

New prolific constructions of strongly regular graphs

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Abstract. We present a purely combinatorial construction of strongly regular graphs with geometric parameters. The construction produces hyperexponentially many graphs, and uses neither linear algebra nor groups. Among the graphs constructed, there are graphs with parameters of a generalized quadrangle $GQ(n-1, n+1)$ which further produce distance regular covers of complete graphs.

In this paper we present an exceptionally prolific construction of strongly regular graphs. We use the word “prolific” informally, but in a rather strict sense. Thus, there are prolific constructions of strongly regular graphs from Latin squares, from Steiner triple systems, from collections of mutually orthogonal Latin squares (of prime power order), etc. What is especially interesting about our construction is that it produces graphs with geometric parameters (in the sense of [1]); in particular, with those of generalized quadrangles $GQ(n-i, n+i)$ for $i = 0, \pm 1$. Moreover, the graphs with parameters of $GQ(n-1, n+1)$, by virtue of the construction, will have spreads of n -cliques; removing the edges of these cliques one obtains distance regular n -covers of the complete graph K_{n^2} . In this way one can find, for instance, distance regular graphs of diameter 3 with the trivial group of automorphisms.

As building blocks of the construction we shall use certain 2-designs: resolvable designs with constant intersection of non-parallel lines. In this paper we shall call them *affine designs*.

Definition 1. An *affine design* is a 2-design with the following two properties:

- (i) every two blocks are either disjoint or intersect in a constant number r of points;
- (ii) each block together with all blocks disjoint from it forms a parallel class: a set of n mutually disjoint blocks partitioning all points of the design.

Examples of affine designs: all lines of an affine plane of order n ($r = 1$); all hy-

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perplanes of a d -dimensional affine space over the field $\text{GF}(n)$ ($r = n^{d-2}$); Hadamard 3-designs ($n = 2$).

All parameters of an affine design can be expressed in terms of r and n .

Lemma 1. *Let (V, \mathcal{L}) be an affine design with parallel classes of size n and block intersections of size r .*

- (i) *The number $s = (r - 1)/(n - 1)$ is an integer.*
- (ii) *The design has the following parameters:*

$$\begin{aligned}
 v &= |V| = n^2r = n^3s - n^2s + n^2; \\
 b &= |\mathcal{L}| = n^3s + n^2 + n; \\
 p &= |\mathcal{L}|/n = n^2s + n + 1 \text{ (the number of parallel classes);} \\
 k &= nr = n^2s - ns + n \text{ (the block size);} \\
 \lambda &= ns + 1.
 \end{aligned}$$

The parameters satisfy the equality $k\lambda = (p - 1)r$.

- (iii) *If $s = 0$ then the design is an affine plane of order n .*

Proof. (i) and (ii). Considering intersections of a parallel class with another block, we get the equality $k = nr$ which immediately implies $v = nk = n^2r$. Let p be the number of parallel classes; then $b = pn$. We have the standard equality

$$b \cdot k(k - 1) = \lambda \cdot v(v - 1)$$

satisfied in all 2-designs. Further, let B be a block, p a point outside it. Counting in two ways the number of pairs $\{(x, L) \mid x \in B, L \in \mathcal{L}, \{p, x\} \subseteq L\}$ we get the equality $k\lambda = (p - 1)r$. Now it is easy to find that

$$\lambda = r + \frac{r - 1}{n - 1}; \quad p = \frac{n^2r - 1}{n - 1},$$

and all the claims follow.

- (iii). For $s = 0$ we get $(v, k, \lambda) = (n^2, n, 1)$. A design with such parameters is necessarily an affine plane of order n . □

Here is our main construction.

Construction 1. *Let $\mathcal{S}_1, \dots, \mathcal{S}_{p+1}$ be arbitrary affine designs with parameters as in Lemma 1; p is the number of parallel classes in each \mathcal{S}_i . Let $\mathcal{S}_i = (V_i, \mathcal{L}_i)$. Let $I = \{1, \dots, p + 1\}$.*

For every i , denote arbitrarily the parallel classes of \mathcal{S}_i by symbols \mathcal{L}_{ij} , $j \in I \setminus \{i\}$. For $v \in V_i$, let $l_{ij}(v)$ denote the line in the parallel class \mathcal{L}_{ij} which contains v .

For every pair i, j , $i \neq j$, choose an arbitrary bijection $\sigma_{ij} : \mathcal{L}_{ij} \rightarrow \mathcal{L}_{ji}$; we only require that $\sigma_{ji} = \sigma_{ij}^{-1}$.

