K}$-module, cf. Chapter 2 of [3].) It is an easy exercise to show that a stable geometry is connected (but the converse is not true).

**Theorem 5.7.** *Assume* $(X,X')$ *is connected and fix a base point* $(o,o') \in M$. *Then the subgroup*

$$G := \langle \tau_V \cup \tilde{\tau}_{V'} \rangle$$

*of* $\mathrm{Int}(X,X')$ *generated by the translation group and the dual translation group, acts transitively on X, on X′ and on M, i.e.*

$$X \cong G/(P\cap G), \quad X' \cong G/(P'\cap G), \quad M \cong G/(\mathrm{Str}(V,V')\cap G).$$

*In particular, X, X′ and M are homogeneous under the action of* $\mathrm{Aut}(X,X')$, *and M is a homogeneous symmetric space in the sense that it is homogeneous under its automorphism group.*

*Proof.* It is enough to show that, if $(x,a) \in M \cap (V\times V')$ is an arbitrary point, then there exists $(g,g') \in G$ such that $(x,a)=(g.o,g'.o')$; the claim then follows by a straightforward induction using connectedness. We can write $(x,a)=(f.o,h'.o')$ with $(f,f')=(\tau_x,\tilde{\tau}_x)$, $(h,h')=(\tilde{\tau}_a,\tau_a) \in G$. Since $(x,a)\in M$, we have

$$\begin{aligned}(x,a) &= (\tilde{\tau}_a,\tau_a).(\tilde{\tau}_{-a}(x),o') \\ &= (\tilde{\tau}_a,\tau_a)\circ(\tau_{\tilde{\tau}_{-a}(x)},\tilde{\tau}_{\tilde{\tau}_{-a}(x)}).(o,o'),\end{aligned}$$

thus we have $(x,a)=(g,g').(o,o')$ with the element

$$(g,g') = (\tilde{\tau}_a,\tau_a)\circ(\tau_{\tilde{\tau}_{-a}(x)},\tilde{\tau}_{\tilde{\tau}_{-a}(x)}) \tag{5.5}$$

of G. We have proved that G, and hence also $\mathrm{Aut}(X,X')$, act transitively on M, X and X′. Since $\mathrm{Aut}(X,X')$ acts by automorphisms of the symmetric space M (cf. Section 3.7, the proof of Theorem 4.2 and Corollary 4.4), $M$ is homogeneous under its automorphism group. □

In [21], the group $G$ is called the *projective elementary group* of $(V, V')$. Transitivity of $\mathrm{Aut}(X, X')$ on $M$ means that up to isomorphism there is only *one* $\mathbb{K}$-module on which a connected geometry $(X, X')$ is modelled, namely the equivalence class of all vectorializations under the automorphism group. For non-connected geometries this is in general no longer true. Note that in the preceding proof we could also have written

$$\begin{aligned}(x,a) &= (\tau_x, \tilde{\tau}_x).(o, \tilde{\tau}_{-x}\tau_a.o') \\ &= (\tau_x, \tilde{\tau}_x) \circ (\tilde{\tau}_{\tilde{\tau}_{-x}(a)}, \tau_{\tilde{\tau}_{-x}(a)}).(o, o'),\end{aligned}$$

and thus we could also have taken the element

$$(e, e') := (\tau_x, \tilde{\tau}_x) \circ (\tilde{\tau}_{\tilde{\tau}_{-x}(a)}, \tau_{\tilde{\tau}_{-x}(a)}) \tag{5.6}$$

of $G$. These two possibilities correspond to two different ways of joining $(o, o')$ and $(x, a)$ by using the double fibration of $M$ mentioned in 4.5.

**Corollary 5.8.** *With respect to a base point $(o, o') \in M$, we have the following expression for the multiplication map $\mu_r$: for all $(a, x) \in M \cap (V \times V')$ and $r \in \mathbb{K}$,*

$$r_{x,a} = \tau_x \tilde{\tau}_{\tilde{\tau}_{-x}(a)} r_{o,o'} (\tau_x \tilde{\tau}_{\tilde{\tau}_{-x}(a)})^{-1},$$

*and dually.*

*Proof.* Just write $(x, a) = (e.o, e'.o')$ with $(e, e')$ given by (5.6) and use that $r_{e.o, e'.o'} = e \circ r_{o,o'} \circ e^{-1}$ since $(e, e')$ is an automorphism. □

**5.9 Non-homogeneous symmetric spaces.** If $(X, X', p)$ is a generalized polar geometry, then the associated symmetric space $M^{(p)}$ is general *not* homogeneous under its automorphism group, even if $(X, X')$ is connected. The orbit structure of $M^{(p)}$ under $\mathrm{Aut}(X, X', p)$ is in general as complicated as the classification of non-degenerate quadratic forms over $\mathbb{K}$ (which is a special case of our general set-up, see Section 5.3 of [3]). However, there may exist polarities "of symplectic type"; for those, the orbit structure is as simple as the classification of symplectic forms (which also is a special case of our general set-up). Theorem 5.7 shows that the "exchange polarity" from Proposition 3.6 is of the latter type.

## 6 Bergman operator and quadratic map

**6.1.** In this section we fix a base point $(o, o') \in M$ and let $V := V_{o'}$, $V' := V'_o$. Following a standard terminology in Jordan theory, the set of remote elements in $V \times V'$,

$$M \cap (V \times V') = \{(x, a) \in M \mid (x, o') \in M, (o, a) \in M\} \tag{6.1}$$

is called the set of *quasi-invertible pairs*. From the definition it is clear that $(x, a)$ is quasi-invertible in $(X, X'; o, o')$ iff so is $(a, x)$ in $(X', X; o', o)$. If $(x, a)$ is quasi-invertible, then we can write $(x, a) = (g, g').(o, o') = (e, e').(o.o')$ according to Equations (5.5) and (5.6). The element $(g, g')^{-1} \circ (e, e') \in \mathrm{Str}(V, V')$ defines the *Bergman operator*: for $(x, a)$ quasi-invertible we define

$$\begin{aligned} B(x, a)^{-1} &:= \tau_{-\tilde{\tau}_{-a}(x)} \tilde{\tau}_{-a} \tau_x \tilde{\tau}_{\tilde{\tau}_{-x}(a)}, \\ B(a, x)^{-1} &:= \tau_{-\tilde{\tau}_{-x}(a)} \tilde{\tau}_{-x} \tau_a \tilde{\tau}_{\tilde{\tau}_{-a}(x)}. \end{aligned} \tag{6.2}$$

Then $(B(x, a), B(a, x)^{-1})$ is an automorphism of $(X, X')$ fixing $(o, o')$ and hence belongs to $\mathrm{Str}(V, V')$.

**6.2.** From the definition it is easily deduced that the Bergman operator depends functorially on the geometry with base point: if $(h, h')$ is a base point preserving homomorphism, then

$$B(hx, h'a) \circ h = h \circ B(x, a), \tag{6.3}$$

and similarly for base point preserving adjoint pairs.

**6.3 The quadratic maps.** If $(x, a)$ or $(-x, a)$ belongs to $M$, then the following expressions are defined:

$$\begin{aligned} Q(x)a &:= -M^{(1/2)}_{x,-x}(a) = -\mu_{1/2}(x, a, -x), \\ Q(a)x &:= -M^{(1/2)}_{a,-a}(x) = -\mu'_{1/2}(a, x, -a). \end{aligned} \tag{6.4}$$

The first expression depends linearly on $a$ in the $\mathbb{K}$-module $V'$ (wherever the expression is defined); in fact, (PG2) and (1.6) together imply that $(M^{(1/2)}_{x,-x})^t = M^{(1/2)}_{x,-x}$, whence $Q(x)^t = Q(x)$, which means that $(g, h) = (Q(x), Q(x))$ is an adjoint pair, and thus $Q(x) : V'_{Q(x)o'} \to V_{o'}$ is affine. Since $Q(x)o' = -\mu_{1/2}(x, o', -x) = o$, it follows that $Q(x) : V'_o \to V_{o'}$ is linear with respect to the zero vectors $o$ and $o'$. Next we show that $Q(x)a$ is homogeneous quadratic in $x$: from the fact that $(r_{x,a}, r_{a,x})$ for $r \in \mathbb{K}$ is an adjoint pair it follows that if $(rx, a)$ is quasi-invertible, then so is $(x, ra)$. Then we have by (PG1) and by linearity in $a$:

$$\begin{aligned} Q(rx)a &= -\mu_{1/2}(rx, a, -rx) = -r\mu_{1/2}(x, ra, -x) \\ &= -r^2\mu_{1/2}(x, a, -x) = r^2 Q(x)a. \end{aligned}$$

It will be shown in 8.6 that $Q$ actually extends to a quadratic polynomial on $V$.

