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Point-line characterizations of Lie geometries

Anna Kasikova and Ernest Shult

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Abstract. There are two basic theorems. Let Γ be a strong parapolar space with these three properties: (1) For each point x and symplecton S, x is collinear to some point of S. (2) The set of points at distance at most two from a point forms a geometric hyperplane. (3) If every symplecton has rank at least three, every maximal singular subspace has finite projective rank. Then Γ is either $D_{6,6}, A_{5,3}$ or $E_{7,1}$, a classical dual polar space of rank three, or a product geometry $L \times P$ where P is a polar space and L is a line. The second theorem concerns parapolar spaces Σ of symplectic rank at least three whose point-collinearity diameter is at least three such that for every point-symplecton pair, (x, S), $x^{\perp} \cap S$ is never just a point. With a mild local condition, one can show that such a geometry has point-diameter three and has a simply connected point-collinearity graph. If singular spaces have finite projective rank, one can show that Σ is $E_{6,4}$, $E_{7,7}$, $E_{8,1}$, a metasymplectic space, or a polar Grassmannian of type $B_{n,2}$, $D_{n,2}$, $n \geqslant 4$. All of these geometries are truncations of buildings. The last case can be modified so that the assumption that singular spaces have finite projective rank can be discarded.

1 Introduction

More than twenty years ago, Bruce Cooperstein initiated the study of Lie incidence geometries in [5]. The important concepts of symplecton and parapolar space unfolded from this work, and opened the door to characterizing geometries of points and lines associated with the exceptional groups, just as had been done for projective spaces and polar spaces.

About 15 years ago, Cohen and Cooperstein produced two theorems which when taken together, allowed one to characterize every Lie group of rank at least three as the automorphism group of at least one geometry of points and lines described by simple axioms [4]. Their theorem has been the central result in this field up to the present.

Their theory requires three basic assumptions: (1) There is a constant finite symplectic rank at least three. (2) Every singular subspace has finite projective rank. (3) The spectrum of possible ranks of the projective spaces $x^{\perp} \cap S$, where x is a point and S is a symplecton not incident with x, experiences some gaps. The spaces $x^{\perp} \cap S$ are either (a) the empty set, a single point, or a maximal singular subspace of S, or (b) the empty set, a line, or a maximal singular subspace of S.

The present paper concentrates on their second theorem (using (3)(b)) which characterizes most of the so-called "long root" geometries. Our version dispenses with the first assumption, and greatly weakens the second assumption (2). We replace Assumption (3)(b) with something weaker and quite different.

In order to make the theorems understandable we insert a few definitions: Following Cohen [3] a *parapolar space* Γ is a connected partial linear gamma space with a family of convex subspaces $\mathscr{S}$ called *symplecta* each isomorphic to a non-degenerate polar space of rank at least 2 such that every line and every 4-circuit lies in a symplecton. (Note that by allowing symplecta of polar rank 2 this definition of "parapolar space" is more general than that appearing in virtually all of the literature preceeding [3].) Γ has *symplectic rank at least k* if the symplecta (whose polar ranks may vary among themselves) have rank at least k. One says Γ has *symplectic rank k* if each symplecton has polar rank exactly k.

In this paper all parapolar spaces have thick lines. We shall always denote the point-collinearity graph of a parapolar space $\Gamma = (\mathscr{P}, \mathscr{L})$ by $\Delta = (\mathscr{P}, \sim)$. Then $\Delta_k^*(p)$ denotes the set of points at distance at most k from p in Δ. A parapolar space is a *strong parapolar space* if and only if each pair of points at distance two is always contained in some symplecton.

Here are the two main theorems:

Theorem 1. *Suppose Γ is a parapolar space of symplectic rank at least three satisfying these axioms*:

1. *Given a point x not incident with a symplecton S, the space $x^{\perp} \cap S$ is never just a point.*
2. *Given a projective plane π and line L meeting π at point p, either* (i) *every line of π on p lies in a common symplecton with L, or else* (ii) *exactly one such line incident with (p, π) has this property.*
3. *Given any line L on a point p, there exists at least one further line N on p such that $L^{\perp} \cap N^{\perp} = \{p\}$.*
4. *If all symplecta have rank at least four, assume every maximal singular subspace has finite projective rank.*

Then Γ is

1. $E_{6,4}$, $E_{7,7}$, *or* $E_{8,1}$,
2. *a metasymplectic space*, *or*
3. *a polar Grassmannian of lines of a non-degenerate polar space of* (*possibly infinite*) *rank at least four. In the case of finite polar rank, these would be classical Lie incidence geometries of type* $(B/C)_{n,2}$ *or* $D_{n,2}$, $n \geqslant 4$.

Theorem 2. *Suppose Γ is a strong parapolar space with these three properties*:

1. *For every point-symplecton pair* (x, S), $x^{\perp} \cap S \neq \emptyset$.
2. *The ball* $\Delta_2^*(p)$ *of radius* 2 *about any point p is a geometric hyperplane of* Γ.
3. *If there is no symplecton of rank two assume every maximal singular subspace has finite projective rank.*

Then Γ *is one of the following*:

1. $D_{6,6}$, $A_{5,3}$, *or* $E_{7,1}$,
2. *a dual polar space of rank three*,
3. *a product geometry*, $L \times P$, *where L is a line, and P is a non-degenerate polar space of rank at least two. (It may have infinite polar rank.)*

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2 Basic concepts

2.1 Point-line geometries and parapolar spaces. For the definitions of point-line geometry, subspace, singular subspace, polar space, polar rank and diagram geometry, one may consult Cohen's survey article in the Handbook of Incidence Geometry [3]. Let $\Gamma_i := (\mathscr{P}_i, \mathscr{L}_i)$, $i = 1, 2$ be two point-line geometries. The *product geometry* $\Gamma_1 \times \Gamma_2$ is a point-line geometry whose set of points is the Cartesian product $\mathscr{P}_1 \times \mathscr{P}_2$ and whose lines are subsets of the form $p_1 \times L_2$ where $p_1 \in \mathscr{P}_1$ and $L_2 \in \mathscr{L}_2$ (a "vertical" line), or of the form $L_1 \times p_2$ where $L_1 \in \mathscr{L}_1$ and $p_2 \in \mathscr{P}_2$ (a "horizontal" line). Thus the product of two lines is the familiar "grid".

A subspace of a point-line geometry $(\mathscr{P}, \mathscr{L})$ is a *geometric hyperplane* if it is a proper set of points meeting each line at exactly one or all of its points.

In the introduction we gave the definition of "parapolar space" and "symplecton". In a parapolar space of symplectic rank at least three (as defined in the introduction), all singular subspaces are projective spaces. But the reader should be warned that that conclusion can fail if some symplecta have polar rank two (Consider a product of two affine planes, for example).

In a parapolar space, for any two distinct points x and y, either (i) they are collinear, (ii) the set $x^\perp \cap y^\perp$ of points collinear with both x and y is empty, (iii) $x^\perp \cap y^\perp$ consists of a single point (then (x, y) is called a *special pair*), or the convex closure of $\{x, y\}$ is a symplecton (then (x, y) is called a *polar pair*). It is called a *strong parapolar space* if no special pairs exist. Always in any parapolar space of symplectic rank at least three, every plane lies in some symplecton.

2.2 Special aspects of graphs.

2.2.1 Subgraphs. Let $G = (V, E)$ be a simple graph. For $X \subseteq V$ let E_X be the set of edges in E whose incident vertices lie in X. A *subgraph* is a pair $H = (X, E')$ where $E' \subseteq E_X$, with the inherited incidence. It is an *induced subgraph* if $E' = E_X$. The intersection of two subgraphs $H_i = (X_i, E_i)$, $i = 1, 2$ is the subgraph $H_1 \cap H_2 := (V_1 \cap V_2, E_1 \cap E_2)$, and of course one can form such intersections over arbitrary families of subgraphs.

A *geodesic path* is a path of minimal length connecting its initial and terminal vertices. A finite subgraph (X, E') is said to be *convex* in (V, E) if, for every geodesic path of (V, E) connecting two vertices of X, the intermediate vertices of the path all belong to X.

The *distance* $d_G(x, y)$ between vertex x and y is the length of a shortest path connecting them, if there is one, or is "∞" if x and y belong to different connected components of G. Given a subgraph $H = (X, E')$, there are two distance metrics d_G and d_H that can be applied to its vertex pairs. We always have $d_G(h_1, h_2) \leqslant d_H(h_1, h_2)$. We say H is *isometrically embedded in G* if and only if these metrics coincide on H.

One observes the following relations among properties of subgraphs:

1. "Isometrically embedded" implies "induced".
2. "Induced and convex" together imply "isometrically embedded".
3. Neither of the two concepts "convex" and "isometrically embedded" alone implies the other.

An important observation:

Lemma 3. *The class of convex induced subgraphs of a graph is closed under arbitrary intersections. Any intersection of connected graphs in this class is connected.*

2.2.2 Strong gatedness in graphs. Now suppose $G = (V, E)$ is a connected graph so the distance metric for G assumes only finite values. A subgraph $H = (X, E')$ is said to be *strongly gated with respect to a vertex v* if and only if there exists a "gate" g_v in X such that for every $x \in X$,

$$d_G(v, x) = d_G(v, g_v) + d_H(g_v, x).$$

The subgraph H is *strongly gated* if and only if it is strongly gated with respect to every vertex. One now has

Lemma 4. *Every strongly gated subgraph is a convex induced* (*and hence isometrically embedded*) *subgraph.*

2.2.3 Graph morphisms. Let $G_i = (V_i, E_i)$, $i = 1, 2$, be a pair of simple graphs. A *graph morphism* $G_1 \to G_2$ is a mapping $V_1 \to V_2$ such that if $\{x, y\}$ is in E_1, then either $f(x) = f(y)$ or else $\{f(x), f(y)\}$ is an edge of E_2. (Of course it may happen that $\{f(x), f(y)\}$ is an edge of E_2 even when x and y are distinct non-adjacent vertices of G_1.) The morphism is *full* if every edge of E_2 with vertices in $f(V_1)$ is the image $\{f(x), f(y)\}$ of some edge $\{x, y\}$ of G_1. The morphism is *vertex injective* or *vertex surjective* according as the induced map $V_1 \to V_2$ is injective or surjective. Of course these morphisms can be composed. For any induced subgraph defined on vertex subset X, we may consider $f|_X$, (f restricted to X) as another graph morphism.

A *fibering* is a vertex-surjective morphism $f : G_1 \to G_2$ which is vertex-bijective (but not necessarily an isomorphism) when restricted to the neighborhood graph $G_1(v)$,

$v \in V_1$. When $f : G_1 \to G_2$ is a fibering, every walk of G_2 possesses a unique lift at every preimage of the initial vertex of the walk. The fibering is called a *$\mathscr{C}$-covering* if and only if G_1 is connected and all circular walks belonging to a family $\mathscr{C}$ of G_2 always lift to circular walks of G_1. We let $\mathscr{T}$ denote the collection of all 3-circuits of G_2. $\mathscr{T}$-covers of point-collinearity graphs play a key role in Sections 3, 5 and 6. Finally, a fibering $f : (V_1, E_1) \to (V_2, E_2)$ is called a *universal $\mathscr{C}$-covering* if it is a $\mathscr{C}$-covering and if, for any other $\mathscr{C}$-covering, $g : (V_3, E_3) \to (V_2, E_2)$ there is a graph morphism $h : (V_1, E_1) \to (V_3, E_3)$ such that $f = g \circ h$. It is well known that for any collection of circuits $\mathscr{C}$ of a connected graph, there exists a universal $\mathscr{C}$-covering (see [1] and [10]).

Most objects of concern here in this paper can be described by graphs that anyone can understand, and so their morphisms can be described as graph morphisms.

2.3 Chamber systems and geometries. A *chamber system* $C = (V, E, \lambda, I)$ over I is a simple graph (V, E) together with an edge-labelling $\lambda : E \to 2^I - \{\emptyset\}$ by non-empty subsets of I, such that if x, y and z are three pair-wise adjacent vertices, then

$$\lambda(x, y) \cap \lambda(y, z) \subseteq \lambda(x, z). \tag{1}$$

For any element i of I, two vertices x and y of (V, E) are said to be *i-adjacent* if and only if $\{x, y\}$ is an edge and $\lambda(x, y)$ contains i. By (1), i-adjacency union the identity relation is an equivalence relation. The chamber system is *connected* if and only the underlying graph (V, E) is connected.

The vertices of a chamber system are typically called *chambers.*

Note that the definition does not require that every label of I be realized as an element of some $\lambda(x, y)$. Thus, if $I \subseteq K$, then any chamber system over I is *a fortiori* a chamber system over K.

Let $C^{(j)} = (V^{(j)}, E^{(j)}, \lambda^{(j)}, I)$, $j = 1, 2$, be two chamber systems over I. A *morphism of chamber systems* is a graph morphism

$$\phi : (V^{(1)}, E^{(1)}) \to (V^{(2)}, E^{(2)})$$

which "preserves labels" in the sense that if $\{x, y\} \in E^{(1)}$ is such that $\{\phi(x), \phi(y)\}$ is an edge in $(V^{(2)}, E^{(2)})$—that is, $\phi(x)$ and $\phi(y)$ are distinct vertices—then

$$\lambda^{(1)}(x, y) \subseteq \lambda^{(2)}(\phi(x), \phi(y)).$$

We denote this morphism by $\phi : C^{(1)} \to C^{(2)}$. Clearly morphisms can be composed when the underlying graph homomorphisms can, and the chamber systems over I form a category with respect to these morphisms.

For any subset J of I let $E_J := \{e \in E \mid \lambda(e) \cap J \neq \emptyset\}$, the set of edges bearing at least one label from J. The connected components of the graph (V, E_J) are called *residues of type J* and can be regarded as a chamber system over J with labelling λ_J where $\lambda_J(x, y) = \lambda(x, y) \cap J$, for each edge $\{x, y\}$ in E_J. If $|J| = 1$, a residue of type J

is called a *panel*. A chamber system is *firm* if and only if every panel contains at least two chambers.

A residue of type J is said to be of *cotype* $I - J$ and *corank* $|I - J|$.