Construct a graph $\Gamma_1 = \Gamma_1((\mathcal{S}_i), (\sigma_{ij}))$ on the vertex set $X = \bigcup_{i \in I} V_i$. The sets V_i

will be independent. Two vertices $v \in V_i$ and $w \in V_j$, $i \neq j$, are adjacent in Γ_1 if and only if $w \in \sigma_{ij}(l_{ij}(v))$ (or, equivalently, $\sigma_{ij}(l_{ij}(v)) = l_{ji}(w)$).

The graph so obtained is strongly regular with parameters (V, K, Λ, M) , $V = n^2r(n^2s + n + 2)$, $K = nr(n^2s + n + 1)$, $\Lambda = M = r(n^2s + n)$.

The proofs of this and the subsequent constructions are quite easy and straightforward; at this point we leave them to the reader. For the sake of completeness, they will be given at the end of the paper.

The parameters of graphs obtained by Construction 1 are geometric; they correspond to a partial geometry with parameters

$$(K, R, T) = (n(ns + 1) + 2, n(ns + 1) - ns, ns + 1)$$

(cf. [1, Ch. 2]). It is outside the scope of the present paper to decide when the graphs constructed are geometric.

When we use in Construction 1 affine d -spaces, we get graphs with parameters

$$(V, K, \Lambda) = \left(n^d \left(\frac{n^d - 1}{n - 1} + 1 \right), n^{d-1} \frac{n^d - 1}{n - 1}, n^{d-1} \frac{n^{d-1} - 1}{n - 1} \right).$$

In particular, when we use affine planes of order n we get strongly regular graphs with parameters $(n^2(n + 2), n(n + 1), n, n)$; that is, graphs with parameters of a generalized quadrangle $\text{GQ}(n + 1, n - 1)$. The special structure of the graphs so constructed enables one to use them to construct other graphs, with parameters of generalized quadrangles $\text{GQ}(n, n)$ and $\text{GQ}(n - 1, n + 1)$.

Construction 2. Let $\Gamma_1 = \Gamma_1((\mathcal{S}_i), (\sigma_{ij}))$ be a graph of Construction 1 with $\mathcal{S}_1, \dots, \mathcal{S}_{n+2}$ affine planes of order n .

Remove all vertices of V_{n+2} , add $n + 1$ new vertices x_1, \dots, x_{n+1} . Add the following edges:

the $(n + 1)$ -clique on the new vertices;

for all i , edges joining x_i to all vertices of V_i ;

for all i , n -cliques on every line of the parallel class $\mathcal{L}_{i, n+2}$.

The resulting graph Γ_2 is strongly regular with parameters $((n^2 + 1)(n + 1), n(n + 1), n - 1, n + 1)$, those of a $\text{GQ}(n, n)$. If Γ_1 is geometric then so is Γ_2 .

Construction 3. Let $\Gamma_1 = \Gamma_1((\mathcal{S}_i), (\sigma_{ij}))$ be a graph of Construction 1 with $\mathcal{S}_1, \dots, \mathcal{S}_{n+2}$ affine planes of order n .

Remove all vertices of $V_{n+1} \cup V_{n+2}$.

For all i , add n -cliques on every line of the parallel classes $\mathcal{L}_{i, n+1}$ and $\mathcal{L}_{i, n+2}$.

The resulting graph Γ_3 is strongly regular with parameters $(n^3, (n - 1)(n + 2), n - 2, n + 2)$, those of a $\text{GQ}(n - 1, n + 1)$. If Γ_1 is geometric then so is Γ_3 .

By a result of Brouwer (cf. [3, p. 146]), if a graph with parameters of $\text{GQ}(s, t)$ has a spread of $(s + 1)$ -cliques then it can be used to construct a distance regular graph of diameter 3. This is precisely what we have in our case.

Construction 4. Let Γ_3 be a graph constructed in Construction 3. On each set V_i it induces the square grid graph. Choose one of the two parallel classes in each grid; together they form a spread of n -cliques. Deleting all edges of these cliques results in a graph Γ_4 which is distance regular of diameter 3 with the intersection array $(n^2 - 1, n^2 - n, 1; 1, n, n^2 - 1)$ – a distance regular n -cover of the complete graph K_{n^2} .

A partial case of Construction 4 was found by de Caen and the author in [2]. Actually, the construction of (q^2, q, q) -covers described there was a starting point to a line of successive generalizations which has ultimately led to this paper.

The question of mutual isomorphisms and automorphisms of the graphs constructed here is a difficult one and requires further study. Here we give only a rough estimate for the number of pairwise non-isomorphic graphs obtained by Construction 1 from affine planes.

