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Another case of the prime power conjecture for finite projective planes

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Abstract. Let G be an abelian collineation group of order $n(n-1)$ of a projective plane of order n. We show that n must be power of a prime p and that the p-part of G is elementary abelian.

1 Introduction

The purpose of this note is a surprisingly elementary proof of the following result.

Theorem 1. *Let G be an abelian collineation group of order $n(n-1)$ of a projective plane of order n. Then n must be a power of a prime p and the p-part of G is elementary abelian.*

Theorem 1 adds to the extremely scarce conclusive results known in the context of the prime power conjecture for projective planes. Let us review some background.

In what follows, Π will denote a finite projective plane of order n, and G will be a large (to be specific, $|G| > (n^2+n+1)/2$) abelian (and hence quasi-regular) collineation group of Π. Such planes have been classified into eight cases by Dembowski and Piper [5], which are usually referred to as types (a) through (h); see also [10, p. 114] for a statement of their result. As a special instance of the prime power conjecture for projective planes in general, it is widely conjectured that planes of any one of those eight types can only exist for prime powers n. This conjecture has been established for abelian groups of order n^2 (types (b) and (c)) in a recent paper by Blokhuis, Jungnickel and Schmidt [2] and for arbitrary groups of type (h) by Ganley and McFarland [7]; the latter case is sporadic and only occurs for $n = 4$.

Our Theorem 1 takes care of one further case, namely that of abelian groups of type (f). Here G has order $n(n-1)$, fixes a double flag $(\infty_A, \infty_B, L_\infty)$ of Π together with a further line L_A through ∞_A and acts regularly on the $n(n-1)$ points not incident with either of the two special lines L_∞ and L_A and on the $n(n-1)$ lines not incident with either of the two special points. In this case, Π is both (∞_A, L_∞)- and (∞_B, L_A)-transitive and therefore at least in Lenz–Barlotti class II.2; conversely, any plane admitting two such transitivities is of type (f); cf. Dembowski [4] and Hughes and Piper [8] for background. We note that the only known examples are provided by the Desarguesian planes $\mathrm{PG}(2,q)$. Indeed, it seems quite reasonable to conjecture that a plane with an abelian group of type (f) must be Desarguesian.

Prior to the present note, the following restrictions on planes with an abelian group of type (f) were known, due to Ganley [6] for even orders and Pott [9] for odd orders, respectively; see also [3] for simpler proofs for the first two parts of the following result.

Result 2. *Let Π be a projective plane of order n admitting an abelian collineation group G of type (f).*

1. *If n is even, then n is a power of 2 and the Sylow 2-subgroup of G is elementary abelian.*
2. *If n is odd, then the Sylow 2-subgroup of G is cyclic.*
3. *If n is odd and not a perfect square, then n is a prime power.*
4. *If $n = p$ is a prime, then Π is the Desarguesian plane $\mathrm{PG}(2,p)$.*

We note that our proof of Theorem 1 does not require that p is an odd prime, so that we also recover Result 2.1. For the proof, it will be convenient to write G multiplicatively (with unit 1) and to use the integral group ring $\mathbb{Z}G$. Let us briefly recall the necessary notation. For $X = \sum x_g g \in \mathbb{Z}G$ and $t \in \mathbb{Z}$ we write $|X| = \sum x_g$ and $X^{(t)} = \sum x_g g^t$. For $r \in \mathbb{Z}$ we write r for the group ring element $r \cdot 1$, and for $S \subseteq G$ we write S instead of $\sum_{g \in S} g$. Group rings are a standard tool in the theory of difference sets; for background, see [1]. They will be useful in our context, as planes with a group of type (f) are equivalent to a certain kind of difference set.

2 The proof

Planes of type (f) may be represented using the *direct product difference sets* (DPDS) introduced by Ganley [6]. In group ring notation, a DPDS of order n may be defined to be a subset D of a group G of order $n(n-1)$ with two subgroups A and B of orders n and $n-1$, respectively, which satisfies the equation

$$DD^{(-1)} = n + G - A - B \tag{1}$$

in $\mathbb{Z}G$; thus every element not in the union of the two *forbidden subgroups* A and B has a unique "difference representation" from D. For our purposes, G is assumed to be abelian, and hence $G = A \times B$; see Pott [9] for examples in semidirect products and [3] for an explicit description of the plane determined by an abelian DPDS (which simplifies the one given by Ganley [6]). In what follows, we will write G multiplicatively and work in the group ring $\mathbb{Z}G$.

We require the following simple lemma which was observed by Ganley [6].

Lemma 3. *Let D be a DPDS for a plane of order n in an abelian group $G = A \times B$. Then D meets every coset of A and all but one coset of B exactly once.*

In particular, we may assume $D \cap B = \emptyset$ in what follows. Thus D may be written in the form

$$D = \sum_{b \in B} bf(b), \tag{2}$$

where $f : B \to A \backslash \{1\}$ is a bijection.

The proof of Theorem 1 will proceed via computing the group ring element $D^{(-1)}D^{(p)}$ modulo p, where p is any prime dividing n. This agrees with a major step in the proof for the case of groups of type (b) given in [2], though the remainder of the argument will require a totally different approach. Let us first note the following result, which follows from Lemma 3 by induction; the rather easy details may be left to the reader.

Lemma 4. *Let D be a DPDS for a plane of order n in an abelian group $G = A \times B$, and let p be a prime dividing n and m any positive integer. Then the following equations hold in the group algebra $\mathbb{Z}_pG$ over the field $\mathbb{Z}_p$ of residues modulo p:*

$$GD^m = (-1)^m G; \tag{3}$$

$$AD^m = GD^{m-1} = (-1)^{m-1} G; \tag{4}$$

$$BD^m = (G - B)D^{m-1} = (-1)^{m-1} mG + (-1)^m B. \tag{5}$$

Proof of Theorem 1. As already mentioned, we first evaluate the group ring element $D^{(-1)}D^{(p)}$ modulo p. Using (2), Lemma 4 and the fact $X^p = X^{(p)}$ (see [1, Lemma VI.3.7]), we compute in $\mathbb{Z}_pG$:

$$\begin{aligned} D^{(-1)}D^{(p)} &= D^{(-1)}D^p \\ &= (D^{(-1)}D)D^{p-1} \\ &= (G - A - B)D^{p-1} \\ &= G + G - [-(p-1)G + B] \end{aligned}$$

and therefore

$$D^{(-1)}D^{(p)} = G - B \quad (\text{in } \mathbb{Z}_pG). \tag{6}$$

But $|D^{(-1)}D^{(p)}| = |G - B| = (n-1)^2$, and so (6) must hold as an identity in $\mathbb{Z}G$. Using (2), we may write this identity as

$$\sum_{b,c \in B} b^{-1}f(b)^{-1}c^p f(c)^p = G - B. \tag{7}$$

Now, if some element $f(c)^p$ equals one of the elements $f(b)$, we get the element

$b^{-1}c^p \in B$ from the sum in (7), which is forbidden. Hence we conclude $f(c)^p = 1$ for all $c \in B$, since $f : B \to A \backslash \{1\}$ is a bijection. This means $a^p = 1$ for all elements $a \neq 1$ of A. Thus A is an elementary abelian p-group and n is a power of p, as claimed. □

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