Let $C = (V, E, \lambda, I)$ be a chamber system over I and let J be a fixed subset of I. Let V/J denote the collection of all residues of C of type J. If R_1 and R_2 are two distinct members of V/J and $i \in I - J$, declare R_1 and R_2 to be i-adjacent if and only if $R_1 \cup R_2$ is contained in a residue of type $J \cup \{i\}$. Letting E/J be all 2-subsets of V/J exhibiting some i-adjacency, and letting

$$\lambda^J(R_1, R_2) = \{i \mid R_1 \text{ is } i\text{-adjacent to } R_2\},$$

then

$$C_J := (V/J, E/J, \lambda^J, I - J)$$

is a chamber system over $I - J$ called the *truncation* of type $I - J$ of C.

A chamber system C is *residually connected* if and only if the following statements hold:

1. C is connected.
2. Let $\{R_\sigma\}$ be any collection residues of C. Then these residues pair-wise intersect non-trivially if and only if they have a non-empty global intersection $\bigcap R_\sigma$.
3. The intersection of any collection of residues is either empty or is itself a residue whose type T is the intersection of the types of the residues of the collection—that is, it is a connected subgraph when restricted to edges bearing labels from T.

In a residually connected chamber system, C, the intersection over $\mathbf{\Gamma}(c)$, the set of all corank-one residues containing c, is just $\{c\}$.

A residue R of type J of a connected chamber system is said to be *strongly gated* if and only if, it is strongly gated as an induced subgraph of (V, E), the underlying graph of $C = (V, E, \lambda, I)$. A strongly gated residue is always a convex isometric subgraph of (V, E). In particular, any two of its vertices which form an edge in E, form one in E_J. The intersection of finitely many strongly gated residues is strongly gated.

Let $M = (m_{ij})$ be a symmetric matrix with diagonal entries 1 and all other entries integers greater than one, or the symbol ∞—a so-called *Coxeter Matrix*. A chamber system is *type M* (or *belongs to diagram M*) if and only if its residues of type $\{i, j\}$ are chamber systems of generalized $m_{i,j}$-gons. Note that "type M" implies "firm" and the property that λ assumes only singleton values on edges. A *building* is a connected chamber system of type M all of whose corank-one residues are strongly gated. (The equivalence of this definition with the traditional one is proved in Shult [12]: see also Scharlau [9].)

Now it follows from Lemmas 3 and 4 that in a building B, the intersection of residues R_i of cotype i for i ranging over J, is itself a residue of type $I - J$ or is empty. If $J \neq I$, it follows that since B is firm, the intersection over a finite set of strongly gated R_i's having pairwise non-trivial intersection is such a residue. Since a building is firm and connected, we have

Lemma 5. *Any building of finite rank is residually connected.*

On the other hand, in [7] the authors of this paper showed that no firm chamber system without multiple edge-labels over an infinite typeset can be residually connected. Thus buildings of infinite rank are not residually connected.

A *geometry over I* is a multipartite graph (V, E) whose (non-empty) parts are indexed by I. One thinks of the underlying partition of the vertices into cocliques as segregating the objects of the geometry according to their type (for example, "points", "lines" "planes" "symplecta" etc) and adjacency in the graph as indicating the incidence relations among objects. Thus we have an onto type function $typ : V \to I$ and two objects of the same type are never incident. A *flag* F is just a set of pairwise incident vertices (that is, a clique), and its *type* is the set $typ(F)$ of types of its vertices. Since the type function is injective when restricted to cliques, $|typ(F)| = |F|$ for all flags. A *chamber flag* is a flag of type I, and so is just a set of pairwise incident objects (that is, a *clique*) with one object of each type.

Let $G^{(j)} = (V^{(j)}, E^{(j)}, typ^{(j)}, I)$, $j = 1, 2$ be geometries over I. A *morphism of geometries*, $\phi : G^{(1)} \to G^{(2)}$, is a graph homomorphism $\phi : (V^{(1)}, E^{(1)}) \to (V^{(2)}, E^{(2)})$ which "preserves types"—that is, for any object v of $G^{(1)}$,

$$typ(v) = typ(\phi(v)).$$

Such a morphism takes flags of type J to flags of type J. Geometries over I form a category with respect to the geometry morphisms.

Suppose F is a flag of type J in geometry G. The *residue in G of F* is the subgraph induced on the set $\mathrm{Res}_G(F)$ of vertices v not in F for which $F \cup \{v\}$ is a flag. Letting I_F be the set of types of vertices in the residue of F, then $\mathrm{Res}_G(F)$ becomes a geometry over I_F under suitable restriction of the type mapping. Clearly $I_F \subseteq I - J$ and F lying in some chamber-flag is a sufficient (but not necessary) condition for equality of these type sets.

A geometry is *residually connected* if and only if the residue of every flag of corank one is non-empty and the residue of every flag of corank at least 2 is non-empty and connected. (The residue of the empty flag is the entire geometry, so as a multipartite graph, it is connected.)

2.4 The functors Γ and C. Let Γ be a geometry over I. Let $\mathbf{C}(\Gamma)$ be the set of chamber flags of Γ. Two chamber flags F and F' are said to be *i-adjacent* if and only if they differ only in their objects of type i. The i-adjacencies define a collection of labelled edges on the set of chamber flags with respect to which $\mathbf{C}(\Gamma)$ is a chamber system over I. It may happen that $\mathbf{C}(\Gamma)$ is empty (that is, Γ has no chamber flags) but as defined, it is still a chamber system over I. It is easy to see that $\mathbf{C}$ is a functor from the category of geometries over I to the category of chamber systems over I.

Similarly let C be a chamber system over I. Let V be the collection of all corank-one residues of C, and let E be the pairs of distinct residues which have a non-empty intersection. Then we have a mapping $typ : V \to I$ which records the cotype of each corank-one residue. Clearly its fibres are non-empty cocliques of the graph (V, E), so it is multipartite. Thus $\mathbf{\Gamma}(C) := (V, E, typ, I)$ is a geometry over I. Then the mapping from the category of chamber systems over I to the category of geometries over I which takes C to $\mathbf{\Gamma}(C)$ is a functor.

2.5 $\mathbf{\Gamma}$-images and residual connectedness. Let $C = (V, E, \lambda, I)$ be a chamber system over I. For each subset X of V let $\mathbf{\Gamma}(X)$ be the collection of all corank-one residues of C which contain X. In particular for a single vertex (or chamber) c, $\mathbf{\Gamma}(c)$ is all corank-one residues which contain c.

Now suppose F is a flag of the geometry $\mathbf{\Gamma}(C)$. Then F is a collection $\{R_i\}_{i \in L}$ of corank-one residues of C which pairwise intersect non-trivially. We may assume these corank-one residues to be indexed by their cotypes so i is the cotype of R_i, and L is the type of the flag F. Such a flag is said to be a $\mathbf{\Gamma}$-*image* if and only if $\bigcap_{i \in L} R_i \neq \emptyset$. In that case, $\bigcap_{i \in L} R_i$ is a union of residues of C of type $I - L$. Also in that case, whenever we choose a chamber c in the intersection of the R_i, $\mathbf{\Gamma}(c)$ is a chamber flag of $\mathbf{\Gamma}(C)$ containing flag F. Conversely, any subflag of a chamber flag that is a $\mathbf{\Gamma}$-image is also a $\mathbf{\Gamma}$-image. It follows that the collection of all $\mathbf{\Gamma}$-images is a subcomplex of the simplicial complex of all flags of $\mathbf{\Gamma}(C)$.

Remarks. 1. Moreover if C is residually connected, every flag of $\mathbf{\Gamma}(C)$ is a $\mathbf{\Gamma}$-image. In fact,

$$\mathbf{C}(\mathbf{\Gamma}(C)) \simeq C.$$

2. Buildings over a finite typeset I are always residually connected (Tits [14]) while buildings over an infinite typeset I are never residually connected (Kasikova and Shult [7]).

3. The functors $\mathbf{\Gamma}$ and $\mathbf{C}$ preserve the properties of residual connectedness for geometries and for chamber systems of finite rank.

2.6 Coverings of chamber systems. A morphism of chamber systems over I, $\alpha : C_0 \to C$, is said to be a *k-covering* if and only if C_0 is connected, α is a fibering (in the sense of Section 2.2) and for any subset J of I of cardinality at most k, the restriction of α to any residue R of type J induces a chamber-system isomorphism $R \to \alpha(R)$ between residues of type J.

Obviously if ℓ is less than k, any k-covering is an ℓ-covering.

Lemma 6. *Let k be any positive integer greater than* 1 *and let C be a chamber-system over I. Then there exists a universal k-covering $\kappa : \hat{C} \to C$—that is, for every k-covering $\alpha : C_0 \to C$, there is a morphism $\phi : \hat{C} \to C_0$ such that $\alpha \circ \phi = \kappa$.*

This is proved by Tits for chamber systems, but is part of a general theorem on covers of graphs proved in Aschbacher–Segev [1]. (See Shult [10] for a full exposition.)

When we use the word "covering" without a prefixed "k", we shall mean a 2-covering. These are then just chamber-surjective morphisms of chamber systems which are isomorphisms when restricted to rank-1 and rank-2 residues.

The major theorem of the field is this:

Theorem 7 (Tits' Local Approach Theorem [15]). *Suppose C is a chamber system of type M, where M is a finite Coxeter matrix, whose residues of rank 3 are 2-covered by buildings. Then the universal 2-cover of C is a building.*

2.7 A vital lemma on 2-coverings of chamber systems. This section contains a lemma which concludes that under certain conditions, the morphism of geometries induced by a 2-covering of chamber systems, is injective when restricted to an appropriate residue.

Lemma 8. *Suppose $f : \hat{C} \to C$ is a 2-covering of chamber systems over I. Suppose $\overline{X}$ is a residue of cotype K in C, and X is any lift of $\overline{X}$ to a cotype K residue of $\hat{C}$. Let $h = \mathbf{\Gamma} f$ be the (functorially induced) morphism of geometries over I:*

$$h : \Delta := \mathbf{\Gamma}(\hat{C}) \to \mathbf{\Gamma}(C) := \bar{\Delta}.$$

Then $\overline{X}$ may be regarded as a flag of type K of the geometry $\bar{\Delta}$ and X may similarly be regarded as a flag X of the geometry Δ, where $h(X) = \overline{X}$.

Now suppose the following:

1. *The residue $\overline{X}$ is a 2-simply connected chamber system over $I - K$.*
2. *For some subset J of I properly containing K, the truncation C_J is a residually connected chamber system over J. If $|K| > 1$, assume also that the truncation $\hat{C}_J$ is residually connected.*

Let Δ_J and $\bar{\Delta}_J$ be the truncations to type J of the respective geometries Δ and $\bar{\Delta}$. Then, the mapping of geometric residues

$$h_X : \mathrm{Res}_{\Delta_J}(X) \to \mathrm{Res}_{\bar{\Delta}_J}(\overline{X}),$$

induced by the restriction of h is injective.

Proof. Suppose by way of contradiction, that y_1 and y_2 are distinct objects of type $i \in J - K$ in the geometry Δ which are incident with X while $h(y_1) = h(y_2)$. Then, by the definition of Δ, each y_k can be regarded as a residue Y_k of chamber system $\hat{C}$, of cotype i such that

1. $Y_1 \cap Y_2 = \varnothing$,
2. $f(Y_1) = f(Y_2)$ is a residue of cotype i of $f(\hat{C}) = C$, and
3. for each $k \in K$, $Y_1 \cap X_k \neq \varnothing \neq Y_2 \cap X_k$,

where X_k denotes the unique residue of cotype k containing X.

But we actually have

3*. $Y_1 \cap X \neq \varnothing \neq Y_2 \cap X$

which we now justify. If $|K| = 1$, $X_k = X$, with corank 1. Then the Statements 3 and 3^* coincide. If $|K| > 1$, the fact that $\hat{C}_J$ is residually connected, together with

Statement 3 just above, implies the intersections $Y_j \cap (\bigcap_K X_k)$, $j = 1, 2$, of residues whose cotypes are subsets of J, are non-empty. But by residual connectedness of $\hat{C}_J$, $\bigcap_{k \in K} X_k = X$. Thus 3^* holds in either case.

By Assumption (i) $\overline{X}$ is simply 2-connected and so as f is a 2-cover, the restriction of f to X induces an isomorphism $X \to \overline{X}$ as chamber systems over $I - \{K\}$. Thus in chamber system C, we have residues $\overline{Y} := f(Y_1) = f(Y_2)$ and $\overline{X}$ of cotypes i and K in C, whose intersection $\overline{Y} \cap \overline{X}$ consists of at least two components. Since all of the "components of the Venn diagram", $C - (\overline{X} \cup \overline{Y})$, $\overline{X} - \overline{Y}$, $\overline{Y} - \overline{X}$, and $\overline{Y} \cap \overline{X}$ are each a union of residues of C of type $I - J$, with $\overline{X}$ and $\overline{Y}$ residues of cotypes K and i of the truncation C_J, we must infer that C_J is not residually connected, contrary to hypothesis.

Thus the lemma holds.

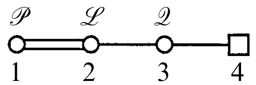
3 Locally truncated geometries: a quick review

For this entire section, D is a finite diagram over the typeset I and J is a subset of I. A connected geometry Γ over J with type function *typ* is said to be *locally truncated of type D (over I)* if and only if

(LT) *for every non-empty flag F of* Γ, *the residue* $\mathrm{Res}_\Gamma(F)$ *is isomorphic to the truncation to* $J - typ(F)$ *of a geometry belonging to the diagram* $\mathrm{Res}_{typ(F)}(D)$ *(that is, the diagram D with the nodes of* $typ(F)$ *suppressed).*

The reader may appreciate our resistance to the temptation to recast Condition (LT) in the form that $\mathrm{Res}_\Gamma(F)$ is isomorphic to the appropriate truncation of a residue of a geometry Ψ belonging to diagram D. No one knows whether such a geometry Ψ exists, so this existence question should not impair the definition above. For further descriptions of locally truncated geometries of type D the reader is referred to Ronan [8], Brouwer and Cohen [2].

A locally truncated geometry Ξ of type D can always be rendered by presenting the diagram D, and then changing all nodes not in the "real-world" set J to square nodes. For example the diagram



refers to a rank-three geometry of points and lines and quadrangles (denoted $\mathscr{P}, \mathscr{L}, \mathscr{Q}$, respectively) with the property that for each point p, the rank-two incidence geometry $\mathrm{Res}_\Xi(p) = (\mathscr{L}_p, \mathscr{Q}_p)$ of all lines and quadrangles on p forms a geometry isomorphic to the "points" and "lines" of a $\mathrm{PG}(3, D)$.