Given a fixed set of $n + 2$ affine planes of order n , with fixed numberings of parallel classes according to Construction 1, and with fixed bijections σ_{1i} and σ_{2i} for all appropriate i , we can choose the bijections σ_{ij} for $i, j \geq 3$ in

$$n! \binom{n}{2}$$

different ways. Form the list of all resulting graphs. Now, given an abstract graph Γ , how many times can it occur within the list?

For any set A of vertices, denote by $\mu(A)$ the set of their common neighbours. The graphs of Construction 1 have the following property:

(*) for any $x, y \in V_i$, the set $\mu(x, y)$ is a line in one of the planes V_j , and $\mu(\mu(x, y))$ is the line in V_i through x and y .

Take three non-collinear points p_1, p_2, p_3 in V_1 . Three vertices x, y, z of Γ corresponding to these points can be chosen in at most V^3 ways. Then, applying (*), we uniquely find the sets of vertices corresponding to the lines $p_i p_j$ of \mathcal{L}_1 . There are at most $(n - 2)!(n - 2)$ ways to assign vertices to all points of $p_1 p_2$, and to one more point of $p_1 p_3$. After this, repeated application of (*) uniquely determines for each point of V_1 a vertex corresponding to it. Also, the vertex sets corresponding to each line of each parallel class \mathcal{L}_{j1} are now determined. In particular, the vertex sets corresponding to each V_i are determined. Further, for all $i \neq j$, the graph induced on $V_i \cup V_j$ determines the partition of V_i into the lines of the parallel class \mathcal{L}_{ij} . Finally, there are at most $n!$ ways to assign vertices to points of the line $\sigma_{12}(p_1 p_2)$. After this, every vertex of V_i , $i > 2$, is uniquely assigned to the intersection of two specified lines from the classes \mathcal{L}_{i1} and \mathcal{L}_{i2} . So, all vertices of the graph get unique assignments, and in particular all permutations σ_{ij} are uniquely determined. Thus every graph can

occur in our list at most $V^3 n!^2$ times. So, the number of pairwise non-isomorphic graphs is at least

$$n!^{n^2/2+o(n^2)} = 2^{(1/2)n^3 \log n(1+o(1))},$$

which is hyperexponential in the number of vertices.

Finally, we give proofs of our constructions. A proof for Construction 4 was given in the text.

Proofs of Constructions 1–3. Construction 1. Every vertex $v \in V_i$ is adjacent to all points of a certain line in each of the V_j , $j \neq i$. If $w \in V_i$ then in λ cases these lines for v and w coincide, and in the remaining cases they are parallel. Thus we get $k\lambda$ common neighbours. If $w \in V_j$, $j \neq i$, then for each $k \neq i$, j these two lines are distinct and not parallel, and we get $(p-1)r$ common neighbours. Now Lemma 1(ii) proves the claim.

Now let Γ_1 be a graph obtained by Construction 1 from affine planes ($\lambda = 1$, $k = n$); and let Γ be obtained from Γ_1 by Construction 2 or 3. Take any two vertices $v, w \in \Gamma$; we need to count their common neighbours.

We shall consider in detail the case of Construction 2. The verification is easy when one or both of the vertices v, w are among the new vertices x_1, \dots, x_{p+1} . Let $v \in V_i$, $w \in V_j$. If $i = j$ and v, w are adjacent in G then their n common neighbours in G_1 belonged to V_{n+2} and have been deleted; instead we get $n-1$ common neighbours in the clique containing v, w , and x_i . If $i = j$ and v, w are not adjacent then x_i is added to their common neighbours.

If $i \neq j$ then one common neighbour of v, w (the one in V_{n+2}) disappears. New common neighbours can appear only within V_i and V_j ; namely, the vertices $\sigma_{ij}(I_{ij}(v)) \cap I_{j,n+2}(w)$ and $\sigma_{ji}(I_{ji}(w)) \cap I_{i,n+2}(v)$. If v, w are adjacent then these vertices coincide with w, v ; otherwise we have two new common neighbours.

The case of Construction 3 is treated similarly. □

A historical remark. After having finished this paper, the author found a paper [5] published in 1971 which contains a construction even more general than Construction 1. It uses affine designs (called there “affine resolvable designs”) in connection with any 2-design with $\lambda = 1$ having the same number of blocks on a point. Construction 1 corresponds to taking the complete design with blocks of size 2 as this 2-design. Unfortunately, it seems that the paper went largely unnoticed; although it is mentioned in the survey paper [4], the parameters of the graphs arising from affine d -spaces do not appear in the tables of parameters there. It is not mentioned in the survey [1], although this survey puts a special emphasis on geometric and pseudo-geometric graphs.

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