**6.4 The symmetry principle.** The expressions $\tilde{\tau}_a(x)$ and $\tilde{\tau}_x(a)$ cannot be directly compared since they belong to different spaces. However, we have for all $a \in V'$ and $x \in V$ the following equality, called the *symmetry principle*:

$$\tau_x Q(x) \tilde{\tau}_x(a) = \tilde{\tau}_a(x). \tag{6.5}$$

In fact, using (PG1) and Equation (1.6) we have

$$\begin{aligned}\tilde{\tau}_a(x) = \tilde{\tau}_{-a}^{-1}(x) &= (2_{o,o'}^{-1} 2_{o,-a})^{-1}(x) = 2_{o,-a}^{-1} 2_{o,o'}(x) \\ &= \mu_{1/2}(o, -a, 2_{o,o'}(x)) = \mu_{1/2}(o, -a, \tau_x(x)) \\ &= \tau_x \mu_{1/2}(-x, \tilde{\tau}_{-x}(-a), x) = \tau_x(-Q(x)\tilde{\tau}_{-x}(-a)) \\ &= \tau_x Q(x) \tilde{\tau}_x(a).\end{aligned}$$

(We have used here that $\tilde{\tau}_{-x}(-a) = -\tilde{\tau}_x(a)$ which follows from the fact that $(-\mathrm{id}_V, -\mathrm{id}_{V'}) = ((-1)_{o,o'}, (-1)_{o',o})$ belongs to the structure group.)

## 7 The tangent bundle

**7.1.** The *tangent bundle* of a generalized projective geometry $(X, X')$ defined over $\mathbb{K}$ is the scalar extension

$$(\Phi, \Phi') : (X, X') \to (TX, TX') := (X_{\mathbb{K}(\varepsilon)}, X'_{\mathbb{K}(\varepsilon)}) \tag{7.1}$$

by the ring $\mathbb{K}(\varepsilon)$ of *dual numbers over* $\mathbb{K} : \mathbb{K}(\varepsilon) = \mathbb{K}[x]/(x^2)$; a model of $\mathbb{K}(\varepsilon)$ is $R = \mathbb{K} \oplus \mathbb{K}$ with elements denoted by $r + \varepsilon s$, $r, s \in \mathbb{K}$ and multiplied by the rule $(r + \varepsilon s)(r' + \varepsilon s') = rr' + \varepsilon(r's + s'r)$. The scalar extension $\phi : \mathbb{K} \to R$ is thus always injective, and so is, for any $\mathbb{K}$-module $V$, the natural map

$$V \to V_{\mathbb{K}(\varepsilon)} := V \otimes_{\mathbb{K}} R \cong V \oplus \varepsilon V$$

given by restriction of $\Phi$. If a base point $(o, o')$ is fixed, we will identify $V$ with $\Phi(V)$ and $V'$ with $\Phi'(V')$. Next we are going to show that $(TX, TX')$ is a fibered space with fibers carrying a natural $\mathbb{K}$-module structure:

**Proposition 7.2.** *Assume* $(Y, Y')$ *is a generalized projective geometry over* $\mathbb{K}(\varepsilon)$, *and let* $(o, a) \in M$. *Then the set*

$$F_o := \varepsilon_{o,a}(V_a) \subset Y$$

*as well as its affine structure over* $\mathbb{K}$ *induced from* $V_a$ *are independent of the point* $a \in V_o'$.

*Proof.* Let $o' \in V_o'$ be another affinization. We have to prove that

$$\varepsilon_{o,a}(V_a) = \varepsilon_{o,o'}(V_{o'}) \tag{7.2}$$

as sets and as $\mathbb{K}$-modules. Write $a = \tau_a^{(o',o)}(o') =: \tau_a(o')$ and note that, since $(\tilde{\tau}_a, \tau_a)$ is an automorphism, $\tilde{\tau}_a : V_{o'} \to V_a$ is a $\mathbb{K}$-module isomorphism and $\varepsilon_{o,a} = \varepsilon_{\tilde{\tau}_a(o),\tau_a(o')} = \tilde{\tau}_a \circ \varepsilon_{o,o'} \circ \tilde{\tau}_{-a}$. It follows that

$$\tilde{\tau}_a : \varepsilon_{o,o'} V_{o'} \to \varepsilon_{o,a} V_a$$

is a $\mathbb{K}$-module isomorphism. Thus (7.2) is proved if we can show that

$$\tilde{\tau}_a(\varepsilon_{o,o'} z) = \varepsilon_{o,o'} z \tag{7.3}$$

for all $z \in V_{o'}$. In order to prove (7.3), we apply first the symmetry principle 6.4 with $x = \varepsilon_{o,o'} z$ and then use that $Q(x)$ is quadratic in $x$ along with $\varepsilon_{o,o'}^2 = 0_{o,o'}$:

$$\begin{aligned} \tilde{\tau}_a(\varepsilon_{o,o'} z) = \tilde{\tau}_a(x) &= \tau_x Q(x) \tilde{\tau}_x(a) \\ &= \tau_x \varepsilon_{o,o'}^2 Q(z) \tau_x(a) \\ &= \tau_x(o) = x = \varepsilon_{o,o'} z \end{aligned}$$

which had to be shown. □

**7.3.** Equation (7.3) shows also that all pairs $(\varepsilon_{o,o'} v, a)$ with $v \in V$, $a \in V'$, are quasi-invertible, and therefore also all pairs $(v, \varepsilon_{o',o} a)$ are quasi-invertible.

**Corollary 7.4.** *Assume $(Y, Y')$ and $(Z, Z')$ are generalized projective geometries over $\mathbb{K}(\varepsilon)$.*

(i) *If $(g, g') : (Y, Y') \to (Z, Z')$ is a homomorphism, then, for all $o \in Y$, $g(F_o) \subset F_{g(o)}$, and $g : F_o \to F_{g(o)}$ is $\mathbb{K}$-affine.*

(ii) *If $g : Y \to Z$, $g^t : Z' \to Y'$ is an adjoint pair of morphisms, then $g(F_o) \subset F_{g(o)}$ for all $o$ such that $(o, g^t(b)) \in M$ for some $b$ in the domain of definition of $g^t$, and $g : F_o \to F_{g(o)}$ is $\mathbb{K}$-affine.*

In particular, the relation

$$\mu_r(F_x, F'_a, F_y) \subset F_{\mu_r(x,a,y)} \tag{7.4}$$

holds for all $(x, a, y) \in D$, and dually.

*Proof.* The first claim is proved by the following calculation:

$$g(F_o) = g(\varepsilon_{o,a} V_a) \subset \varepsilon_{g(o),g'(a)} V_{g'(a)} = F_{g(o)},$$

and since $g : V_a \to V_{g'(a)}$ is affine, so is is the map $g : F_o \to F_{g(o)}$ obtained by restriction to the affine subspace $\varepsilon_{o,o'} V_a \subset V_a$. For adjoint pairs, the calculation is similar, replacing $a$ by $g^t(b)$. Applying (ii) to the adjoint pairs given by right, left and middle multiplications, we get (7.4). (The assumption of (ii) is fulfilled since we assume $\mu_r(x, a, y)$ to be defined.) □

**7.5.** Since $F_x = \tau_x(F_o)$, the sets $F_o$ with $o \in V_{o'}$ define a partition of $V_{o'}$, and since they do not depend on the affinization, the sets $F_o$ with $o \in Y$ define a partition of $Y$, and dually for $Y'$. Denote by $(Y, Y')/\sim$ the corresponding sets of equivalence classes; then relation (7.4) implies that there is a well-defined structure of generalized projective geometry over $\mathbb{K}$ on $(Y, Y')/\sim$, and the projection

$$(\pi, \pi') : (Y, Y') \to (Y, Y')/\sim \tag{7.5}$$

is a homomorphism. (This is a special case of the construction mentioned in Section 2.7 (5).)

**7.6.** We apply the preceding results to the tangent bundle $(TX, TX')$. The composition

$$(\pi, \pi') \circ (\Phi, \Phi') : (X, X') \to (TX, TX')/\sim, \quad (x, a) \mapsto (\Phi(x), \Phi'(a))/\sim \tag{7.6}$$

is a homomorphism which, in every affinization, is naturally identified with the identity map; one may say that it is a sort of "covering". Using the minimality required in the definition of a scalar extension (see 1.10), one can show that (7.6) actually is a bijection; we omit the technicalities since in the sequel we will actually only need that (7.6) is a bijection when restricted to affine parts. For simplicity of notation we will consider $(\Phi, \Phi')$ to be an inclusion of $(X, X')$ in $(TX, TX')$ and $(\pi, \pi')$ to be a homomorphism onto $(X, X')$.