Like diagram geometries, locally truncated diagrams are a device for axiomatizing a geometry, except that now residues can be certain proper truncations of diagram geometries.

3.1 The idea of sheaves. When is a locally truncated geometry realizable as a truncation of a geometry belonging to a diagram D? This question was first answered by

Ronan [8], whose proof was replaced by a more transparent one due to A. Brouwer and A. Cohen [2]. Since this theory will be needed to show that certain homomorphic images of geometries are isomorphisms, we are forced to review it.

Let Γ be a geometry over J which is locally truncated of type D (whose type set is I). From our definitions, $J \subseteq I$. The definition means that there is an "overall diagram" D over I, such that if F is a flag of Γ of type K (necessarily K is a subset of J), then the residue

$$\mathrm{Res}_{\Gamma}(F)$$

is the truncation to $J - K$ of a geometry belonging to diagram $\mathrm{Res}_K(D)$ (the diagram remaining when the nodes of K are suppressed.) Note the extra property that if $F \subseteq G$ is a containment of flags in Γ, then $typ(F) = K$ is contained in $typ(G) = L$, and that in that case one has

Lemma 9. 1. *$G - F$ is a flag of type $L - K$ in* $\mathrm{Res}_{\Gamma}(F)$.
2. *The residue of $G - F$ in* $\mathrm{Res}_{\Gamma}(F)$ *is naturally isomorphic to* $\mathrm{Res}_{\Gamma}(G)$ *where "naturally" records the correspondence between any super-flag $H - F$ of $G - F$ and superflag H of G.*

3.2 Sheaves. Suppose now that Γ is a geometry over J which is locally truncated of type D (over I). A *sheaf* is a function $\mathscr{F}$ which assigns to each non-empty flag F (whose type is rendered by $typ(F)$), a geometry $\mathscr{F}(F)$ over $I - typ(F)$, whose truncation to $J - typ(F)$ is the geometry $\mathrm{Res}_{\Gamma}(F)$. We must also have "connecting morphisms" in the flag poset of Γ: For any containment of flags of Γ, $F_1 \subseteq F_2$, one has an embedding $e(F_1, F_2) : \mathscr{F}(F_2) \to \mathscr{F}(F_1)$ which induces the identity map on the objects in Γ and whose image is the residue of $F_2 - F_1$ in the codomain, thus inducing an isomorphism

$$\mathscr{F}(F_2) \simeq \mathrm{Res}_{\mathscr{F}(F_1)}(F_2 - F_1), \qquad (*)$$

as geometries over $I - typ(F_2)$. It is required that these morphisms respect compositions: If $F_1 \subseteq F_2 \subseteq F_3$ is a chain of flags, then

$$e(F_1, F_3) = e(F_2, F_3) \circ e(F_1, F_2).$$

If such a function $\mathscr{F}$ exists, we say that *a sheaf exists*.

By convention, for each object x of Γ, we regard x as also denoting the rank-one flag $\{x\}$, so that we may write $\mathscr{F}(x)$ instead of $\mathscr{F}(\{x\})$.

The existence of a sheaf has been worked out for several important cases in the seminal papers of Ronan [8] and Brouwer–Cohen [2]:

Theorem 10. *Assume Γ is a locally truncated geometry over J of type D over I. Then there exists a sheaf in each of the cases depicted by the truncated diagrams in Figure* 1.

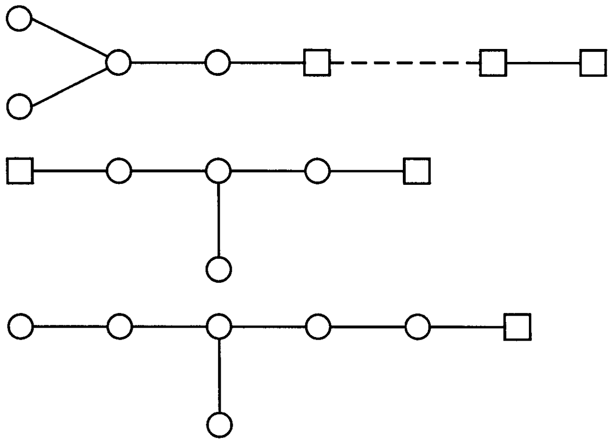


Figure 1. Locally truncated diagrams with sheaves

3.3 The chamber system associated with a sheaf. Suppose that $\mathscr{F}$ is a sheaf for the locally truncated geometry Γ with truncated diagram D. We have $J \subseteq I$ where J is the typeset for Γ and I indexes the nodes of D. Fix a flag F of Γ and select chamber flag c_F of $\mathscr{F}(F)$. Now c_F has type $I - typ(F)$ and so is a sequence of objects with those objects whose type belongs to $J - typ(F)$ forming a flag F_1 of Γ of that type. We can always write any desired subsequence of objects with type in J segregated, and written first, followed by the others. This segregation can be indicated by the "$\oplus$" sign. Thus c_F can be written as $F_1 \oplus c'$ where c' is a chamber flag in $\mathrm{Res}_{\mathscr{F}(F)}(F_1) = \mathscr{F}(F \cup F_1)$. In this way we can produce a sequence involving an object of every type, with more than one way to indicate it. Thus

$$F \oplus c_F = (F \cup F_1) \oplus c',$$

would be such a sequence. We call such a sequence (where an object of each type in I occurs) an *$\mathscr{F}$-chamber* or simply a *chamber of $C(\mathscr{F})$*. Each such chamber can always be written in the extremal form where the segregation puts *all* objects having type in J first. The right-hand term of the expression above is of that form.

We can now convert $C(\mathscr{F})$ into a chamber system as follows: Suppose $F_1 \oplus c_1$ and $F_2 \oplus c_2$ are two $\mathscr{F}$-chambers in $C(\mathscr{F})$, where $F_i \in \mathbf{C}(\Gamma)$ and $c_i \in \mathbf{C}(\mathscr{F}(F_i))$. If $j \in J$, we say that these two $\mathscr{F}$-chambers are *j-adjacent* if and only if F_1 is j-adjacent to F_2 in $\mathbf{C}(\Gamma)$ and $c_1 = c_2$ (as flags of type $I - J$ in $\mathscr{F}(F_1 \cap F_2)$). (Note that if $|J| > 1$, then $F_1 \cap F_2$ is non-empty, so this makes sense.) But if, on the other hand, $i \in I - J$, then these two $\mathscr{F}$-chambers are i-adjacent if and only if $F_1 = F_2$ and c_1 and c_2 are i-adjacent chamber flags of the common geometry $\mathscr{F}(F_1)$ of type $I - J$. With these adjacencies, $C(\mathscr{F})$ becomes a chamber system over I.

Of course, it may happen that for some flag F the geometry $\mathscr{F}(F)$ possesses no flag chambers at all. In that case there are no $\mathscr{F}$-chambers of the form $F \oplus c_F$. In the worse scenario, $C(\mathscr{F})$ itself might be empty. Usually this is prevented by the nature of the diagram D. Such a diagram specifies rank-two residues, and we would like these to be residually connected geometries, so that a diagram can be attached to residues of the chamber system of the same type.

Theorem 11. *Suppose Γ is a locally truncated diagram geometry of type J over a diagram D of type I where $|J| \geqslant 3$. Suppose a sheaf $\mathscr{F}$ exists. Then there is a canonically defined chamber system $C(\mathscr{F})$ over I. Suppose, for each object $x \in \Gamma$, that every rank-two residue of $\mathscr{F}(x)$ is connected. Then the chamber system $C(\mathscr{F})$ also belongs to the diagram D.*

Proof. Let $K = \{i, j\} \subseteq I$. Let R be a residue of $C(\mathscr{F})$ of type K. We must show that the rank-two residue R belongs to the diagram $\mathrm{Res}_{I-K}(D)$ (the relation in D between the nodes i and j). Since R is by definition non-empty, and $|J| > 2$, there exists a non-empty flag F of Γ of type $J - (K \cap J)$, such that

$$R \subseteq R_F := \{F \oplus c_F \mid c_F \in C(\mathscr{F}(F))\},$$

showing that R is a residue of type K of the chamber system $C(\mathscr{F}(x))$ for some $x \in F$. Then R belongs to $\mathrm{Res}_{I-K}(D)$ since $\mathscr{F}(x)$ belongs to the diagram $\mathrm{Res}_{typ(x)}(D)$ and has connected rank-two residues.

3.4 The effect of local isomorphisms on the existence of sheaves. A sheaf purports to assign a type to each of the phantom objects appearing in $\mathscr{F}(F)$ where F is a flag of the original geometry Γ. But in deriving the existence of a sheaf from a local knowledge of residues, one may encounter a problem.

Suppose $\Gamma = (\mathscr{P}, \mathscr{L})$ is a parapolar space of symplectic rank at least three. Given a point p, all singular subspaces over p are visible as singular subspaces of the residue geometry $\mathrm{Res}(p) = (\mathscr{L}_p, \Pi_p)$ and the symplecta on p are symplecta of $\mathrm{Res}(p)$. Now we can assemble these subspaces as an enrichment of the "point"-"line" geometry $\mathrm{Res}(p) = (\mathscr{L}_p, \Pi_p)$. Normally that leads one to believe that one is dealing with a locally truncated geometry. The problem is that the diagram D in that definition assigns a type to the subspaces making their debut in a residue.

There are infinitely many examples showing that this is not justified. Consider the "symmetric" Grassmannian of type $A_{2n-1,n}$, $n \geqslant 8$. This Grassmannian has the same point-residue as does its factor-geometry, $A_{2n-1,n}/\langle\sigma\rangle$ where σ is a polarity of Witt index at most $n-4$. In the latter geometry there are not actually two distinct types of maximal singular subspaces—the classes are fused. That means there is no global way to assign types to all the objects that appear in a residue. So one cannot immediately conclude that one has a locally truncated diagram, let alone a sheaf.

However there is a way around this. Although the procedure is general, we will describe it in the particular case that a point-residue has its incident singular subspaces and symplecta assembled as a Grassmannian $A_{2n-1,n}$ since these are the cases that concern us here.

In this case, for each point p, the point residue $\mathrm{Res}(p)$ contains exactly two classes

A_p and B_p of maximal singular subspaces so that any plane on p lies in exactly one from each class.

Let $\hat{\mathscr{P}}$ be the set of all pairs (p, X_p) where p is a point and X_p is one of the two symbols A_p or B_p. Similarly we let $\hat{\mathscr{L}}$ be the collection of pairs (L, X_L) where L is a line and X_L is a class of maximal singular subspaces containing L. We say that (p, X_p) is incident with (L, X_L) if and only if p is incident with L and $X_L \subseteq X_p$. Then $\hat{\Gamma} := (\hat{\mathscr{P}}, \hat{\mathscr{L}})$ is a point-line geometry and the projection onto the first coordinates of the pairs induces a geometry morphism

$$\gamma : \hat{\Gamma} \to \Gamma,$$

which is a fibering of bipartite graphs. The mapping γ also induces a vertex-surjective morphism of point-collinearity graphs

$$\delta : \hat{\Delta} := (\hat{\mathscr{P}}, \sim) \to \Delta := (\mathscr{P}, \sim),$$

which restricts to an isomorphism $\hat{x}^{\perp} \to x^{\perp}$ on each induced neighborhood graph. In particular δ is a $\mathscr{T}$-covering of graphs in the sense of Section 2.

Now any connected component Y of $\hat{\Gamma}$ (or of $\hat{\Delta}$) is mapped vertex-surjectively by γ (or δ) as either a one-to-one mapping or a two-to-one mapping depending on whether Y is a proper subgeometry (or induced proper subgraph) or not. In either case, Y is a geometry (or graph) which is locally identical with Γ (or Δ) and in Y the local classes are not fused—precisely that means there can be no sequence (y_0, X_0), $(y_1, X_1), \dots,$ (y_n, X_n) where

— X_i is one of the two classes of maximal singular subspaces on $y_i \in Y$.
— y_i is collinear with y_{i+1} by a line lying on a maximal singular space in both X_i and X_{i+1}.
— $y_n = y_0$ so $(y_0, \dots y_n)$ is a circular walk in $\hat{\Delta}$, and X_n is a different class than X_0 in the point-residue of y_0—thus fusing the classes.

Of course, this approach can be generalized whenever one has a connected gamma space Γ whose local diagrams for a point-residue possess proper automorphisms. One normally obtains canonical morphisms γ and δ which are isomorphisms on point-residues and neighborhood graphs, respectively. The point is that in an enrichment of the point-residues of the geometry $Y \subseteq \hat{\Gamma}$, it is perfectly legitimate to assign distinct types to objects of distinct classes within a point-residue. In that case a locally truncated diagram over I exists to support the definition of a sheaf.

We conclude at least the following:

Theorem 12. *Suppose $\Gamma = (\mathscr{P}, \mathscr{L})$ is a parapolar space with symplectic rank at least three, with the property that the collection of all symplecta and singular subspaces on a point p are the singular subspaces and symplecta of the point-line geometry* Res$(p) =$ *$(\mathscr{L}_p, \Pi_p)$ forming in that way the subspaces of a Grassmannian of type $A_{2n-1,n}$. Then there is a locally truncated connected geometry Y with respect to the diagram D derived* *by inserting a branch from the node labelled $\mathscr{P}$ at the middle node of $A_{2n-1,n}$, and a fibering morphism of geometries $\gamma|_Y : Y \to \Gamma$ which induces a* 1- *or* 2-*fold* $\mathscr{T}$*-covering* $\delta|_Y : \Delta_Y \to \Delta = (\mathscr{P}, \sim)$ *of the point-collinearity graph of* Γ.

3.5 Recovering Γ from $C(\mathscr{F})$. We say that a geometry X over K is *strongly chamber-connected* if and only if

1. every flag lies in a chamber flag (a flag of type K), and
2. the geometry is chamber connected—that is, the chamber system of chamber flags $\mathbf{C}(X)$ is connected.

This property of a geometry is implied by residual connectedness; but examples (even at rank three) show that it is weaker.