Using these conventions, we define the *tangent space $T_oX$ of $X$ at $o$* to be the $\mathbb{K}$-module $\varepsilon_{o,o'} V_{o'}$, that is,

$$T_oX = \pi^{-1}(o) \subset TX,$$

and dually, we define the tangent space of $X'$ at $o'$. Thus the tangent bundle is a fibred space over the basis $(X, X')$:

$$TX = \bigcup_{p \in X} T_pX, \quad TX' = \bigcup_{a \in X'} T_aX',$$

and we can write any element of $TX$ as $\delta_p \in T_pX$ with a unique $p \in X$, and dually. If $(g, g') : (X, X') \to (Y, Y')$ is a homomorphism, then we define its *tangent map at $o$* by

$$T_og : T_oX \to T_{g.o}Y, \quad \varepsilon_{o,o'}v \mapsto \varepsilon_{g.o,g'.o'}gv;$$

one easily verifies that this is well-defined and linear; the map $(Tg, Tg') : (TX, TX') \to (TX, TX')$ thus obtained coincides with the extension $(g, g')_{\mathbb{K}(\varepsilon)}$ over

$\mathbb{K}(\varepsilon)$ of $(g, g')$ whose existence is required by 1.10. Similarly, for an adjoint pair $(g, g^t)$ we let

$$T_o g : T_o X \to T_{go} Y, \quad \varepsilon_{o, g^t o'} v \mapsto \varepsilon_{g.o, o'} gv$$

provided there exists $o'$ in the domain of $g^t$ with $(o, g^t(o')) \in M$.

## 8 Infinitesimal automorphisms

**8.1.** An *infinitesimal automorphism* of $(TX, TX')$ is an automorphism $(g, g')$ of $(TX, TX')$ preserving all tangent spaces, that is, we have $\pi \circ g = \pi$, and dually. Since any automorphism of $(TX, TX')$ permutes the tangent spaces (Corollary 7.4), it is actually enough to require that $g(0_p) \in T_p X$, $g'(0_{p'}) \in T_{p'} X'$ for all $p \in X$, $p' \in X'$ (where $0_x$ denotes the zero vector in a tangent space at $x$). The composition of $(g, g')$ with the zero section,

$$(\xi, \xi') : (X, X') \to (TX, TX'), \quad (p \mapsto g(0_p), p' \mapsto g'(0_{p'}))$$

is a homomorphism; we call it the *associated vector field.*

**Lemma 8.2.** *If $(X, X')$ is connected, then the following pairs $(g, g')$, defined with respect to a base point $(o, o') \in M$, are infinitesimal automorphisms of $(TX, TX')$:*

(i) $(g, g') = ((1 + \varepsilon)_{o, o'}, (1 - \varepsilon)_{o', o})$,

(ii) $(g, g') = (\tilde{\tau}_{\varepsilon w}, \tau_{\varepsilon w})$ *with* $w \in V'$,

(iii) $(g, g') = (\tau_{\varepsilon v}, \tilde{\tau}_{\varepsilon v})$ *with* $v \in V$.

*Proof.* All three pairs are automorphisms of $(TX, TX')$: for the last two this follows directly from the definition of $\tilde{\tau}$, and for the first one, this follows from the fact that $(1 + \varepsilon)(1 - \varepsilon) = 1 - \varepsilon^2 = 1$.

In order to prove that $(\pi, \pi') \circ (g, g') = (\pi, \pi')$, we verify this property first on the trivialization of $(TX, TX')$ given by $(V, V')$; then since $(TX, TX')$ is algebraically generated by $(TV, TV')$ (Theorem 5.7), both sides, being homomorphisms, must coincide on all of $(TX, TX')$. For the following calculations, note that $x + \varepsilon v \in T_x X$; in fact

$$\pi(x + \varepsilon v) = \pi((1 - \varepsilon)x + \varepsilon(x + v)) = \pi(\varepsilon_{x, o'}(x + v)) = x. \tag{8.1}$$

Thus for the pair from (i) we have

$$g(x) = (1 + \varepsilon)_{o, o'}(x) = x + \varepsilon x \in T_x X \tag{8.2}$$

and dually. (The corresponding vector field is the *Euler vector field* corresponding to the trivialization $(V, V')$.) For the second pair we use the symmetry principle (6.5) twice (which is possible since all pairs $(x, \varepsilon w)$ are quasi-invertible, by 7.3).

$$\begin{aligned} g(x) = \tilde{\tau}_{\varepsilon w}(x) &= x + Q(x)\tilde{\tau}_x(\varepsilon w) \\ &= x + Q(x)\varepsilon w = x + \varepsilon Q(x)w \in T_x X \end{aligned} \tag{8.3}$$

and

$$g'(a) = \tau_{\varepsilon w}(a) = a + \varepsilon_{o',o} w \in T_a X'. \tag{8.4}$$

For the third pair the same argument applies. □

**Proposition 8.3.** *If $(g, g')$ is an infinitesimal automorphism of $(TX, TX')$, then $g$ and $g'$ act on all tangent spaces by translations, that is, with $(\xi, \xi')$ as in* 8.1, *we have*

$$g(\delta_p) = \delta_p + \xi(p), \quad g'(\delta_{p'}) = \delta_{p'} + \xi'(p')$$

*for all $\delta_p \in T_p X$, $\delta_{p'} \in T_{p'} X'$. In particular, the infinitesimal automorphisms form an abelian subgroup of* $\mathrm{Aut}(TX, TX')$.

*Proof.* Let $o := p$ and choose $o' \in V_o'$ arbitrary; then since $g(o) \in T_o X$ and $T_o X \subset V_{o'}$ we can decompose, using Lemma 5.3,

$$(g, g') = (\tau_{g.o}, \tilde{\tau}_{g.o}) \circ (h, h') \circ (\tilde{\tau}_w, \tau_w) \tag{8.5}$$

with $w = (g')^{-1} o' \in T_{o'} X'$. By Lemma 8.2, the first and the last and hence also the middle factor are then infinitesimal automorphisms. Since the first factor clearly acts by a translation on $T_o X$ and the last factor acts trivially on $T_o X$ (Equation (7.3)), it only remains to be shown that $h$ acts by a translation on $T_o X$. Now, writing $h = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, $\varepsilon = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$ with respect to the decomposition $TV = V \oplus \varepsilon V$, the condition $h \circ \varepsilon = \varepsilon \circ h$ yields

$$h = \begin{pmatrix} a & 0 \\ c & a \end{pmatrix}. \tag{8.6}$$

The condition $\pi \circ h = \pi$ implies $x = \pi(h(x)) = a(x)$ and thus $a = \mathrm{id}_V$, and $h$ acts by a translation on $T_o X$ as had to be shown. □

**8.4.** We denote by $\mathrm{InfAut}(TX, TX') \subset \mathrm{Aut}(X, X')$ the subgroup of infinitesimal automorphisms; by the preceding proposition, its group law is simply given by addition in tangent spaces and will therefore be written additively. We let further, with respect to a base point $(o, o')$,

$$\mathrm{InfStr}(TV, TV') := \mathrm{Str}(TV, TV') \cap \mathrm{InfAut}(TX, TX'); \tag{8.7}$$

this is the subgroup of elements of $\mathrm{Str}(TV, TV')$ of the form (8.6) with $a = \mathrm{id}$. Then the decomposition (8.5) reads, additively,

$$\mathrm{InfAut}(TX, TX') \cong V \oplus \mathrm{InfStr}(TV, TV') \oplus V'. \tag{8.8}$$

The infinitesimal automorphisms form a normal subgroup of $\mathrm{Aut}(TX, TX')$; in particular, for all $(h, h') \in \mathrm{Int}(TX, TX')$ the conjugation

$$(h, h')_*(g, g') := (hgh^{-1}, h'g'(h')^{-1})$$

is an automorphism of $\mathrm{InfAut}(TX, TX')$.

**8.5.** The following theorem is a main result of this chapter: we will prove that infinitesimal automorphisms are in an affinization represented by *polynomial quadratic* maps. Recall that $(g, g') \in \mathrm{InfAut}(TX, TX')$ is uniquely determined by its associated vector field $(\xi(p), \xi'(a)) = (g(0_p), g'(0_a))$, and in $(TV, TV')$, identifying $p$ with $0_p$ and $a$ with $0_a$, we can write

$$g(p) = p + \varepsilon G(p), \quad g'(q) = q + \varepsilon G'(q) \tag{8.9}$$

with functions $G : V \to V$, $G' : V' \to V'$.

**Theorem 8.6.** *For all* $(g, g') \in \mathrm{InfAut}(TX, TX')$, *the maps* $G : V \to V$, $G' : V' \to V'$ *are quadratic polynomial. More precisely, the map G is*

(i) *constant, if* $(g, g') = (\tau_{\varepsilon v}, \tilde{\tau}_{\varepsilon v})$ *belongs to the first term in the decomposition* (8.8),

(ii) *linear, if* $(g, g') \in \mathrm{InfStr}(TV, TV')$,

(iii) *homogeneous quadratic polynomial, if* $(g, g') = (\tilde{\tau}_{\varepsilon w}, \tau_{\varepsilon w})$ *with* $w \in V'$ *is an element of the third term in* (8.8).

*Proof.* It is enough to show that $G$ is quadratic polynomial; by duality the corresponding statement for $G'$ then follows. Part (i) is clear since $\tau_{\varepsilon v}(p) = p + \varepsilon v$ and Part (ii) follows from (8.6) since $h(p) = p + \varepsilon c(p)$. Thus only Part (iii) remains to be proved. Comparing with Equation (8.3), we see that (iii) is equivalent to saying that $Q(x)\varepsilon w$ is quadratic polynomial in $x$. We have already seen (see 6.3) that this expression is homogeneous quadratic in $x$. In order to prove that the term

$$Q(x, z) := Q(x + z) - Q(x) - Q(z) \tag{8.10}$$

is linear in $x$ and in $z$, we use the Bergman operator of $(TX, TX')$ associated to the base point $(o, o')$: we claim that $(B(x, \varepsilon a), B(\varepsilon a, x)^{-1})$ is an infinitesimal automorphism. In fact, replace $a$ by $\varepsilon a$ in the definition of $B(x,a)$ (Section 6.1) and recall that $\tilde{\tau}_{-x}(\varepsilon a) = \varepsilon a$ (Equation (7.3)); thus

$$\begin{aligned} B(x,\varepsilon a)^{-1} &= \tau_{-\tilde{\tau}_{-\varepsilon a}(x)}\tilde{\tau}_{-\varepsilon a}\tau_x\tilde{\tau}_{\tilde{\tau}_{-x}(\varepsilon a)} \\ &= \tau_{-x}\tau_{\varepsilon Q(x)a}\tilde{\tau}_{-\varepsilon a}\tau_x\tilde{\tau}_{\varepsilon a}, \\ B(\varepsilon a, x)^{-1} &= \tau_{-\tilde{\tau}_{-x}(\varepsilon a)}\tilde{\tau}_{-x}\tau_{\varepsilon a}\tilde{\tau}_{\tilde{\tau}_{-\varepsilon a}(x)} \\ &= \tau_{-\varepsilon a}\tilde{\tau}_{-x}\tau_{\varepsilon a}\tilde{\tau}_{\varepsilon Q(x)a}\tilde{\tau}_x. \end{aligned} \tag{8.11}$$

These equations show that $(B(x,\varepsilon a), B(\varepsilon a, x)^{-1})$ is a composition of two infinitesimal automorphisms, where the first factor comes after a conjugation by $(\tau_x, \tilde{\tau}_x)$. More explicitly, evaluating at a point $z \in V$, we get for the first component (note that if $G$ is defined by (8.9), then $(\tau_{-x} g \tau_x)(p) = p + \varepsilon G(x+p)$)

$$\begin{aligned} B(x,\varepsilon a)^{-1} z &= (\tau_{-x}\tau_{\varepsilon Q(x)a}\tau_x)(\tau_{-x}\tilde{\tau}_{-\varepsilon a}\tau_x)\tilde{\tau}_{\varepsilon a}(z) \\ &= z + \varepsilon(Q(x) + Q(z) - Q(z+x))(a) = z - \varepsilon Q(x,z)a. \end{aligned} \tag{8.12}$$

This expression is linear in $z$ since $B(x,\varepsilon a)$ is a linear operator, and since $Q(x,z)$ is symmetric in $x$ and $z$, it is also linear in $x$. □

In the situation of the preceding proof, let

$$\begin{aligned} T(x,\varepsilon a, z) &:= T(x,\varepsilon a)z := (Q(x+z) - Q(x) - Q(z))\varepsilon a \\ T(a,\varepsilon x, b) &:= T(a,\varepsilon x)b := (Q(a+b) - Q(a) - Q(b))\varepsilon x. \end{aligned} \tag{8.13}$$

Equation (8.12) and its dual can be written in matrix form

$$B(x,\varepsilon a)^{-1} = \begin{pmatrix} \mathrm{id} & 0 \\ -T(x,\varepsilon a) & \mathrm{id} \end{pmatrix}, \quad B(a,\varepsilon x)^{-1} = \begin{pmatrix} \mathrm{id} & 0 \\ -T(a,\varepsilon x) & \mathrm{id} \end{pmatrix}.$$

**8.7.** In the preceding proof, we have seen that $V \to \mathrm{Hom}(V', T_o X)$, $x \mapsto (a \mapsto Q(x)\varepsilon a = -\mu_{1/2}(x, \varepsilon a, -x))$ is a homogeneous quadratic polynomial. Although this will not be used in the sequel, we remark here that one can deduce from (PG2) that this polynomial satisfies the identity $Q(Q(x)y) = Q(x)Q(y)Q(x)$ known in Jordan theory as the "fundamental formula".

**8.8 Derivations.** This section will not be needed in the sequel, but for the sake of completeness, we explain the relation between infinitesimal automorphisms and derivations: a *derivation of* $(X, X')$ is a homomorphism $(\xi, \xi') : (X, X') \to (TX, TX')$ such that $(\pi, \pi') \circ (\xi, \xi') = \mathrm{id}_{(X,X')}$. We have already seen that infinitesimal automorphisms are uniquely determined by the associated vector field which is a derivation. Conversely, we have:

**Theorem 8.9.** *There is a canonical bijection between derivations and infinitesimal automorphisms, given by*

$$\xi(x) := g(x), \quad g(\delta_x) := \delta_x + \xi(x),$$

*and dually. The derivations form a* $\mathbb{K}$*-module w.r.t. pointwise addition and multiplication by scalars.*

*Proof.* Using the infinitesimal automorphisms from Lemma 8.2, one proves first that for $\delta_p \in T_pX$, $\delta_q \in T_qX$, $\delta_{o'} \in T_{o'}X'$,

$$\begin{aligned}\mu_r(\delta_p, \delta_{o'}, \delta_q) &= \mu_r(\delta_p, 0_{o'}, 0_q) + \mu_r(0_p, \delta_{o'}, 0_q) + \mu_r(0_p, 0_{o'}, \delta_q) \\ &= T_p(R_{o',q}) \cdot \delta_p + T_q(L_{p,o}) \cdot \delta_q + T_{o'}(M_{p,q}) \cdot \delta_{o'}. \qquad (8.14)\end{aligned}$$

Thus $(\xi, \xi')$ is a derivation iff

$$\xi(\mu_r(p, a, q)) = T_p(R_{a,q}) \cdot \xi(p) + T_a(M_{p.q}) \cdot \xi'(a) + T_q(L_{p,a}) \cdot \xi(q), \qquad (8.15)$$

and dually. Since tangent maps are linear, this description shows that the derivations of $(X, X')$ form a $\mathbb{K}$-module which we denote by $\mathrm{Der}(X, X')$. The zero vector is the canonical imbedding $(X, X') \to (TX, TX')$. Moreover, a direct calculation now shows that if $(\xi, \xi')$ is a derivation, then the pair $g := \xi + \mathrm{id} : TX \to TX$, $\delta_p \mapsto \delta_p + \xi(p)$, $g' := \xi' + \mathrm{id} : TX' \to TX'$, $\delta_a \mapsto \delta_a + \xi'(a)$ is an automorphism of $(TX, TX')$. □

The preceding results on infinitesimal automorphisms can now be rewritten in terms of derivations; see [16] for similar formulas in the case of symmetric spaces. In particular, an analog of [16, Theorem II.2.2] follows now from Theorem 8.6: $(g, g') = (-\mathrm{id}_V, -\mathrm{id}_{V'})$ is an automorphism of $(X, X')$ which induces an involution of $\mathrm{Der}(X, X')$; if we denote by $\mathrm{Der}(X, X') = \mathfrak{h} \oplus \mathfrak{q}$ the corresponding $\pm 1$-eigenspace decomposition, then the evaluation map at $(o, o')$ induces a bijection $\mathfrak{q} \to V \oplus V'$, and $\mathfrak{h}$ is its kernel.