We continue with the hypothesis of this section that Γ is a geometry of type K which is a locally truncated geometry with respect to the finite diagram D over I. We suppose a sheaf $\mathscr{F}$ exists.

Lemma 13. *The following statements hold*:

1. *Suppose F is a flag of type K in the geometry* Γ. *Set*

$$R_F := \{F \oplus c_F \mid c_F \in \mathbf{C}(\mathscr{F}(F))\}.$$

If $\mathscr{F}(F)$ is strongly chamber connected, then R_F is a residue of cotype K in $C(\mathscr{F})$.

2. *Suppose $\mathscr{F}(F')$ is strongly chamber-connected for any flag F' of* Γ *of type K. (Note that K is a fixed subset of J.) Let R be a residue of $C(\mathscr{F})$ of cotype K. Then there exists a flag F of type K such that $R = R_F$.*

Proof. For the first part, the $\mathscr{F}$-chambers in R_F are connected under the i-adjacency relations as i ranges over $I - K$, for the reason that $\mathscr{F}(F)$ is chamber connected. Suppose now, an $\mathscr{F}$-chamber $r := F_1 \oplus c_1$ in R_F were i-adjacent to some $\mathscr{F}$-chamber $c = F_2 \oplus c_2$ in $C(\mathscr{F})$, for some $i \in I - J$ (we suppose $typ(F_i) = K$). Then $F = F_1 = F_2$ whence $c \in R_F$. Thus no chamber of R_F is i-adjacent to any chamber of $C(\mathscr{F}) - R_F$, for any $i \in I - K$. It follows that R_F is a residue of type $I - K$.

For the second part, we let R be an arbitrary residue of $C(\mathscr{F})$ of cotype K (i.e. it is a residue of type $I - K$). The definition of adjacency in $C(\mathscr{F})$ shows that there exists a flag F of type K such that $R \subseteq R_F$. By Part 1 R_F is already a residue of type $I - K$, so $R = R_F$.

Theorem 14. *Suppose the sheaf $\mathscr{F}$ is strongly chamber connected—that is, for each object x of the geometry* Γ, *the geometry $\mathscr{F}(x)$ is strongly chamber-connected. Then there is an isomorphism of geometries*:

$$\phi : \Gamma \to \mathbf{\Gamma}(C(\mathscr{F}))_J,$$

where the right side is the truncation of type J of the geometry functorially defined by the chamber system $C(\mathscr{F})$.

Proof. For each object x of the geometry Γ, set $\phi(x) := R_x$ as defined in Lemma 13 (recall that x is regarded as a flag of rank one, so we don't have to write $\{x\}$ here). By the hypothesis on $\mathscr{F}(x)$ and Lemma 13, R_x is a residue of cotype $typ(x) \in J$ and hence is an object of type $typ(x)$ in the geometry $(\mathbf{\Gamma}(C(\mathscr{F})))_J$. Conversely, any object of $\mathbf{\Gamma}(C(\mathscr{F}))_J$ is a residue R of $C(\mathscr{F})$ of cotype j, for some $j \in J$. Now by Lemma 13, Part 2, R has the form $R = R_x$ for some object x of type j. Thus ϕ is a surjective mapping.

Now suppose (x, y) is an incident pair of (necessarily distinct) objects of Γ. By strong chamber-connectedness of $\mathscr{F}(x)$, there is a flag-chamber of $\mathscr{F}(x)$ containing y, and hence an $\mathscr{F}$-chamber $c \in R_x \cap R_y$. Thus the images of x and y under ϕ are incident objects of $(\mathbf{\Gamma}(C))_J$. Thus ϕ is a morphism of geometries.

It is also a full morphism, for if $R_x \cap R_y \neq \emptyset$ for $x \neq y$, then $R_x \cap R_y$ contains an $\mathscr{F}$-chamber of the form $\{x, y\} \oplus c'$, when $\{x, y\}$ is a flag of rank two, whence x is incident with y.

Finally, suppose x and x' are distinct objects of Γ of the same type. Then $R_x \neq R_y$ since they contain no $\mathscr{F}$-chamber in common. Thus $\phi(x) \neq \phi(x')$. So ϕ is injective. Now ϕ is an isomorphism.

3.6 Residual connectedness of Γ and $(C(\mathscr{F}))_J$. We shall say that a sheaf $\mathscr{F}$ has a geometric property P if and only if each of its values $\mathscr{F}(F)$ have property P, as F ranges over the non-empty flags of Γ. (We did this for strongly chamber-connected sheaves in Theorem 14.) Thus we say that a sheaf is chamber connected (residually connected) if and only if each geometry $\mathscr{F}(F)$ is chamber-connected (residually connected, respectively) for each non-empty flag F of Γ.

Lemma 15. *Assume the sheaf $\mathscr{F}$ is residually connected. Then the following statements hold*:

1. *The geometry Γ is residually connected.*
2. *There is an isomorphism*

$$\mathbf{C}(\Gamma) \simeq C(\mathscr{F})_J,$$

as chamber systems over J.

3. *The chamber systems of the preceeding statement are residually connected chamber systems.*

Proof. Let F be any fixed non-empty flag of Γ. From the definition of a sheaf,

$$\mathrm{Res}_\Gamma(F) \simeq (\mathscr{F}(F))_J. \tag{2}$$

Moreover, $\mathscr{F}(F)$ is a geometry over $I - t(F)$, and, for any object $x \in F$, the residue

$$\mathrm{Res}_{\mathscr{F}(x)}(F - \{x\}),$$

is a residue of the residually connected geometry $\mathscr{F}(x)$, and so is itself residually connected. As a result

(a) $\mathscr{F}(F)$ is non-empty if $I - typ(F)$ is non-empty.
(b) $\mathscr{F}(F)$ is strongly chamber-connected—that is, all flags lie in a chamber flag and the chamber system $\mathbf{C}(\mathscr{F}(F))$ is connected. (This includes the cases where it is rank 1 or is empty (only if $typ(F) = J = I$).)
(c) Any truncation of $\mathscr{F}(F)$ of rank at least two is also residually connected.

Now, if F is a flag of Γ of corank 1 so $\{J - t(F)\} = \{j\}$ for some $j \in J$, then $\mathscr{F}(F)$ has chambers (by (b)), and so contains an object of type j, so $\mathrm{Res}_\Gamma(F)$ is non-empty. On the other hand, if F is non-empty of corank at least two, then, by Equation (2) and (c), $\mathrm{Res}_\Gamma(F)$ is connected. Finally, if F is empty, its residue is Γ itself, of rank at least three, and connected by the initial hypothesis. Thus Γ is a residually connected geometry.

The chamber system $C(\mathscr{F})$ has as its chambers elements of the form $c = F \oplus c_F$ where F is a flag of Γ of type J (that is, a flag-chamber of Γ) and c_F is a flag-chamber of $\mathscr{F}(F)$. The residue of $C(\mathscr{F})$ of type $I - J$ which contains chamber c must consist of chambers of the form $F \oplus c'_F$ where c'_F wanders over a connected component of the chamber system $\mathbf{C}(\mathscr{F}(F))$ containing c'_F. Thus the mapping

$$\psi : C(\mathscr{F}) \to \mathbf{C}(\Gamma),$$

which takes each chamber $F \oplus c_F$ to F (that is, it reads off the J-part of each chamber flag) satisfies this important property:

(P) If chamber c is i-adjacent to chamber c' in $C(\mathscr{F})$, then either $\psi(c) = \psi(c')$ and $i \in I - J$, or else $\psi(c)$ and $\psi(c')$ are distinct and i-adjacent for some i in J.

Notice that (b) above implies that every chamber flag F of Γ is the ψ-image of a chamber $F \oplus c_F$, and so ψ is surjective. But in particular (P) implies ψ is a morphism of chamber systems and that the kernel of ψ is a partition of $C(\mathscr{F})$ into fibers which are unions of $(I - J)$-residues—that is, the map ψ factors through a morphism $\bar{\psi} : C(\mathscr{F})_J \to C(\Gamma)$. Now, finally, (b) above tells us that in fact each of these fibers is a single $(I - J)$-residue of $C(\mathscr{F})$. Thus the induced mapping $\bar{\psi}$ is an injective morphism of chamber systems over J.

It only remains to show that $\bar{\psi}$ is a full morphism. Suppose A and B are distinct elements of $C(\mathscr{F})_J$ (that is, residues of type $I - J$ in $C(\mathscr{F})$), such that $F := \psi(A)$ is j-adjacent to $G := \psi(B)$ in $C(\Gamma)$. Then as $|J| \geqslant 2$, $H := F \cap G$ is a non-empty flag of cotype j in Γ. Then since $\mathscr{F}(H)$ is residually connected, $F - H$ and $G - H$ lie in respective flag chambers c_F and c_G of $\mathscr{F}(H)$, and $a := H \oplus c_F = F \oplus c'_F$ and $b := H \oplus c_G = G \oplus c'_G$ are connected by a gallery whose type is a word in $(I - J) \cup \{j\}$ (this gallery corresponds to one in $C(\mathscr{F}(H))$). But $R_F = A$ and $R_G = B$ contain a and b, respectively, and so lie in a common residue T of type $(I - J) \cup \{j\}$. Thus $\bar{\psi}$ is a full bijective morphism, and so is an isomorphism of chamber systems. This proves the second statement.

The third conclusion is immediate for if Γ is residually connected, $\mathbf{C}(\Gamma)$ is residually connected as a chamber system. The proof is complete.

3.7 Configurations produced by a 1-covering of $C(\mathscr{F})$. This section concerns what happens when we have a 1-covering of the chamber system $C(\mathscr{F})$. We must standardize both the notation and the hypotheses:

1. (The locally truncated geometry) As has been standard so far, Γ is a geometry over J which is locally truncated with respect to the diagram D over I.
2. (Existence of a sheaf) We assume there is a sheaf $\mathscr{F}$ defined for this local truncation, and we let $\bar{C}$ denote its associated chamber system over I. If each $\mathscr{F}(x)$ is strongly chamber-connected, we know from Theorem 14 that there is an isomorphism $\phi : \Gamma \to \mathbf{\Gamma}(C(\mathscr{F}))_J = \mathbf{\Gamma}(\bar{C}_J)$.
3. (The covering of the chamber system of the sheaf) We assume there is a 1-covering $\kappa : C \to \bar{C}$.
4. (The morphisms of geometries) We let $\bar{\Delta} := \mathbf{\Gamma}(C(\mathscr{F}))$ and $\Delta := \mathbf{\Gamma}(C)$ be the geometries over I defined by the chamber systems $\bar{C}$ and C, respectively. (Recall that a flag in $\mathbf{\Gamma}(C)$ is just a collection of pairwise intersecting corank-one residues of C. Such a flag is called a $\mathbf{\Gamma}$-*image* if all these residues lie on a common chamber. So some flags of $\mathbf{\Gamma}(C)$ are $\mathbf{\Gamma}$-images, and some might not be. Of course all flags are $\mathbf{\Gamma}$-images if C is residually connected as a chamber system. A similar distinction applies to the flags of $\bar{\Delta} := \mathbf{\Gamma}(\bar{C})$.) We let $h := \Gamma(\kappa)$ be the functorially defined geometry morphism $\Delta \to \bar{\Delta}$. We let $h_J : \Delta_J \to \bar{\Delta}_J$ be the morphisms induced by h on their truncations of type J.

Lemma 16. *Assume these hypotheses*:

(a) $\kappa : C \to \bar{C} = C(\mathscr{F})$ *is a* 1-*covering of chamber systems.*
(b) *The sheaf* $\mathscr{F}$ *is strongly chamber-connected.*

Then the following statements hold:

1. *The induced* $h_J : \Delta_J \to \bar{\Delta}_J$ *is a full epimorphism. Thus there is a full epimorphism of geometries*

$$f = \phi^{-1} \circ h_J : \Delta_J \to \Gamma.$$

2. *Every flag of* Γ *is the image under* f *of a flag of* Δ_J *which is a* $\mathbf{\Gamma}$-*image with respect to* $\mathbf{\Gamma}(C)$.
3. (The flag-lifting property) *Suppose* X' *is a flag of* Δ_J *which is a* $\mathbf{\Gamma}$-*image such that* $f(X') = X$. *Suppose further that* F *is any flag of* Γ *such that* $X \subseteq F$. *Then there exists a flag* F' *of* Δ_J *(also a* $\mathbf{\Gamma}$-*image) containing* X' *such that* $f(F') = F$.

Proof. Consider a containment of non-empty flags, $F_1 \subseteq F_2$ of Γ, and an object $x \in F_1$. Since $\mathscr{F}$ is strongly chamber-connected, by Lemma 13, $R_{F_2} \subseteq R_{F_1} \subseteq R_x$ is a containment of residues of $\bar{C}$ of cotypes $typ(F_2)$, $typ(F_1)$ and $typ(x)$, respectively. Let c be a chamber in R_{F_2}. Since κ is surjective, there is a preimage c' of c in C. Then let $R_2 \subseteq$

$R_1 \subseteq R(x)$ be the residues of C of respective cotypes $typ(F_2)$, $typ(F_1)$ and $typ(x)$, containing c'. Then, since κ is a 1-cover, R_2, R_1 and $R(x)$ are the lifts of R_{F_2}, R_{F_1} and R_x at c', respectively.

We apply the developement of this paragraph to prove the first two parts of the lemma.

Taking $F_1 = F_2$, we see that $F_1' := \{R(x) \mid x \in F_1\}$ is a flag of Δ of type $typ(F_1)$ which is a $\mathbf{\Gamma}$-image and which maps onto F_1. This proves Part 2. But it also proves Part 1, since $x = \phi^{-1}(R_x)$ for all $x \in \Gamma$, and any incident pair of objects in Γ lies in a flag F_1 which is the image of the flag F_1' just described.

Part 3. Suppose X' is a $\mathbf{\Gamma}$-image of Δ_J mapping to a flag X of Γ which lies in a larger flag F of Γ. Since X' is a $\mathbf{\Gamma}$-image, it is a collection of corank-one residues $R(x)$ of C, whose intersection contains a residue R_1 of cotype $typ(X)$ which maps via κ onto R_X. Choose a chamber $c \in R_F$. It possesses a preimage $c' \in R_1$. Now let R_2 be the residue of cotype $typ(F)$ on c'. Now we have

$$c' \in R_2 \subseteq R_1.$$

Then the full set of corank-one residues of C containing R_2 is the desired $\mathbf{\Gamma}$-image flag F' of Part 3.