## 9 Kantor–Koecher–Tits algebra and Jordan functor

**9.1.** As in Lie theory, we have to derive *twice* in order to linearize identities which take account of *non-commutativity*—the reason is simply that the group of infinitesimal automorphisms of $(TX, TX')$ is abelian, and thus we lose all information on non-commutativity if we derive only once. Therefore we introduce the "double tangent bundle" $(TTX, TTX') := (T(TX), T(TX'))$. It is isomorphic to the scalar extension of $(X, X')$ by

$$(\mathbb{K}(\varepsilon_1))(\varepsilon_2) = \mathbb{K} \oplus \varepsilon_1 \mathbb{K} \oplus \varepsilon_2 \mathbb{K} \oplus \varepsilon_3 \mathbb{K} \qquad (9.1)$$

with $\varepsilon_3 = \varepsilon_1\varepsilon_2 = \varepsilon_2\varepsilon_1$ and $\varepsilon_1^2 = \varepsilon_2^2 = \varepsilon_3^2 = 0$. Fixing as usual a base point $(o, o') \in M$, the corresponding linearization of $(TTX, TTX')$ is denoted by

$$TTV = V \oplus \varepsilon_1 V \oplus \varepsilon_2 V \oplus \varepsilon_3 V, \quad TTV' = V' \oplus \varepsilon_1 V' \oplus \varepsilon_2 V' \oplus \varepsilon_3 V'. \tag{9.2}$$

The space $(TTX, TTX')$ is fibered over $(X, X')$, and every fiber decomposes in the way indicated by (9.2). Let $\pi_1$, $\pi_2$ be the projections with kernels $\varepsilon_1(V \oplus \varepsilon_2 V)$, resp. $\varepsilon_2(V \oplus \varepsilon_1 V)$, and $\pi_3 = \pi_1 \circ \pi_2$ (kernel $\varepsilon_1\varepsilon_2 V$).

**9.2.** The group of infinitesimal automorphisms of $(TX, TX')$ can be injected into $\mathrm{Aut}(TTX, TTX')$ in three different ways such that fibers over $(X, X')$ are preserved. However, it is no longer true that these actions commute—in fact, their commutators give rise to the *Lie bracket*, to be defined in the following theorem. Recall that, if the integers are invertible in $\mathbb{K}$, the space $\mathrm{Pol}(V, V)$ of polynomial mappings from $V$ to $V$ is a Lie algebra over $\mathbb{K}$ with respect to the Lie bracket (see e.g. [15])

$$[X, Y](x) = \mathrm{D}X(x) \cdot Y(x) - \mathrm{D}Y(x) \cdot X(x), \tag{9.3}$$

where $\mathrm{D}F$ is the ordinary total differential, defined in terms of dual numbers by the condition

$$F(x + \varepsilon h) = F(x) + \varepsilon \mathrm{D}F(x) \cdot h. \tag{9.4}$$

If 2 and 3 are invertible in $\mathbb{K}$ (which we assume from now on), this formula still serves to define a bracket defined on quadratic polynomials having values in polynomials of degree at most 3.

**Theorem 9.3.** *There is a unique structure of a Lie algebra over* $\mathbb{K}$ *on the space* $\mathrm{InfAut}(TX, TX')$ *such that, if elements are realized as quadratic polynomials according to Theorem* 8.6, *then their bracket is given by Formula* (9.3). *The group* $\mathrm{Aut}(X, X')$ *acts by automorphisms of this Lie algebra structure.*

*Proof.* The following definition of the Lie bracket follows the one given by Faulkner [6] in a different context: let $(g, g'), (h, h') \in \mathrm{InfAut}(TX, TX')$. We realize these elements on $(TTX, TTX')$ in two different ways: we extend $(g, g')$ to an element $(g_1, g_1') \in \mathrm{Aut}(TTX, TTX')$ such that $(\pi_1, \pi_1') \circ (g_1, g_1') = (\pi_1, \pi_1')$ and $(h, h')$ to an element $(h_2, h_2')$ such that that $(\pi_2, \pi_2') \circ (h_2, h_2') = (\pi_2, \pi_2')$. Consider now the element

$$(f, f') := (g_1 h_2 g_1^{-1} h_2^{-1}, (g_1 h_2 g_1^{-1} h_2^{-1})') \tag{9.5}$$

of $\mathrm{Aut}(TTX, TTX')$. Since infinitesimal automorphisms with respect to $\varepsilon_1$ or with respect to $\varepsilon_2$ form normal subgroups in $\mathrm{Aut}(TTX, TTX')$, we see that $(f, f')$ belongs to both of them and hence satisfies $(\pi_3, \pi_3') \circ (f, f') = (\pi_3, \pi_3')$. Therefore $(f, f')$ preserves fibers of the subbundle $(X, X')_{\mathbb{K}(\varepsilon_3)}$ of $(TTX, TTX')$ and can thus be identified with an infinitesimal automorphism of this subbundle which we denote by $([g, h], [g', h'])$. It is clear that $\mathrm{Aut}(X, X')$ acts by automorphisms of this bracket.

Next let us prove that with respect to an affinization the bracket is indeed described by formula (9.3): we write $g_1(x) = x + \varepsilon_1 G(x)$, $h_2(x) = x + \varepsilon_2 H(x)$ and apply (9.4) in the following calculation to $G$ with respect to $\varepsilon_1$ and to $H$ with respect to $\varepsilon_2$:

$$\begin{aligned} g_1 h_2 g_1^{-1} h_2^{-1}(x) &= \varepsilon_1 G(x) + h_2 g_1^{-1}(x - \varepsilon_2 H(x)) \\ &= \varepsilon_1 G(x) + h_2(x - \varepsilon_2 H(x) - \varepsilon_1 G(x - \varepsilon_2 H(x))) \\ &= \varepsilon_1 G(x) + h_2(x - \varepsilon_1 G(x) - \varepsilon_2(H(x) - \varepsilon_1 \mathrm{D}G(x) \cdot H(x))) \\ &= \varepsilon_1 G(x) + (x - \varepsilon_1 G(x)) + \varepsilon_2 H(x - \varepsilon_1 G(x)) - \varepsilon_2(x - \varepsilon_1 \mathrm{D}G(x) \cdot H(x)) \\ &= x - \varepsilon_1 \varepsilon_2 (\mathrm{D}H(x) \cdot G(x) - \mathrm{D}G(x) \cdot H(x)), \end{aligned}$$

and thus $[g, h]$ is described by the polynomial $[G, H]$ given by (9.3) (*à priori*, this polynomial is of degree at most 3, but Theorem 8.6 tells us that it is actually of degree at most 2). We conclude that $\mathrm{InfAut}(TX, TX')$ is a Lie algebra: the defining identities of a Lie algebra are satisfied over the affinization $(V, V')$; every point is contained in some affinization; therefore they hold everywhere. □

**9.4.** The Lie algebra defined in the preceding theorem is called the *Kantor–Koecher–Tits algebra* associated to $(X, X')$. This Lie algebra is 3-graded with grading $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{-1}$ given by Equation (8.8), corresponding to the homogeneous polynomial components with respect to a fixed base point $(o, o')$: since the bracket of two homogeneous polynomials of degree $r$ and $s$ is of degree $r + s - 1$ and $\mathfrak{g}$ contains only quadratic polynomials, we have $[\mathfrak{g}_1, \mathfrak{g}_1] = 0 = [\mathfrak{g}_{-1}, \mathfrak{g}_{-1}]$ and $[\mathfrak{g}_1, \mathfrak{g}_{-1}] \subset \mathfrak{g}_0$. To any 3-graded Lie algebra over $\mathbb{K}$, one associates two $\mathbb{K}$-trilinear maps

$$T_\pm : \mathfrak{g}_{\pm 1} \times \mathfrak{g}_{\mp 1} \times \mathfrak{g}_{\pm 1} \to \mathfrak{g}_{\pm 1}, \quad (x, a, z) \mapsto T_\pm(x, a)z := [[x, a], z]. \tag{9.6}$$

It is easily verified that then the following identities are satisfied:

(LJP1) $T_\pm(x, a, z) = T_\pm(z, a, x)$

(LJP2) $[T_\pm(x, a), T_\pm(y, b)] = T_\pm(T_\pm(x, a, y), b) - T_\pm(y, T_\mp(a, x, b))$

By definition, a *linear Jordan pair over* $\mathbb{K}$ is a pair $(V^+, V^-)$ of $\mathbb{K}$-modules together with $\mathbb{K}$-trilinear maps $T_\pm$ such that the identities (LJP1) and (LJP2) hold (cf. [17, Proposition 2.2. (b)]); equivalently, linear Jordan pairs can be defined as the *polarized linear Jordan triple systems*, see [17, 1.12], or as the *twisted polarized Lie triple systems*, see [3, Ch. III.3] for the real case. Note that in the present case the maps $T_\pm$ can be expressed by the quadratic maps $Q$: we write elements of $\mathfrak{g}_1$, respectively of $\mathfrak{g}_{-1}$, as

$$\tau_{\varepsilon v}(x) = x + \varepsilon v =: x + \varepsilon \xi_v, \quad \tilde{\tau}_{\varepsilon a}(x) = x + \varepsilon Q(x)a =: x + \varepsilon \tilde{\xi}_a;$$

then

$$\begin{aligned} [\xi_v, \tilde{\xi}_a](x) &= -\varepsilon \mathrm{D}(Q(\cdot)a)(x)v = -2T(v, \varepsilon a, x), \\ [[\xi_v, \tilde{\xi}_a], \xi_w] &= \xi_{-2T(v, \varepsilon a, w)}. \end{aligned}$$

Thus the maps $(T_+, T_-)$ are essentially given by $(-2T(x, \varepsilon a, y), -2T(a, \varepsilon x, b))$. We use the notation $\varepsilon T(x, a, y) := T(x, \varepsilon a, y)$ (that is, we extend $T$ to a $\mathbb{K}(\varepsilon)$-trilinear map). Then we can summarize the preceding results:

**Theorem 9.5.** *The pair of* $\mathbb{K}$*-modules* $(V, V')$ *with the trilinear maps* $(x, a, y) \mapsto T(x, a, y)$, $(a, x, b) \mapsto T(a, x, b)$ *is a linear Jordan pair over* $\mathbb{K}$.