3.8 A covering of a point-collinearity graph derived from a locally truncated geometry. In this subsection we assume the notation of Items 1–4 at the beginning of Section 3.7. In addition we assume

1. D is a Coxeter diagram over the finite set I whose rank-three residues are covered by buildings.
2. $\kappa : C \to \bar{C}$ is a universal 2-covering.
3. The sheaf $\mathscr{F}$ is residually connected.

Since D is a Coxeter diagram with rank-three residues covered by buildings, C is a chamber system over I which is a building and so is residually connected. Accordingly, the associated geometry $\Delta := \mathbf{\Gamma}(C)$ is a building geometry over I.

Now Assumption 3 implies $\mathscr{F}(F)$ is strongly chamber-connected for every non-empty flag F of Γ. Thus hypotheses (a) and (b) of Lemma 16 hold, so

$$f : \Delta_J \to \Gamma, \tag{3}$$

is a full epimorphism of geometries over J.

Again, since the sheaf is residually connected, Part 3 of Lemma 16 forces the Morphism (3) to possess the flag-lifting property:

(FL) *If X' is a flag of Δ_J that is a $\mathbf{\Gamma}$-image such that $f(X') = X$, and F is a flag of Γ incident with X (i.e. $X \cup F$ is a flag), then there exists a flag F' of Δ_J with $f(F') = F$ and $F' \cup X'$ a flag of Δ_J.*

Finally, our hypothesis that Γ is connected and the Assumption 3 that $\mathscr{F}$ is residually connected imply three statements: (1) that Γ is a residually connected geometry,

(2) that $\mathbf{C}(\Gamma) \simeq C(\mathscr{F})_J$ as chamber systems over J, and (3) that $C(\mathscr{F})_J$ is residually connected as a chamber system. All three conclusions are a direct application of Lemma 15.

We are now ready to introduce the main theorem of this section.

Theorem 17. *Suppose $t_{\mathscr{P}}$, $t_{\mathscr{L}}$, and t_{Π} are three pairwise disjoint collections of subsets of J. Let $(\mathscr{P}_\Gamma, \mathscr{L}_\Gamma, \Pi_\Gamma)$ and $(\mathscr{P}_\Delta, \mathscr{L}_\Delta, \Pi_\Delta)$ be the triples of collections of all flags of these respective types in the geometries Γ and Δ_J, respectively. We assume*

1. *For $E \in \{\Gamma, \Delta\}$, and each object $X \in \Pi_E$, $\mathrm{Res}_E(X) \cap \mathscr{P}_E$ is a singular subspace of $P_E := (\mathscr{P}_E, \mathscr{L}_E)$, regarded as a point-line geometry.*
2. *For any triple $(a, b, c) \in \mathscr{P}_\Gamma \times \mathscr{P}_\Gamma \times \mathscr{P}_\Gamma$ of pairwise collinear points of Γ, there exists an object $X \in \Pi_\Gamma$ such that $\{a, b, c\} \subseteq \mathrm{Res}_\Gamma(X) \cap \mathscr{P}_\Gamma$.*
3. *If $X \in \mathscr{P}_\Gamma \cup \mathscr{L}_\Gamma$, then $\mathscr{F}(X)$ is a 2-simply connected geometry.*
4. *$(\mathscr{P}_\Gamma, \mathscr{L}_\Gamma)$ is a partial linear space.*

Then the graph morphism

$$\phi : (\mathscr{P}_\Delta, \sim) \to (\mathscr{P}_\Gamma, \sim)$$

of point-collinearity graphs of $(\mathscr{P}_\Delta, \mathscr{L}_\Delta)$ and $(\mathscr{P}_\Gamma, \mathscr{L}_\Gamma)$ induced by the truncated mapping

$$f : \Delta_J \to \Gamma,$$

is a $\mathscr{T}$-covering of graphs, where $\mathscr{T}$ is the collection of all 3-circuits in $(\mathscr{P}_\Gamma, \sim)$.

Remarks. (i) The part of Condition (1) concerning Γ (but not necessarily Δ), and Conditions (2) and (4) hold, for example, if $(\mathscr{P}_\Gamma, \mathscr{L}_\Gamma)$ is a parapolar space and Π_Γ is the full collection of all its projective plane subspaces.

(ii) Condition (3) is satisfied if $\mathscr{F}(X)$ is a building geometry over $I - typ(X)$ for all $X \in \mathscr{P}_\Gamma \cup \mathscr{L}_\Gamma$.

Proof. For the first two steps below, we need to establish the hypotheses of Lemma 8. We have on hand the 2-covering of chamber systems: $\kappa : C \to \bar{C} = C(\mathscr{F})$ belonging to diagram D.

Let X be a flag of Γ whose type is an element of $\{t_{\mathscr{P}}, t_{\mathscr{L}}\}$—that is, X is either a point or a line of the geometry $(\mathscr{P}_\Gamma, \mathscr{L}_\Gamma)$. By Lemma 13

$$R_X := X \oplus C(\mathscr{F}(X)) = \{X \oplus c_X \mid c_X \in \mathbf{C}(\mathscr{F}(X)\}$$

is a residue of cotype $typ(X)$. By Hypothesis 3 of the theorem, $\mathbf{C}(\mathscr{F}(X))$ is 2-simply connected. Since the mapping $X \oplus c_X \to c_X$ induces an isomorphism of $\mathbf{C}(\mathscr{F}(X))$ and R_X, we see that

(A-1) *R_X is 2-simply connected as a chamber system over $I - typ(X)$.*

But by property (3) just preceeding the statement of this theorem,

(A-2) *$C(\mathscr{F})_J$ is residually connected.*

We have now assembled all of the hypotheses of Lemma 8, where C, $C(\mathscr{F})$ and $typ(X)$ fulfill the roles of $\hat{C}$, C, and K of that lemma. Thus the lemma produces this conclusion:

(LI) *If $\hat{X}$ is a flag of Δ_J with $f(\hat{X}) = X$, then the mapping*

$$f_{\hat{X}} : \mathrm{Res}_{\Delta_J}(\hat{X}) \to \mathrm{Res}_{\Gamma}(X),$$

induced by the restriction of $f : \Delta_J \to \Gamma$, is injective on objects. It is therefore injective when restricted to the sets of flags of any prescribed type of these residues.

If X is a point, then we have

Step 1. (Local injectivity of lines) *If L and N are distinct lines of $\mathscr{L}_\Delta$ incident with a common point p of $\mathscr{P}_\Delta$ then $f(L)$ is not equal to $f(N)$, that is ϕ, our graph morphism, is injective when restricted to the lines incident with a common point.*

If X is a line we have

Step 2. (Local injectivity of points) *If p and q are distinct points of $\mathscr{P}_\Delta$ incident with a common line L of $\mathscr{L}_\Delta$, then $\phi(p)$ is not equal to $\phi(q)$, that is, ϕ is point-injective when restricted to a line.*

We conclude

Step 3. $\phi : (\mathscr{P}_\Delta, \sim) \to (\mathscr{P}_\Gamma, \sim)$ *is a fibering of graphs, that is,*

1. *Each fiber $\phi^{-1}(x)$ of a vertex $x \in \mathscr{P}_\Gamma$ is a coclique of $(\mathscr{P}_\Delta, \sim)$.*
2. *Moreover, ϕ induces a bijective mapping*

$$P_\Delta(x) \to P_\Gamma(\phi(x))$$

when restricted to neighborhood subgraph of a point $x \in \mathscr{P}_\Delta$.

3. *ϕ is both point and edge surjective.*

Thus for every path in $(\mathscr{P}_\Gamma, \sim)$ and any specified point x in the fiber above the initial point of this path, there is a unique lift in $(\mathscr{P}_\Delta, \sim)$ of this path beginning at x.

Proof of Step 3. Part 1 follows from Step 2. For Part 3 we use the flag-lifting property (FL) to conclude that every point and edge of $(\mathscr{P}_\Gamma, \sim)$ is the image of a point or edge of $(\mathscr{P}_\Delta, \sim)$.

To prove Part 2, we invoke for the first time the hypothesis that $(\mathscr{P}_\Gamma, \mathscr{L}_\Gamma)$ is a partial linear space. Suppose (y, y_1) and (y, y_2) are distinct edges of $(\mathscr{P}_\Delta, \sim)$. Then there are (not necessarily unique) lines N_i of $\mathscr{L}_\Delta$ containing $\{y, y_i\}$, $i = 1, 2$. Suppose by way of contradiction that $\phi(y_1) = \phi(y_2)$. By Step 2, this point is distinct from $\phi(y)$ but collinear with it. Then $f(N_1)$ and $f(N_2)$ are lines of $\mathscr{L}_\Gamma$ on the distinct points y and $\phi(y_1) = \phi(y_2)$. Since $(\mathscr{P}_\Gamma, \mathscr{L}_\Gamma)$ is a partial linear space, we have $f(N_1) = f(N_2)$. But since the N_i lie in $\mathrm{Res}_{\Delta_J}(y)$, Step 1 forces $N_1 = N_2$. But in that case y_1 and y_2 are distinct points of N_1 mapping to a common point of $f(N_1)$, against Step 2. Thus $\phi(y_1) = \phi(y_2)$ is impossible, and so ϕ restricts to a vertex-injective mapping of the set of neighborhood vertices of a given vertex. That ϕ restricted to the neighborhood of y in $(\mathscr{P}_\Delta, \sim)$ maps onto the neighborhood of $\phi(y)$ in $(\mathscr{P}_\Gamma, \sim)$ follows from the fact that any line of $\mathscr{L}_\Gamma$ on $\phi(y)$ lifts to a line on y in $\mathscr{L}_\Delta$ by Condition (FL). Thus the induced mapping on vertex neighborhoods is bijective on vertices as required.

Step 4. *Every lift of a* 3*-circuit in* $(\mathscr{P}_\Gamma, \sim)$ *is a* 3*-circuit of* $(\mathscr{P}_\Delta, \sim)$.

Proof of Step 4. Suppose (a, b, c) is a 3-circuit of $(\mathscr{P}_\Gamma, \sim)$. By Hypothesis 2, of the theorem, there exists an object $X \in \Pi_\Gamma$, incident with the three points a, b and c. Since $f : \Delta_J \to \Gamma$ has the flag-lifting property (FL), for every point a' in the fiber $\phi^{-1}(a)$, there exists a flag X' of Δ_J incident with a' such that $f(X') = X$. Also, by (FL), each point $x \in \{a, b, c\}$ has a preimage $x' \in \mathscr{P}_\Delta$ incident with X'. Now by Hypothesis 1 of this theorem, $\mathrm{Res}_{\Delta_J}(X') \cap \mathscr{P}_\Delta$ is a singular subspace of $(\mathscr{P}_\Delta, \mathscr{L}_\Delta)$, and so the three points a', b', and c' are pairwise collinear. Thus the 3-circuit (a', b', c') is the unique lift of the 3-circuit (a, b, c) at the point a' of the fiber above a.

The proof that ϕ is a $\mathscr{T}$-covering is complete.

4 The Cohen–Cooperstein theory revisited and updated

Nearly seventeen years ago Cohen and Cooperstein [4] proved the following theorem:

Theorem 18 (Cohen and Cooperstein). *Suppose* $\Gamma = (\mathscr{P}, \mathscr{L})$ *is a strong parapolar space of symplectic rank exactly k where* $k \geqslant 3$ *such that each maximal singular subspace has finite projective rank. Suppose the following*:

$(\mathrm{CC})_1$ *If* $(x, S) \in \mathscr{P} \times \mathscr{S}(\Gamma)$ *is a non-incident point-symplecton pair and* $x^\perp \cap S$ *contains at least two points, then* $x^\perp \cap S$ *is a maximal singular subspace of the symplecton S.*

Then:

(CCC) *At least one of the following statements is true*:

1. Γ *is a polar space.*
2. $k = 3$ *and*
   (a) Γ *is a Grassmannian* $A_{n,d}$, $1 < d < n - 1$.
   (b) Γ *is* $A_{2n-1,n}/\langle \rho \rangle$, *a quotient of a Grassmannian by a polarity* ρ *of Witt index at most* $n - 4$.

3. $k = 4$ *and* Γ *is the point-line truncation to types* $\{n, n-2\}$ *of a locally truncated geometry of type J over the Dynkin diagram of type* D_n *over* $I = \{1, 2, \ldots, n\}$ *satisfying the conditions of Theorem* 17 *such that D and J correspond to the top diagram of Figure* 1. *In particular*, Γ *is a homomorphic image of a truncation of a building. Moreover the point-collinearity graph of the half-spin geometry* $D_{n,n-2}(K)$ *is a* $\mathscr{T}$*-cover of the point-collinearity graph of* Γ.
4. $k \geqslant 5$: *Here* Γ *is one of the exceptional Lie incidence geometries* $E_{6,1}$, *or* $E_{7,1}$ ($k = 5$ *and* 6 *respectively*). *No example with k larger than* 6 *can exist.*

Remark. This differs from their original theorem in only two minor details: namely by Cohen's up-dated definition of "parapolar space" and the added remarks in (CCC) about the case $k = 4$. The latter follow from Theorem 17 of the previous section.

The next result was an attempt to view this theorem without the hypothesis of constant symplectic rank. It involves this hypothesis:

(WH) (The weak hexagon hypothesis) *Suppose* $(x_0, x_1, \ldots, x_5, x_0)$ *is a* 6*-circuit in the point-collinearity graph of the gamma space* Γ. *If the distance* $d(x_0, x_3) = 3$, *then there exists a point* y *in* $x_0^{\perp} \cap x_2^{\perp} \cap x_4^{\perp}$. (*Of course if* $x_2 = x_4$, *we may take* $y = x_1$.)

The adjective "weak" comes from the distance-three requirement.

Let $\mathscr{E}_3$ be the collection of all strong parapolar spaces $\Gamma = (\mathscr{P}, \mathscr{L})$ with point-diameter three, such that

1. for any point p, $\Delta_2^*(p)$, the collection of all points at distance at most two from p forms a *geometric hyperplane* of Γ, that is, a proper subspace which meets every line non-trivially;
2. every geodesic path of length two in the point-collinearity graph extends to one of length three.

The reader may check that these assertions are equivalent to the axioms (E1)–(E4) for $\mathscr{E}_n$ in [6] when $n = 3$.

The class $\mathscr{F}$ consists of all gamma spaces with these properties:

1. Every geodesic path of length two in the point-collinearity graph of Γ extends to one of length three.
2. For any two points x and y at distance three in the point-collinearity graph of Γ, their convex closure (the smallest convex subspace containing them) is a member of $\mathscr{E}_3$.

Remark. It follows that any member of $\mathscr{F}$ is a strong parapolar space.

Theorem 19 (El-Atrash and Shult [6]). *Suppose* $\Gamma = (\mathscr{P}, \mathscr{L})$ *is a geometry in* $\mathscr{F}$ *with all lines thick having point-diameter at least three and all symplecta of polar rank at least three, such that*

1. *the Condition* (WH) *holds,*
2. *every symplecton has finite polar rank (necessarily at least three), and*
3. *all singular subspaces of* Γ *possess finite projective rank.*

Then Γ *satisfies the conclusion* (CCC) *of Cohen and Cooperstein's theorem.*

Basically the hypotheses imply constant symplectic rank and Condition $(\mathrm{CC})_1$. We will use this theorem to prove Theorem 1.

5 A special class of strong parapolar spaces

In this section Γ is a strong parapolar space satisfying the following axioms:

(P1) If x is a point and S is a symplecton, then $x^\perp \cap S \neq \emptyset$.
(P2) For every point p, $\Delta_2^*(p)$ is a geometric hyperplane of Γ.
(P3) If every symplecton has rank at least three, all singular subspaces are assumed to have finite projective rank.

Notice that if all symplecta have rank at least three, then all singular subspaces of Γ are projective spaces. But without this assumption, it is conceivable, for the time being, that singular subspaces are not projective. Indeed, we shall prove that they are projective in a later corollary.

Our objective is to prove the second main theorem by showing that Γ is one of the following:

1. $D_{6,6}$, $A_{5,3}$ or $E_{7,1}$.
2. A classical dual polar space of rank three.
3. A product geometry $L \times P$, where L is a line and P is a polar space of arbitrary rank.

This is accomplished by a series of theorems:

Theorem 20. *Suppose a point x is distance three from a point p in a symplecton S. Then* $x^\perp \cap S = \{r\}$, *a single point, and* $\Delta_2^*(x) \cap S = r^\perp \cap S$. *The symplecton S is gated with respect to x.*

Proof. Clearly $x^\perp \cap S$ is a non-empty singular subspace of S. Since S is a polar space and contains a point at distance three from x, $x^\perp \cap S$ is a single point set $\{r\}$. Then $r^\perp \cap S \subseteq \Delta_2^*(x) \cap S$, and equality now follows from the fact that in a polar space of rank at least two having thick lines, all geometric hyperplanes are maximal subspaces.

Theorem 21. *In the point-collinearity graph* $\Delta = (\mathscr{P}, \sim)$ *of* Γ, *every geodesic path of length two extends to a geodesic path of length three.*

Proof. Let (p, a, b) be a geodesic path of length two in $(\mathscr{P}, \sim)$. We wish to extend it to a geodesic path (p, a, b, c) of length three. Let z be any element in $\Delta_3(p)$ (exists by (P2)) and let S be the unique symplecton containing the geodesic (p, a, b). Then

$z^{\perp} \cap S = \{r\}$ by (P1) and by Theorem 20 r is the gate of S with respect to z. We need only show that b is collinear to a point of $\Delta_3(p)$. This is implied by the following claim

(Claim #1) *Every point of* $S - p^{\perp}$ *is collinear to a point of* $\Delta_3(p)$.

But since S has thick lines, the induced point-collinearity graph on $S - p^{\perp}$ is connected. Since r is a point of $S - p^{\perp}$ which is connected to a point of $\Delta_3(p)$, it suffices to prove

(Claim #2) *If* r_1 *and* x *are collinear points of* $S - p^{\perp}$ *and* r_1 *is collinear with a point of* $\Delta_3(p)$ *then* x *is also collinear with a point of* $\Delta_3(p)$.

Proof of Claim #2. By hypothesis r_1 is collinear with a point z_1 of $\Delta_3(p)$. Then x is distance two from z_1 and so there is a symplecton T containing $\{x, z_1\}$. Since S is a polar space, p is collinear with a point g on line xr_1 and g is distinct from both x and r_1 by hypothesis. Now, since T contains a point z_1 at distance three from p, T is gated with respect to p so $\Delta_2^*(p) \cap T = g^{\perp} \cap T$. But as T is a non-degenerate polar space g is the unique deep point of $g^{\perp} \cap T$, so there is a point of $x^{\perp} \cap T$ not contained in $g^{\perp} \cap T = \Delta_2^*(p) \cap T$. Thus x is collinear with a point of $\Delta_3(p)$ and the Claim #2 is proved.

So, as remarked, the entire theorem is proved.

Since we cannot assume that singular subspaces are projective, we must be careful about what we call a "plane". We know that Γ is a partial linear gamma space. We say that a singular subspace is a *plane* if it is generated by a non-incident point-line pair—that is, it has the form $\langle p, L\rangle$. Conceivably, one plane could properly contain another. But because of our strong hypotheses, we can show that all planes are projective planes.

Theorem 22. *Every plane* $\pi := \langle a, L\rangle$ *lies in some symplecton.*

Proof. We begin by assuming that π lies in no symplecton.

Choose a point $b \in L$ and select a symplecton R on line ab (one exists by the parapolar hypothesis). Since $a^{\perp} \cap R$ is not a clique, there is a point x in R with $x^{\perp} \cap ab = \{a\}$. Now if $u \in x^{\perp} \cap L$, then the symplecton $\ll x, b \gg$ contains b, a, and $u \in L - \{b\}$ and so contains L (Γ is a partial linear space), and so contains π, contrary to assumption. Thus

$$x^{\perp} \cap L = \varnothing. \tag{4}$$

Now the geodesic (x, a, b) extends to one of length three, say (x, a, b, z). Clearly if $z^{\perp}$ contained L, then the symplecton $\ll z, a \gg$ would contain L and a and hence would contain π, a contradiction. Thus $z^{\perp} \cap L = \{b\}$.

We observe that any symplecton on L is a rank-two symplecton, or "quad". For otherwise, if such a symplecton Q_0 had polar rank at least three, $L^{\perp} \cap Q_0$ would not be a clique, and so would contain a point v not in $a^{\perp}$, and then the symplecton $\ll v, a \gg$ would contain π, against our assumption.

Now let $c \in L - \{b\}$ and let $Q = \ll z, c \gg$. By the previous paragraph, Q is a quad. Then by Theorem 20 there is a unique point $w \in x^{\perp} \cap Q$ and $w^{\perp} \cap Q = \Delta_2^*(x) \cap Q$ contains L. But as observed, L is a maximal singular subspace of the quad Q, and so $w \in L$. But this contradicts Equation (4) and completes the proof.

Corollary 23. *All singular subspaces of* Γ *are projective spaces.*

Proof. By Theorem 22, all planes of such a subspace are projective.

Corollary 24. *Every pair of distinct intersecting lines lies in a symplecton. If the two lines do not lie in a singular subspace, the symplecton containing them is unique.*

Theorem 25. *If an intersection of two symplecta contains a point, it also contains a line.*

Proof. Suppose by way of contradiction, that S_1 and S_2 are distinct symplecta whose intersection is a single point p. Choose a geodesic (p, b, x) in S_1 and extend it to a geodesic (p, b, x, q) (permitted by Theorem 21). Now by hypothesis, $x^{\perp} \cap S_2$ contains a point y. If $x^{\perp} \cap S_2$ contained a line, this line would contain a point of $p^{\perp} - \{p\}$ lying in $S_1 \cap S_2$, contrary to our assumption. Thus $x^{\perp} \cap S_2 = \{y\}$.

Now either (y, x, q) is a geodesic of length two, or $\{y, x, q\}$ lies in a plane. Using Theorem 22 in the last case, we see that in all cases, $\{y, x, q\}$ lies in a symplecton R. Now R is gated with respect to p, so $p^{\perp} \cap R = \{t\}$, and $\Delta_2^*(p) \cap R = t^{\perp} \cap R$. But the latter set contains both x and y, whence

$$t \in p^{\perp} \cap x^{\perp} \cap y^{\perp} \subseteq S_1 \cap S_2,$$

by the convexity of these symplecta. This forces $t = p$, which is impossible since $d(p, q) = 3$.

Theorem 26. *If* $x^{\perp} \cap S$ *is a single point for some symplecton* S*, then* S *contains a point at distance three from* x*. In particular,* S *is strongly gated with respect to point* x*.*

Proof. Suppose, for some point x and symplecton S, that $x^{\perp} \cap S = \{y\}$. If S were not strongly gated with respect to x, there would be a point z in S at distance two from y which was also distance two from x. In that case there is a symplecton R on $\{x, z\}$. Now by Theorem 25 the intersection of R and S contains a line L on z. Then as L and x are in symplecton R, $x^{\perp}$ meets L at a point v. But then $v \in x^{\perp} \cap S = \{y\}$, so $v = y$. But that contradicts $d(y, z) = 2$. Thus all points of $S - y^{\perp}$ are at distance three from x and S is gated with respect to x.

Theorem 27. Γ *possesses the weak hexagon property.*

Proof. Suppose $(x_0, x_1, \ldots, x_6)$, $x_6 = x_0$, is a six-circuit in the point-collinearity graph of Γ and that $d_\Gamma(x_0, x_3) = 3$. Let R and S be the symplecta on $\{x_0, x_2\}$ and $\{x_0, x_4\}$, respectively. Then by Theorem 25, $R \cap S$ contains a line L on point x_0. Now $x_2^\perp \cap L = \{a\}$ and $x_4^\perp \cap L = \{b\}$, for points a and b lying in $\Delta_2^*(x_3) \cap L$. Since $\Delta_2^*(x_3)$ is a subspace and x_0 is distance three from x_3, we must have $a = b \in x_0^\perp \cap x_2^\perp \cap x_4^\perp$. Similarly, there is a point in $x_1^\perp \cap x_3^\perp \cap x_5$. So the weak hexagon property holds.

We require a minor lemma:

Lemma 28. *If all symplecta have rank at least three, then the point-collinearity graph of* Γ *is simply connected.*

Proof. Axiom (P2) of the hypotheses introduced at the beginning of this section shows that the point-collinearity graph has diameter three. Moreover, since symplecta have rank at least three, any circuit within a symplecton is contractible. Thus we need only show that any circuit of length seven or less decomposes into circuits of length three or four.

Suppose $c = (x_0, x_1, \ldots, x_5)$, $x_5 = x_0$, is a circuit of length five not decomposable into circuits of length three or four. Then x_0 is distance two from both x_2 and x_3, and the intersection of the two symplecta R and S on $\{x_0, x_2\}$ and $\{x_0, x_3\}$ contains a line L on x_0. Now if $R = S$, c is contractible. So we may assume x_2 is not in S. Then $x_2^\perp \cap S$ is a clique containing x_3 and a point a on L. Now c decomposes into the circuits (x_0, x_1, x_2, a, x_0), (x_3, a, x_2, x_3) and (x_0, a, x_3, x_4, x_0) of lengths 4, 3 and 4, a contradiction. Thus all five-circuits are decomposable into circuits of length three or four.

If c is a six-circuit with an antipodal pair of vertices at distance two, it decomposes into two five circuits. Otherwise, an antipodal pair is at distance three, and the weak hexagon property shows that this decomposes into three circuits of length four. Thus all six-circuits are decomposable.

Suppose now that $c = (x_0, x_1, \ldots, x_7)$, $x_7 = x_0$, is an indecomposable seven-circuit. Then we must have $d_\Gamma(x_0, x_3) = d_\Gamma(x_0, x_4) = 3$. Now the hyperplane $\Delta_2^*(x_0)$ meets the line N on x_3 and x_4 at a point b. Choosing a in $x_0^\perp \cap b^\perp$, we see that c decomposes into the two six-circuits, $(x_0, x_1, x_2, x_3, b, a, x_0)$ and $(x_0, a, b, x_4, x_5, x_6, x_0)$. This contradicts the indecomposability of c.

Theorem 29. *If all the symplecta of* Γ *have rank at least three, then* Γ *is isomorphic to one of the following Lie incidence geometries*:

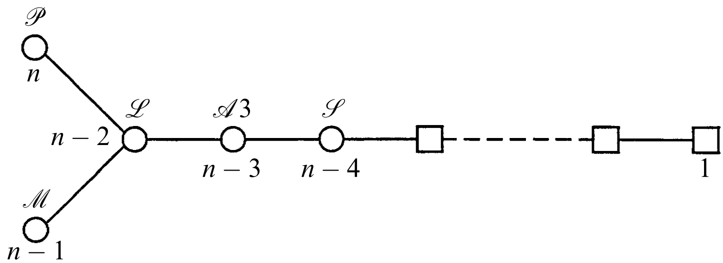
$$D_{6,6}, \quad A_{5,3} \quad \text{or} \quad E_{7,1}.$$

Proof. We have noted that the point-collinearity graph of Γ has diameter three. The axioms (P1) and (P2) at the beginning of this section together with Theorem 21 show that Γ belongs to the family $\mathscr{E}_3$ of polarized spaces introduced in [11] and studied in [6]. Since all symplecta have rank at least three, our initial Assumption (P3) requires all maximal singular subspaces to have finite projective rank. By Theorem 27, the weak hexagon property holds, and so the main theorem in [6] (that is, Theorem 19) shows that Γ is one of the following:

1. A Grassmann space $A_{n,k}$,
2. $A_{2k-1,k}/\langle\sigma\rangle$, a homomorphic image of a Grassmannian, $A_{2k-1,k}$ where σ is a polarity of Witt index at most $k-4$.
3. A homomorphic image of a classical halfspin geometry $D_{n,n}$.
4. The Lie incidence geometry $E_{7,1}$.

Now we have the additional properties that Γ has point-diameter at least three, and that for any point-symp pair (x,S), $x^{\perp} \cap S$ is non-empty. In the first case listed this eliminates all but the Grassmannian $A_{5,3}$. The second case is eliminated altogether since the geometry there has point diameter at least four.