**9.6.** The Jordan pair structure can be written in terms of the Bergman operator of $(TTX, TTX')$: we rewrite (8.11) with $\varepsilon_1 := \varepsilon$ and $x$ replaced by $\varepsilon_2 x$:

$$\begin{aligned} B(\varepsilon_2 x, \varepsilon_1 a)^{-1} &= \tau_{-\varepsilon_2 x}\tilde{\tau}_{-\varepsilon_1 a}\tau_{\varepsilon_2 x}\tilde{\tau}_{\varepsilon_1 a} \\ B(\varepsilon_1 a, \varepsilon_2 x)^{-1} &= \tau_{-\varepsilon_1 a}\tilde{\tau}_{-\varepsilon_2 x}\tau_{\varepsilon_1 a}\tilde{\tau}_{\varepsilon_2 x}. \end{aligned} \tag{9.7}$$

These equations show that $(B(\varepsilon_2 x, \varepsilon_1 a), B(\varepsilon_1 a, \varepsilon_2 x)^{-1})$ is a commutator in the group $\mathrm{Aut}(TTX, TTX')$ corresponding to a Lie bracket in $\mathrm{InfAut}(TX, TX')$. From (8.12) we get

$$\begin{aligned} B(\varepsilon_2 x, \varepsilon_1 a)^{-1} z &= z - \varepsilon_1\varepsilon_2 Q(x, z)a = z - \varepsilon_1\varepsilon_2 T(x, a, z), \\ B(\varepsilon_1 a, \varepsilon_2 x)^{-1} b &= b - \varepsilon_2\varepsilon_1 Q(a, b)x = b - \varepsilon_1\varepsilon_2 T(a, x, b). \end{aligned} \tag{9.8}$$

**Theorem 9.7.** *The associated Jordan pair depends functorially on the generalized projective geometry* $(X, X')$ *with base point* $(o, o')$.

*Proof.* The Jordan pair is induced by the Bergman operator, and by 6.2 the Bergman operator depends functorially on the pointed space. □

Note that in general a homomorphism of geometries with base point induces only a homomorphism of associated Jordan pairs, but not a homomorphism of the whole Kantor–Koecher–Tits algebra—this corresponds to the fact that in general homomorphisms of Lie triple systems do not induce homomorphisms of their standard imbeddings, cf. [5, Ch. V].

**Theorem 9.8.** *The functor from Theorem* 9.7 *induces functors of the following kind*:

(i) *a functor associating to a generalized polar geometry over* $\mathbb{K}$ *with base point a Jordan triple system over* $\mathbb{K}$,

(ii) *a functor associating to a pointed symmetric space constructed from a polar geometry a Lie triple system over* $\mathbb{K}$.

*Proof.* (i) A polarity with $p(o) = o'$ defines an involution on the Jordan pair $(V, V')$ which is thus turned into a Jordan triple system.

(ii) Any Jordan triple system $T$ defines by Formula (0.1) a Lie triple system $R_T$ which by definition is the Lie triple system associated to the symmetric space $M^{(p)}$

with base point $o$. (In the real finite-dimensional case this is the curvature tensor of the symmetric space, see [5].) □

**9.9 Isotopy.** Recall (Section 5.9) that the automorphism group $G$ of a generalized polar geometry $(X, X', p)$ does in general *not* act transitively on the set $M^{(p)}$ of non-isotropic points. Therefore, the Jordan triple systems associated to different base points $o_1, o_2 \in M^{(p)}$ may be non-isomorphic. However, since $(o_1, p(o_1))$ and $(o_2, p(o_2))$ are conjugate in $M$ (Theorem 5.7), the underlying Jordan pairs of both Jordan triple systems will be isomorphic, and the two Jordan triple systems will be *isotopic* in the Jordan theoretic sense (cf. [17]; this corresponds to the *conformal equivalence* from [5, Ch. XI.5]). If they are isomorphic, then Theorem 10.1 (see below) implies, essentially, that there exists $g \in G$ with $g.o_1 = o_2$. Thus the set of isomorphism classes of Jordan triple systems in a given isotopism class corresponds to the set of $G$-orbits in $M^{(p)}$. (The classification of non-degenerate quadratic forms over $\mathbb{K}$ is a special case of this, cf. 5.9.)

## 10 Existence theorem

**Theorem 10.1.** *For every Jordan pair $(V^+, V^-)$ over a ring $\mathbb{K}$ with $2 \in \mathbb{K}^*$ there exists a generalized projective geometry $(X, X')$ with base point $(o, o')$ whose associated Jordan pair is $(V^+, V^-)$ and such that every Jordan pair homomorphism extends to a homomorphism of the associated generalized projective geometries.*

*Proof.* Given a Jordan pair $(V^+, V^-)$ with trilinear maps $T_\pm$ and quadratic maps $Q_\pm$ over a ring $\mathbb{K}$, let its *Kantor–Koecher–Tits algebra*

$$\mathfrak{g} := V^+ \oplus \mathfrak{h} \oplus V^-$$

be defined as in [21, Section 1.1], with $\mathfrak{h}$ generated by the brackets

$$[v, w] = (-T_+(v, w), T_-(w, v)), \quad v \in V^+, w \in V^-,$$

and by the Euler operator $\zeta := (\mathrm{id}_{V^+}, -\mathrm{id}_{V^-})$. We represent elements of $\mathrm{End}_{\mathbb{K}}(\mathfrak{g})$ by $3 \times 3$-matrices in the obvious way. The following elements (for $v \in V^+$, $w \in V^-$) are automorphisms of $\mathfrak{g}$ (see [21, Lemma 1.2]):

$$\exp_+(v) := e^{\mathrm{ad}(v)} = \begin{pmatrix} 1 & \mathrm{ad}(v) & Q(v) \\ & 1 & \mathrm{ad}(v) \\ & & 1 \end{pmatrix},$$

$$\exp_-(w) := e^{\mathrm{ad}(w)} = \begin{pmatrix} 1 & & \\ \mathrm{ad}(w) & 1 & \\ Q(w) & \mathrm{ad}(w) & 1 \end{pmatrix},$$

and $\exp_\pm : V^\pm \to \mathrm{Aut}(\mathfrak{g})$ are injective group homomorphisms onto groups called $U^+$ and $U^-$. An automorphism $h = (h^+, h^-)$ of $(V^+, V^-)$ is identified with the element

$$\tilde{h} := \begin{pmatrix} h_+ & & \\ & h_0 & \\ & & h_- \end{pmatrix}$$

of $\mathrm{Aut}(\mathfrak{g})$, with $h_0$ given by conjugation with $h$. In particular, we have a homomorphism

$$\mathbb{K}^* \to \mathrm{Aut}(\mathfrak{g}), \quad r \mapsto r\tilde{\mathrm{id}} = \begin{pmatrix} r\,\mathrm{id} & & \\ & \mathrm{id} & \\ & & r^{-1}\,\mathrm{id} \end{pmatrix}.$$