In the third case above, the parapolar space $\Gamma = (\mathscr{P}, \mathscr{L})$ is enriched to a rank-five geometry $\Gamma = (\mathscr{P}, \mathscr{L}, \mathscr{M}, \mathscr{A}3, \mathscr{S})$ over $J = \{n, n-1, \dots n-4\}$, which is a locally truncated geometry relative to the diagram



where $\mathscr{M}$ and $\mathscr{A}3$ are two classes of maximal singular subspaces and $\mathscr{S}$ is the class of symplecta. (This much is in [4] and [6].)

Now by the Remark (1) following Theorem 17, Hypotheses 1, 2, and 4 of that theorem are in place, where Π is the collection of all projective subplanes. Since the residues of any point or line in Γ are truncations of buildings, Condition 3 of Theorem 17 holds (Remark (2) following Theorem 17).

Thus by Theorem 17, the geometry epimorphism $f : \Delta_J \to \Gamma$ obtained from Lemma 16 induces a $\mathscr{T}$-covering

$$\phi : G \to \bar{G}$$

of their point-collinearity graphs. But since the point collinearity graph $\bar{G}$ is simply connected, ϕ is a graph isomorphism. Thus Γ has the point-collinearity graph of the half-spin geometry. But all objects in the rank-n building geometry Δ are induced subgraphs of its half-spin collinearity graph, and so the morphism f is also a geometry isomorphism. Then Γ is a bonafide half-spin geometry. Now the two special properties of Γ force Γ to be isomorphic to $D_{6,6}$.

In the last case all the axioms hold, so this case survives.

Theorem 30. *If Γ contains a symplecton which is a grid, and a symplecton which is not, then Γ is a product geometry $L \times P$, where L is a line and P is a polar space of arbitrary rank.*

We begin the proof of Theorem 30 with a series of lemmas.

Lemma 31. *Suppose G, S_1 and S_2 are three symplecta whose intersection $G \cap S_1 \cap S_2$ is a point p. We suppose that G is a grid. Then at least one of the S_i is also a grid.*

Proof. From Theorem 25 and the hypotheses, $G \cap S_i = L_i$, $i = 1, 2$ are the two distinct lines of G on p and $S_1 \cap S_2$ is a singular subspace meeting the L_i at p and containing a line N on p. Suppose by way of contradiction that neither S_i was a grid. Then the lines and planes of S_i which are incident with point p is either a rank one polar space (or coclique) with at least three points, or is a polar space of rank at least two. In either case, it is not the union of a clique and the perp of a point. Thus each symplecton S_i contains a line N_i on p which is not in $L_i^{\perp}$ or in the singular space $S_1 \cap S_2$. By Corollary 24, there is a symplecton R containing $N_1 \cup N_2$. But then by Theorem 25, $R \cap G$ must be one of the two lines of G on p, L_1 or L_2. But if $R \cap G$ were the line L_1, then R would be the unique symplecton containing $L_1 \cup N_1$, namely S_1, and that would imply that line N_2 was in S_1, contrary to choice. But also $R \cap G$ cannot be the other line L_2 by the complete symmetry of the indices $i = 1, 2$ in the face of the hypotheses and the choice of the lines N_i. This contradiction infirms the assumption that neither S_i was a grid, and the proof of the lemma is complete.

Lemma 32. *Suppose G is a symplecton which is a grid, and that G intersects a second symplecton Q, which is not a grid, at a line L_1. Choose a point p on L_1, let L_2 be the other line of G on p distinct from L_1, and choose any point q in $L_2 - \{p\}$. The following statements must hold*:

1. *With the exception of the line L_2, every line on p lies in Q.*
2. $q^{\perp} \cap Q = \{p\}$.
3. *Any symplecton containing p which is distinct from Q, is a grid.*
4. *Let $\mathscr{L}_p$ and $\mathscr{L}_q$ be the collections of all lines of Γ on points p and q, respectively. Then there is a bijection*

$$\beta : \mathscr{L}_q - \{L_2\} \to \mathscr{L}_p - \{L_2\},$$

such that for corresponding lines $L \in \mathscr{L}_q - \{L_2\}$ and $\beta(L)$, there is a second bijection $L \to \beta(L)$ taking each point r of L to the unique point of Q to which it is collinear.

Proof. 1. Suppose M were a line on p distinct from L_2 and not lying in Q. Then by Corollary 24, $M \cup L_2$ lies in some symplecton S. By Theorem 25 the intersection $S \cap Q$ contains a line N on p which is necessarily distinct from L_1. But now, Q and S meet G at distinct lines, and so the intersection of all three is just the single point p.

Since G is a grid and Q is not, Lemma 31 implies that S is a grid. But that is impossible since L_2, M and N are three distinct lines of S on point p. Thus no such line M exists, which proves the result.

2. Suppose by way of contradiction that $q^{\perp} \cap Q$ is contained a line N and let π be the projective plane generated by q and N. Clearly $\pi \cap Q = N$. It follows that there is a line of π on p distinct from N and L_2, against Part 1. Thus $q^{\perp} \cap Q$ can only contain the point p.

3. Suppose S is a symplecton on p distinct from Q. Then $p^{\perp} \cap S$ consists of the singular subspace $S \cap Q$, which at least contains a line on p, and the unique line L_2 (Part 1) on p which is not in Q. Precisely, the collinearity graph induced on $p^{\perp} \cap S$ is the union of two cliques. It follows that S is a grid.

4. We define the mapping $\beta : \mathscr{L}_q - \{L_2\} \to \mathscr{L}_p - \{L_2\}$ as follows. For each line L on q, there is a symplecton R on $L \cup L_2$ (Corollary 24). By Part 3, R is a grid, so L is not in $L_2^{\perp}$ and so R is the unique symplecton on L and L_2. Thus, using Theorem 25, $R \cap Q$ is a line $\beta(L)$ uniquely determined by L. Note that L and $\beta(L)$ are opposite lines of the grid R, and so the desired bijection $L \to \beta(L)$ exists.

Suppose now that $\beta(L) = \beta(L')$ and let R and R' be the unique grids on $L \cup L_2$ and $L' \cup L_2$ as in the previous paragraph. Then R and R' are the unique symplecta on $\beta(L) \cup L_2$, forcing $R = R'$ and $L = L'$. Thus β is injective.

If N is any line of $\mathscr{L}_p - \{L_2\}$, then by Part 1, N is not in $L_2^{\perp}$ and the unique symplecton T on $N \cup L_2$ is a grid by Part 3. Then $N = \beta(N')$ where N' is the unique line of T on q which is distinct from L_2. Thus β is onto. The proof of 4 is complete.

Lemma 33. *Again let G, Q, L_1 be as in Lemma 32. Every line of Q lies in exactly one further symplecton which is a grid. Consequently, every point of Q is incident with exactly one line which is not in Q.*

Proof. Choose point p in L_1 and let L_2 be the other line of G on p as in Lemma 32. Let L be any line of Q. If L is incident with p then the unique grid containing $L \cup L_2$ is the only symplecton on L besides Q. Suppose then L is not incident with p. Then there is a point r in L collinear with p. Without loss of generality, L_1 can be taken to be the line on p and r and G to be the grid on $L_2 \cup L_1$. Then r enjoys the same hypotheses that p did in Lemma 32. So there is a unique line L_3 on r which is not in Q. Then the symplecta on L are Q and the unique symplecton on $L \cup L_3$, which, by Part 3 of Lemma 32, is a grid.

The uniqueness of the out-going lines follows.

Now we can complete the proof of Theorem 30. Suppose the polarized space Γ contains a symplecton which is a grid and one which is not. Since the point-collinearity graph is connected and every line of Γ lies in at least one symplecton (Corollary 24 for example), there must be an instance in which a symplecton G which is a grid intersects a symplecton Q which is not a grid at some point. Then by Theorem 25, $G \cap Q$ is a line L_1.

Now choose a point p on L_1, let L_2 be the unique second line of G on point p and choose point q in $L_2 - \{p\}$. We now have the situation of Lemma 32.

We know that we can choose at least two distinct lines, N_1 and N_2, in $\mathscr{L}_q - \{L_2\}$. Let Q_q be any symplecton containing $N_1 \cup N_2$. We make three claims:

1. $Q_q \cap Q = \emptyset$.
2. The symplecton Q_q is isomorphic to Q, and so is not a grid.
3. Q_q contains all lines on q except L_2.

First suppose Q_q and Q had a nonempty intersection. Then by Lemma 33, Q_q would be a grid intersecting Q in a line L. Moreover, since $q^\perp \cap L$ is nonempty, L must contain p, the unique point of $q^\perp \cap Q$ (Lemma 32, Part 2). It follows that Q_q contains L_2, and so L_2, N_1 and N_2 comprise three distinct lines of Q_q on q. That is impossible since Q_q was a grid in this case division.

Now by our basic hypothesis on Γ, each point of Q_q is collinear with at least one point of Q, and, since $Q_q \cap Q = \emptyset$ and each point of Q lives on only one out-going line, this point must be unique. Thus there is an injective mapping $\phi : Q_q \to Q$ taking each point of Q_q to the unique point of Q with which it is collinear. But also by the fundamental hypothesis, each point of Q is collinear with at least one point of Q_q and so ϕ is a bijection. Using the presence of the unique system of interlocking grids, it is easy to see that ϕ and ϕ^{-1} both preserve the collinearity relation on points. Thus ϕ induces a bijection β of the lines of Q_q with those of Q, extending the bijection β of Lemma 32 Part 4. This establishes the second and third claims.

Now set $Q = Q_p$, and for each point x of symplecton Q, let L_x be the unique line on x not in Q. (In this notation, L_2 is now L_p.) From what we have established, each of these "out-going" lines L_x meets each Q_q at a single point, each point in any Q_y lies on a unique one of these L_x's, and has all its remaining lines in Q_y. It follows that the union of the disjoint Q_y's, as y ranges over the points of L_p, is a connected component of the collinearity graph of Γ and hence covers all of the points. Thus every point of $\mathscr{P}$ can be coordinatized as (x, y) where the point x of $Q = Q_p$, indexes the unique line L_x connecting it to Q (or is the point itself, if it already is in $Q = Q_p$), while the coordinate y is the point of L_p which indexes the unique Q_y in which the point lies. All lines are now either the "horizontal" lines of one of the symplecta Q_y which partition the points, or one of the "vertical" lines L_x, $x \in Q_p$. Thus we have a product geometry $L_p \times Q_p$. The proof is complete.

Remark. Note that in this case it is possible for the vertical lines to possess a different cardinality than that for the horizontal lines.

Theorem 34. *If all symplecta are generalized quadrangles*, Γ *is a dual polar space of rank three or the product geometry* $L \times Q$ *of a line L and a generalized quadrangle Q*. (Of course if Q is itself a grid, Γ *is just the* "*Hamming cube*"*—that is*, *the product of three lines* $L_1 \times L_2 \times L_3$, *where three line cardinalities are possible.*) *In each case*, Γ *is a near hexagon of classical type*.

Proof. By [13] it suffices to show that Γ is a near hexagon with all quads classical. If there were a plane in Γ, by Theorem 22, it would lie in some symplecton. But that would be impossible since each symplecton is a generalized quadrangle. So there are no planes. Yet, by hypothesis, for each non-incident point-symplecton pair (p, Q), the intersection $p^{\perp} \cap Q$ is not empty. Since there are no planes, the intersection $p^{\perp} \cap Q$ is always a single point. By Theorem 26, the symplecton Q is strongly gated with respect to x. Thus we see that any symplecton of Γ is a quadrangle with the property that it is strongly gated with respect to every exterior point, and that every such point is collinear with exactly one of its points. This makes Γ a near hexagon of classical type.

Theorem 35. *If Γ contains no grids, and at least one symplecton has rank at least three, then all symplecta have rank at least three and the conclusion of Theorem* 29 *holds.*

We first prove the following technical lemma:

Lemma 36. *Suppose Γ contains no grids. Suppose (a_1, x, y) and (a_2, x, y) are two geodesics, $i = 1, 2$. Then there is a point b in $y^{\perp}$ which is simultaneously distance three from both a_1 and a_2.*

Proof. By Theorem 21 there is a point b_i such that (a_i, x, y, b_i) is a geodesic of length three, for $i = 1, 2$. If $b_1 = b_2$ we are done, so assume the b_i are distinct. By either the strong parapolar hypothesis or Theorem 22, there is a symplecton R on $\{b_1, y, b_2\}$. Now by Theorem 20, for $i = 1, 2$, there exist points r_i such that

$$a_i^{\perp} \cap R = \{r_i\} \quad \text{and} \quad \Delta_2^*(a_i) \cap R = r_i^{\perp} \cap R.$$

It may happen that $r_1 = r_2$, but in any case, both are distinct from y since $d(a_i, y) = 2$, for both values of i. In any case, the set $\mathscr{L}_i$ of lines of R on point y which lie in $\Delta_2^*(a_i)$ are just those in $(r_i y)^{\perp}$.

If R has rank at least three, the lines and planes of R on y form a polar space with thick lines, $\mathrm{Res}(y) \cap R$, of which the two sets $\mathscr{L}_i$ form hyperplanes. Since no polar space with thick lines is the union of two hyperplanes, there is a line yb in $\mathrm{Res}(y) \cap R$ which is in neither of these two hyperplanes.

If, on the other hand R is a generalized quadrangle, one has $(r_i y)^{\perp} \cap R = r_i y$. In this case, since R is not a grid, there exists a line yb not in either $\Delta_2^*(a_i) \cap R$.

In all cases $d(b, a_i) = 3$, $i = 1, 2$, as required.

Proof of Theorem 35. Under the hypothesis of no grids we shall show that any symplecton which intersects a symplecton of rank at least three non-trivially must itself have rank at least three. It will then follow from the connectedness, that all symplecta have rank at least three.

So suppose S_1 is a symplecton of rank at least three and S_2 is a second symplecton intersecting S_1 non-trivially. Our objective is to show that S_2 has rank at least three. By Theorem 25, $S_1 \cap S_2$ contains a line. If the intersection contains a plane, S_1 has rank at least three and we are done. So we may assume that $S_1 \cap S_2$ is a exactly a line L. Choose distinct points x and y on line L, and points a_i in $x^{\perp} \cap S_i - y^{\perp}$, $i = 1, 2$. Then (a_i, x, y) are geodesics of length two which, by the previous lemma, can be extended in a common way to geodesics (a_i, x, y, b), $i = 1, 2$. Note that these metric requirements show that b cannot be in either S_1 nor S_2.