Let $G \subset \mathrm{Aut}(\mathfrak{g})$ be the group generated by $U^+$, $U^-$ and the elements $r\tilde{\mathrm{id}}$, $r \in \mathbb{K}^*$ (this is a slightly extended version of the *projective elementary group of* $(V^+, V^-)$ defined in [21]). Set

$$H := G \cap \mathrm{Aut}(V^+, V^-)$$

Because of $h \exp_\pm(x) h^{-1} = \exp(h_\pm(x))$, $H$ normalizes $U^+$ and $U^-$. Define 'parabolic subgroups' and the 'big cell' in $G$ by

$$P^\pm = HU^\pm = U^\pm H, \quad \Omega = U^- H U^+ = P^- P^+.$$

Now we are ready to define the generalized projective geometry $(X, X')$: let

$$X := X^+ := G/P^-, \quad X' := X^- := G/P^+;$$

denote by $g.x := g^+(x)$, $g'.x := g^-(x)$ the action of $G$ on $X$, respectively on $X'$; let $(o, o') := (o_+, o_-)$ be the base point in $X^+ \times X^-$ and define

$$M := G.(o_+, o_-) \cong G/(P^+ \cap P^-) = G/H.$$

(Since $(g, g')$ fixes the Euler operator iff $g$ and $g'$ are linear, a geometric model of $M$ is the orbit of the Euler operator in $\mathfrak{g}$.) As usual we let $V_a = \{x \in X \mid (x, a) \in M\}$, $V'_x = \{a \in X' \mid (x, a) \in M\}$. Then the sets $V_{o'}$ and $V^+$ are naturally identified since

$$V_{o_-} = \{g.o_+ \mid g \in G, g.o_- = o_-\} = P^+.o_+ = U^+.o_+ \cong V^+,$$

and dually. Therefore, by transport of structure, the spaces $V_{o'}$ and $V'_o$ carry natural $\mathbb{K}$-module structures. Since, by its definition, $H$ acts linearly on $V^+ \times V^-$, these

$\mathbb{K}$-module structures are invariant under the stabilizer group $H$ of the origin in $M = G/H$, and thus we can transport them in a well-defined way to any point $(x,a) \in M$. Summing up, we have defined the multiplication map $\mu_r : X \times X' \times X \supset D \to X$ and its dual, with $D$ given by Equation (1.4). Moreover, by construction $G$ acts as a group of automorphisms of $(\mu_r, \mu_r')$; in particular, all $(r_{x,a}, r_{a,x}^{-1})$ with $r \in \mathbb{K}^*$ are of the form $(g, g')$ with $g \in G$ and thus define automorphisms. This means that the identity (PG1) (in its extended version) holds for all $r \in \mathbb{K}^*$.

Let us show that (PG1) holds in fact for all scalars $r \in \mathbb{K}$. We will need the notion of the *quasi-inverse* in a Jordan pair: in [21, Theorem 1.4] it is shown that, for $(x, y) \in V^+ \times V^-$, the condition $(x, y) \in M$ is equivalent to the *quasi-invertibility* of $(x, y)$, and then

$$\exp_-(y)\exp_+(x).o_+ = \exp_+(x^y).o_+ = x^y$$

is the usual quasi-inverse in a Jordan pair. This means that the operator $\tilde{\tau}_y$ defined by Equation (2.9) is, in terms of the Jordan pair $(V^+, V^-)$, given by the formula

$$\tilde{\tau}_y(x) = x^y.$$

Since $(\tilde{\tau}_y, \tau_y)$ is an automorphism, we see that

$$\tilde{\tau}_y : V_{o'} \to V_y \tag{10.1}$$

is a $\mathbb{K}$-module isomorphism. Before proving the general case of (PG1), note that the identity (T) from 2.3 holds (see [17, Theorem 3.7]). Now let us show that, for $(x, a) \in M$ and $r \in \mathbb{K}$,

$$r_{x,a} : V_{r_{a,x}b} \to V_b$$

is $\mathbb{K}$-linear for all $b \in V_x'$. We may assume that $(x, a) = (o, o')$. Now, it is easily verified that for all $r \in \mathbb{K}$ the following relation holds:

$$(rx)^y = r(x^{ry}) \tag{10.2}$$

(see [20, Proposition 1.2]) which together with (10.1) implies that

$$r_{o,o'} : V_{rb} \to V_b$$

is a $\mathbb{K}$-module homomorphism. We have proved (PG1).

Next we verify (PG2). Let $r \in \mathbb{K}$ and assume $x, y \in V_c$, $c \in X'$. We choose notation such that $c = o'$ and $o = \mu_r(y, o', x) = M_{y,x}(o')$, i.e.

$$(1 - r)y + rx = 0. \tag{10.3}$$

Then we have to show that

$$M_{x,y} : V_o' \supset U \to V_{o'}, \quad a \mapsto M_{x,y}(a) = \mu_r(x,a,y), \tag{10.4}$$

where $U = \{a \in V_o' \mid (x,a) \in M\} = V_o' \cap V_x'$, has an extension to an affine map $V_o' \to V_{o'}$. We need the following expression for $\mu_r(x,a,y)$: for $(a,x) \in M \cap (V \times V')$,

$$r_{x,a} = \tau_x \tilde{\tau}_{\tilde{\tau}_{-x}(a)} r_{o,o'} (\tau_x \tilde{\tau}_{\tilde{\tau}_{-x}(a)})^{-1}. \tag{10.5}$$

In fact, this is proved in the same way as Corollary 5.8 (its proof uses only (PG1) which has already been established.) We evaluate at $y$ and mind Equation (10.2):

$$r_{x,a}(y) = x + r\tilde{\tau}_{(r-1)\tilde{\tau}_{-x}(a)}(y-x). \tag{10.6}$$

Next, using that $\tilde{\tau}_u(v) = B(v,u)^{-1}(v - Q(v)u)$,

$$r_{x,a}(y) = x + rB(y-x,(1-r)\tilde{\tau}_{-x}(a))^{-1}(y - x - Q(y-x)(r-1)\tilde{\tau}_{-x}(a)). \tag{10.7}$$

We transform the $B$-operator appearing in this expression using first the following identity JP35 from [17]: $B(v,u)^{-1} = B(-v, \tilde{\tau}_v(u))$, and then taking account of (10.3):

$$\begin{aligned} B(y-x,(1-r)\tilde{\tau}_{-x}(a))^{-1} &= B(y-x, \tilde{\tau}_{y-x}((r-1)\tilde{\tau}_{-x}(a))) \\ &= B(y-x,(r-1)\tilde{\tau}_{(r-1)(y-x)-x}(a)) \\ &= B(y-x,(r-1)a) \\ &= B((r-1)(y-x),a) = B(-x,a). \end{aligned}$$

Using this, (10.7) reads

$$r_{x,a}(y) = x + rB(-x,a)(y-x) - r(r-1)B(-x,a)Q(y-x)\tilde{\tau}_{-x}(a). \tag{10.8}$$

The second term of the right hand side of (10.8) equals

$$(\mathrm{id} - T(-x,a) + Q(x)Q(a))(r(y-x)) = y + T(x,a)y + Q(x)Q(a)y,$$

where we use that $r(y-x) = y$. In order to calculate the third term we use that, by the identity JP23 of [17],

$$\begin{aligned} B(-x,a)Q(y-x) &= B(x-y,(r-1)a)Q(y-x) \\ &= Q(y-x)B((r-1)a, x-y) \\ &= Q(y-x)B(a,-x), \end{aligned}$$

and we obtain for the third term in (10.8):

$$\begin{aligned} r(r-1)&B(-x,a)Q(y-x)\tilde{\tau}_{-x}(a) \\ &= r(r-1)Q(y-x)B(a,-x)B(a,-x)^{-1}(a-Q(a)(-x)) \\ &= r(r-1)Q(y-x)a + (r-1)Q(y-x)Q(a)rx \\ &= r(r-1)Q(y-x)a + (r-1)^2Q(y-x)Q(a)y \\ &= r(r-1)Q(y-x)a + Q(x)Q(a)y. \end{aligned}$$

Thus (10.8) finally gives

$$r_{x,a}(y) = x + y + T(x,a)y - r(r-1)Q(y-x)a.$$

This is clearly affine in $a$, as has to be shown. (In case $r = \frac{1}{2}$, this expression reduces to $-Q(x)a$; in this case the result can be proved more directly using the symmetry principle [17, Proposition 3.3].)

Let us prove now that a Jordan pair homomorphism $g^{\pm} : V^{\pm} \to W^{\pm}$ extends to a homomorphism of the associated pointed generalized projective geometries. First of all, $(g^+, g^-)$ defines on the affine parts belonging to the base points indeed a local homomorphism of the generalized projective geometry: this follows by using the explicit formula (10.5) for the multiplication maps together with the relation

$$g^+(x^y) = g^+(x)^{g^-(y)} \tag{10.9}$$

(see [20, Equation I.1.(7)]). Since $(X^+, X^-)$ is connected and thus algebraically generated by $(V^+, V^-)$, there is at most one extension to a global homomorphism. We have to prove existence of such an extension. For simplicity, let us first assume that $(V^+, V^-)$ is *stable* in the sense of 5.4. This means that

$$V^- \times V^+ \to X^+, \quad (a, y) \mapsto \tilde{\tau}_a \tau_y.o$$

is surjective, and dually. The fibers of this map define an equivalence relation on $V^- \times V^+$, called *projective equivalence*, and $X^+ = X(V)$ is called the *projective space of* $V$ (see [20]). From (10.9) it can be deduced that $g^+ \times g^-$ passes to the quotient as a well-defined map $X(g) : X(V) \to X(W)$, see [20, Section 1.3], which is then a homomorphism of generalized projective geometries. In the general non-stable case essentially the same argument applies: since $(V, V')$ is generating, there are surjective maps

$$V' \times V \times \cdots \times V' \times V \to X_n, \quad (a, y, \ldots, b, z) \mapsto \tilde{\tau}_a \tau_y \ldots \tilde{\tau}_b \tau_z(o),$$

($2n$ factors) such that $\bigcup_n X_n = X$, and dually. The fibers of these maps define equivalence relations which can be explicitly described in terms of the Jordan pair, see [7, Theorem 1]. The explicit formula is fairly complicated, but one can conclude as above that $\times^n(g^+ \times g^-)$ passes to the quotient as a well-defined map $X(g) : X(V) \to X(W)$ which then is a homomorphism of generalized projective geometries.

Finally, if $V$ is a Jordan pair over $\mathbb{K}$ and $\phi : \mathbb{K} \to R$ is a scalar exension, then $V \otimes_{\mathbb{K}} R$ is a Jordan pair over $R$, and the associated space $(X_R, X_R')$ is a scalar extension of $(X, X')$ in the sense of 1.10: in fact, it is clear on the level of Jordan pairs that homomorphisms defined over $\mathbb{K}$ extend to homomorphisms defined over $R$, and by the preceding arguments, this carries over to the level of spaces. By connectedness the extension thus obtained is unique.

Summing up, $(X, X')$ is a connected generalized projective geometry over $\mathbb{K}$. Its associated Jordan pair is nothing but the Jordan pair we started with; in fact, the operators $B(x, y)$ and $T(x, y)$ we introduced in Chapters 6 and 8 are precisely the operators associated to the given Jordan pair. □

**10.2.** If one wants to announce Theorem 10.1 in the form of an *equivalence of categories*, then one has to introduce a notion of "simply connectedness" for generalized projective geometries. More precisely, if $(X, X, o, o')$ is a connected generalized projective geometry with base point, $V = (V^+, V^-)$ the associated Jordan pair and $X(V)$ the geometry associated to $V$, then the identity map of $V^+ \times V^-$ is a local homomorphism $X(V) \to (X, X')$. By the preceding argument, it extends to a homomorphism $X(V) \to (X, X')$ which is surjective since $(X, X')$ is connected. Thus it is a covering in a sense extending the corresponding notion of [6, Chapter 3]; we do not know whether it has to be always injective. (Rationality arguments show that this is so in the finite-dimensional case over a field.)

**10.3.** An analogue of Theorem 10.1 in the category given by generalized projective geometries and adjoint pairs of morphisms is true: an adjoint pair preserving base points defines a so-called *structural transformation* of the associated Jordan pairs; conversely, an analogue of (10.9) holds for structural transformations (see [21, Proposition 1.2.(e)]), which implies that structural transformations are adjoint pairs of morphisms of the associated geometry, defined on the affine part $(V, V')$. They do in general not extend to the whole of $(X, X')$, as shows already the example of ordinary projective geometry over $\mathbb{K}$ where any non-zero linear map together with its transposed defines an adjoint pair on the quotient.

**10.4.** Since Jordan triple systems are the same as Jordan pairs with involutions, Theorem 10.1 implies that we can associate to every Jordan triple system a generalized polar geometry in a functorial way; taking the associated symmetric space, we get a functor from Jordan triple systems over $\mathbb{K}$ into symmetric spaces over $\mathbb{K}$.

## 11 Problems and further results

**11.1 Algebraic equations of hyperplanes and quadrics.** In ordinary projective geometry over $\mathbb{K}$, the hyperplanes $H_a = X \setminus V_a$ and the various quadrics are given by algebraic equations. The analogue of these equations in our setting is as follows: one proves that the Bergman operator extends to a biquadratic map, given by the formula

$$B(x,a) = \mathrm{id} - T(x,a) + Q(x)Q(a) \tag{11.1}$$

(cf. [17, I.2.11]). Then, with respect to a fixed base point $(o,o')$, the complements of hyperplanes and of quadrics are sets defined by a non-degeneracy condition on the Bergman operator; more precisely, one can show (cf. [21, Theorem 1.4]):

(i) $V_a \cap V_{o'} = \{x \in V \mid B(x,a) \in \mathrm{Gl}(V)\}$,

(ii) $M \cap (V \times V') = \{(x,a) \in V \times V' \mid B(x,a) \in \mathrm{Gl}(V)\}$,

(iii) for a given polarity, $M^{(p)} \cap V = \{x \in V \mid B(x,p(x)) \in \mathrm{Gl}(V)\}$.

These conditions can be formulated more intrinsically in terms of sections of certain vector bundles over $(X,X')$; see [5] for the real finite-dimensional case. Note that in the finite-dimensional case over a field the conditions from (i)–(iii) are polynomial since they can be written in terms of $\det B(x,a)$. More difficult is the task to find the equations of hyperplanes and quadrics passing through the origin ("parabolic realization"); it is closely related to determining the *incidence structure* of $(X,X')$, see below. Here, in the finite-dimensional case, the *rank* of the quadratic operators $Q(a)$ plays an important rôle.

**11.2 "Jordan theoretic analog of the Campbell–Hausdorff formula".** One can show that the operators $\tilde{\tau}_a$ are given by the usual formula for the quasi-inverse in Jordan theory (see [17, I.3]),

$$\tilde{\tau}_a(x) = x^a = B(x,a)^{-1}(x - Q(x)a); \tag{11.2}$$

more generally, all automorphisms $g$ can be written as $g(x) = d_g(x)^{-1} n_g(x)$ $(x \in V \cap g^{-1}(V))$ with a quadratic *denominator* $d_g$ and a quadratic *numerator* $n_g(x)$ (see [5] for the finite-dimensional real case). Together with Corollary 5.8 this gives an explicit formula for the maps $\mu_r$ in terms of the associated Jordan pair. In the finite-dimensional case over a field the inverse in $\mathrm{Gl}(V)$ is rational and thus also our formulas are rational; thus also the associated symmetric spaces will be "algebraic over $\mathbb{K}$"—see [5, Section X.3] for the real case. Since the explicit formula describes the multiplication maps in a canonical chart, we may consider it as a Jordan analogue of the Campbell–Hausdorff formula.

**11.3 Jordan algebras.** We have described the geometric objects associated to Jordan pairs and Jordan triple systems, but not yet the geometric object corresponding to *(unital) Jordan algebras*. It is known that unital Jordan algebras are the same as Jordan pairs containing *invertible elements* (see [17, I.1.6]). This property can be translated to our context by requiring the existence of *inner polarities*, but it is also closely related to the existence of *canonical null-systems* which explains the somewhat special rôle of the Jordan inverse in a Jordan algebra ([4]; see also Chapter 4 of [3]).

**11.4 Case of $2 \notin \mathbb{K}^*$.** As mentioned in 2.6, if $2 \notin \mathbb{K}^*$, the theory has to be based on maps of four arguments. The main problem here is to find a good set of identities satisfied by the maps defined by (2.10).

**11.5 Case of non-commtative base fields or rings.** We have used commutativity of the base ring $\mathbb{K}$ in an essential way. Thus our theory applies to the quaternionic projective space $X = \mathbb{HP}^n$, considered as a geometry over the center $Z(\mathbb{H}) = \mathbb{R}$, but not as a "geometry over $\mathbb{K} = \mathbb{H}$". However, it should be interesting to have also a formalism of generalized projective geometries, over, say, $\mathbb{K} = \mathbb{H}$, since it seems that the *quaternionic symmetric spaces* are related to such geometries. From a Jordan theoretic point of view, the latter correspond to certain *non-commutative Jordan structures*, called *balanced Freudenthal–Kantor pairs*, cf. [2].

**11.6 Incidence structure.** There are two structures associated to a generalized projective geometry which are related to what one might call the associated *incidence* or *remoteness structure*: on the one hand, we have the distribution of the "hyperplanes" $H_a$, $a \in X'$ (see 11.1); on the other hand, there exist subspaces which appear linearly in every affinization (inner ideals, see 2.7 (4)). In ordinary projective geometry these two structures are almost the same; in general, the situation is much more complicated, and one would like to have a good Jordan theoretic description. In the finite-dimensional and non-degenerate case over a field these structures seem to be related to *buildings* in the sense of J. Tits; therefore a general theory of the correspondence between algebra and incidence structure is an important topic for further investigations. In a final step one has to study the action of a polarity on these structures and to describe the new structure on the associated symmetric spaces thus obtained—here one will get a vast generalization of the concept of a *generalized conformal structure* proposed by S. Gindikin, S. Kaneyuki and others (cf. [10]).

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