Now by hypothesis there is a plane π in S_1 on line L, and so there is a point z in $\pi - L$. Now if b were collinear with z, we would have $b^\perp \cap S_1$ containing the line yz meeting $a_1^\perp$, against $d(b, a_1) = 3$. Thus $d(b, z) = 2$, and as Γ is a strong parapolar space, there is a symplecton S on $\{z, y, b\}$. Then $S \cap S_2$ contains a line M.

Obviously, as b is distance three from a_2, $S \cap S_2$ cannot contain a plane. Thus $S \cap S_2 = M$ exactly.

Now the unique point r on $a_2^\perp \cap M$ is distinct from y, and by Theorem 20 is the unique point of $a_2^\perp \cap S$ (the gate) and

$$\Delta_2^*(a_2) \cap S = r^\perp \cap S.$$

Why can't $M = L$, so that $r = x$? In that case $S \cap S_1$ contains the plane $\langle L, z\rangle = \pi$ forcing $d(b, a_1) = 2$, a contradiction. Thus M is not in S_1, for otherwise $M \subseteq S_2 \cap S_1 = L$. Thus $M \cap S_1 = \{y\}$, and r is not in S_1.

Since M is a thick line, there is a point s in M distinct from r and y. Moreover, since S is a non-degenerate polar space, there is a point b' in S collinear only with the point s of M. Since b' is not in $r^\perp$, we have that $d(a_2, b') = 3$. Thus we see that

$$S_2 \cap \Delta_2^*(b') = S_2 \cap s^\perp. \qquad (5)$$

Now $x^\perp \cap S$ contains the line yz and so $d(b', x) = 2$. It follows from Equation (5) that s is collinear with x. We now see that the clique $\{s, x, y\}$ spans a plane in S_2, and so S_2 must have rank at least three. The proof is complete.

6 Proof of Theorem 1

In this section, Γ is a parapolar space of polar rank at least three satisfying these hypotheses:

(A1) Given a point x not incident with a symplecton S, the space $x^\perp \cap S$ is never just a point.

(A2) Given a projective plane π and line L meeting π at point p, either (i) every line of π on p shares a symplecton with L, or else (ii) exactly one such line incident with (p, π) has this property.

(A3) If L is a line on point p, then there exists at least one further line N on point p such that

$$L^\perp \cap N^\perp = \{p\}.$$

(A4) If all symplecta have rank at least four, then every maximal singular subspace has finite projective rank.

Remark. Note that axiom (A3) prevents Γ from being a strong parapolar space.

6.1 Simple-connectedness of the point-collinearity graph of Γ. It is useful at the beginning to show that the axioms (A1), (A2) and (A3) alone force the point-collinearity graph $G = (\mathscr{P}, \sim)$ of Γ to be simply $\mathscr{T}$-connected—that is, every cycle of the graph can be deformed to a single point by some iteration of the processes of either replacing an edge representing one side of a triangle by the other two sides, or the reverse of this process.

We define the *angle* between lines L_1 and L_2 of $\mathscr{L}_p$ as the integer $\alpha(L_1, L_2)$ where

$$\alpha(L_1, L_2) = \begin{cases} 0 & \text{if } L_1 = L_2 \\ 1 & \text{if } L_1 \neq L_2 \text{ but both are in a plane} \\ 2 & \text{if } L_1 \neq L_2 \not\subseteq L_1^{\perp} \text{ but both lie in a symplecton} \\ 3 & \text{if } L_1 \text{ and } L_2 \text{ lie together in no symplecton.} \end{cases}$$

(Note that in the last case, if $y_i \in L_i - \{p\}$, then $y_1^{\perp} \cap y_2^{\perp} = \{p\}$, that is (y_1, y_2) is a special pair.)

Suppose now $w = (x_0, x_1, \dots, x_n)$ is a path in the point-collinearity graph G. Let L_i be the unique line on $\{x_i, x_{i+1}\}$, and let α_{i+1} be the angle between L_i and L_{i+1}. Then the sequence $(\alpha_1, \alpha_2, \dots, \alpha_n)$ is the *angle type* of the path w.

Theorem 37. *The point-collinearity graph $\Gamma = (\mathscr{P}, \sim)$ is simply connected.*

Proof. We must show that every circuit in G is $\mathscr{T}$-contractible where $\mathscr{T}$ is the collection of all triangles of G. Assume $c = (x_0, x_1, \dots, x_n = x_0)$ is a circuit of minimal length n subject to being non-contractible. We can assume $n > 3$, and since any 4-circuit lies in a symplecton (whose circuits are easily seen to be contractible), we can assume $n > 4$ as well.

We can also assume that x_i is not collinear with x_j for $n - 1 > |i - j| > 1$. If R were a symplecton on $\{x_1, x_2, x_3\}$, then $x_0^{\perp} \cap R$ would contain a line L, and $x_3^{\perp} \cap L$ would contain a point z. Then c would decompose as a circuit $c' = (x_0, z, x_3, \dots, x_n = x_0)$ of length $n - 1$, a triangle, and a circuit of length 4, all of which are contractible by the conditions on n. Since this would make c contractible, no such symplecta can lie on $\{x_1, x_3\}$—or on $\{x_i, x_{i+2}\}$ for that matter. Thus c has angle type $(3, 3, \dots, 3)$.

Now choose a plane π on $\{x_1, x_2\}$. Now by axiom (A2) there is a line L of π on x_1 such that L and line x_0x_1 lie in a symplecton R. A second application of (A2) similarly produces a line N of π on x_2, sharing a symplecton S with line x_2x_3. Since both L and N are lines of π they must intersect at a point z. Now by (A1), $x_4^{\perp} \cap S$ contains a line B on x_3, which by the polar space property for S, bears a point b of $z^{\perp}$. Similarly $x_{n-1}^{\perp} \cap R$ contains a line A on x_0, which in turn also bears a point a of $z^{\perp}$ as $z \in R$. (Note that if $n = 5$ then $x_4 = x_{n-1}$; but this doesn't hurt anything.) Now it is clear that c decomposes into a circuit

$$c' = (a, z, b, x_4, \dots, x_{n-1}, a)$$

of length $n - 1$, two circuits (x_0, a, z, x_1, x_0) and (x_3, b, z, x_2, x_3) of length at most

4 and three triangles $(x_{n-1}, x_0, a, x_{n-1})$, (x_1, z, x_2, x_1) (in π), and (x_4, b, x_3, x_4), all of which are contractible. Hence c is contractible.

Thus no non-contractible circuits exist, and the theorem is proved.

6.2 The uniform structure of the point-residues. As usual, we let $\mathscr{P}, \mathscr{L}, \Pi$ and $\mathscr{S}$ be the set of points, lines, planes and symplecta of Γ. Then the sets $\mathscr{L}_p, \Pi_p$ and $\mathscr{S}_p$ are the lines, planes and symplecta incident with a point p.

Recall that for each point p of Γ, the geometry $\mathrm{Res}(p) := (\mathscr{L}_p, \Pi_p)$ of lines and planes on p is a "point"-"line" geometry which is a strong parapolar space with all singular subspaces projective, whose "symplecta" are the lines and planes incident with a flag $(p, S) \in \mathscr{P} \times \mathscr{S}$. By (A1), $\mathrm{Res}_\Gamma(p)$ satisfies the property that each "point" is collinear with at least one point of any "symplecton" which does not contain it. Similarly (A2), (A3) and (A4) force $\mathrm{Res}_\Gamma(p)$ to satisfy the rest of the hypotheses of Theorem 2. It follows that $\mathrm{Res}_\Gamma(p)$ is isomorphic to one of the geometries appearing in Theorem 2. But is it the same geometry for each point p?

Suppose $L = pq$ is a line. Then both of the geometries $\mathrm{Res}(p)$ and $\mathrm{Res}(q)$ look the same above a "point" L. Thus when $\mathrm{Res}(p)$ is (a) $D_{6,6}$, (b) $A_{5,3}$, (c) $E_{7,1}$, (d) a dual polar space of rank three, or (e) the product $L \times P$ of a line and a polar space, the subgeometries of symps and singular spaces containing L are respectively, (a') $A_{5,2}$, a Grassmannian, (b') $A_2 \times A_2$, the product geometry of two planes, (c') the exceptional geometry $E_{6,1}$, (d') a projective plane A_2 of lines and symps, or (e') the disconnected union of a point and the point-residue of a symp. So each case is distinctive. That means that $\mathrm{Res}(p)$ is of the same type and defined by the same parameters as $\mathrm{Res}(q)$, except possibly in the last case. (If the line pq is the isolated "point" of each point-residue, the symplecta forming a bouquet over p might conceivably be of a different isomorphism type than the symplecta forming a bouquet over q. But even here, the uniformity holds and is discussed fully in Lemma 40 of the next section.)

So we have:

Lemma 38. *For any two points of* Γ, *the point-residue geometries are uniformly isomorphic.*

From this point onward (with a minor abuse of notation) we regard Γ as a higher rank "enriched" geometry over a typeset J singling out points, lines, symplecta, and all singular subspaces. Some of these isomorphism types are sorted into further classes by the nature of the uniform point-residues (See 6.4 for details).

6.3 The case of finite singular rank. Assume now that every maximal singular subspace of Γ has finite projective rank.

It follows from Lemma 38 and Theorem 1 that Γ is a geometry belonging to one of the following locally truncated diagrams over I (the set J which is the recipient of the truncation is indicated by the round nodes in the figure below).

However, as remarked at the beginning of Section 3.4, in order to define a sheaf we must be sure that one can define types to the objects one sees in a point-residue. Any fusion of types would require an automorphism of the locally truncated diagram

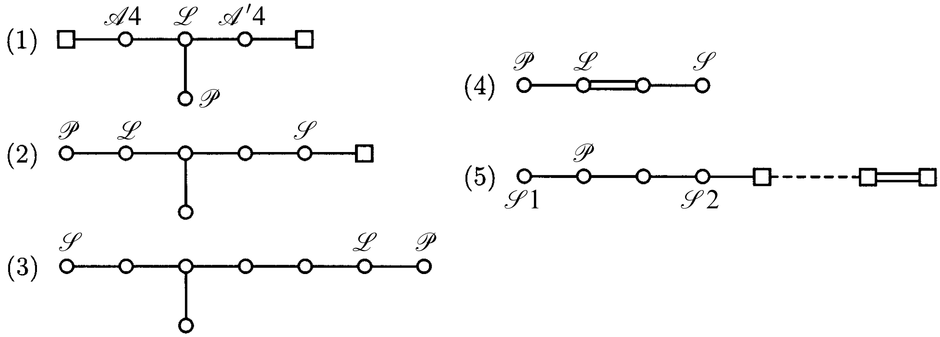


Figure 2

fixing the node "$\mathscr{P}$", and this is only possible in the case that the enriched Res(p) is a truncation of $A_{5,3}$ as in (1) of Figure 2. Here it is conceivable that the two local classes of PG(4) (denoted $\mathscr{A}4$ and $\mathscr{A}'4$) are fused in the global geometry Γ. If so, one can invoke Theorem 12 (with $Y = \hat{\Gamma}$) to conclude that there is a geometry morphism $\gamma : \hat{\Gamma} \to \Gamma$ such that (i) the point-residues of $\hat{\Gamma}$ are mapped isomorphically onto the point-residues of Γ so that $\hat{\Gamma}$ also belongs to the locally truncated diagram (1) of Figure 2, and (ii) in $\hat{\Gamma}$ the two classes $\mathscr{A}4$ and $\mathscr{A}'4$ are not fused. But Theorem 12 also asserts that γ induces a $\mathscr{T}$-covering $\delta = \delta|_Y$ of the point-collinearity graph of the Γ, and by Theorem 37 that graph covering is an isomorphism. Thus γ induces a bijective mapping on points and since all other objects of $\hat{\Gamma}$ are uniquely determined by their point-shadows, γ is an isomorphism of geometries. Thus from property (ii) above one can assign a distinct type to the objects of the locally truncated diagram for Γ just as in all the other cases depicted in Figure 2. Thus, because there is an unambiguous assignment of types, Γ is a locally truncated diagram geometry with respect to the diagram D.

Now, by Theorem 10, there exists a sheaf, and in each case the ambient diagram is a Dynkin diagram, there exists a building geometry Δ over I and a vertex-surjective morphism $h : \Delta \to \bar{\Delta}$ of geometries such that Γ is isomorphic to $\bar{\Delta}$ truncated to the typeset J: the typeset of Γ which includes $\{\mathscr{P}, \mathscr{L}, \mathscr{S}\}$ and all singular subspaces. Thus h induces a morphism

$$f : \Delta_J \to \Gamma$$

as in Lemma 16.

Now if we truncate to $\{\mathscr{P}, \mathscr{L}\}$ we recover the original parapolar space Γ (sans enrichment) and a truncation of a building geometry $(\mathscr{P}_\Delta, \mathscr{L}_\Delta)$, and both of these point-line geometries are parapolar spaces of polar rank at least three. In each case, let Π and Π_Δ be the full sets of projective planes in these respective geometries. Then Hypothesis 1 of Theorem 17 holds, just from our choice of Π. Hypotheses 2 and 4 hold because they are parapolar spaces. We need to check Hypothesis 3 only when X is a line, having already established that the five cases listed above for point-residues are truncations of rank at least three of buildings. But the same conclusions hold for a line L. This is because the lines are a set of flags which isolate the points $\mathscr{P}$ from all other nodes of the diagram. Thus the sheaf-value $\mathscr{F}(L)$ at a line L is a geometry belonging to a diagram of type

$$A_1 \times Y,$$

where Y is the diagram of a line-residue in the building $\mathscr{F}(p)$ for a point p. Since the latter is a building, so is its residue Y. Thus Condition 3 of Theorem 17 is verified.

Now Theorem 17 applies to show that

$(*)$ *The morphism f induces a graph morphism*

$$(\mathscr{P}_\Delta, \sim) \to (\mathscr{P}_\Gamma, \sim)$$

of the point-collinearity graph of Δ onto the point-collinearity graph of Γ, which is a $\mathscr{T}$-covering of graphs.

Now by Theorem 37 the latter graph is simply connected, so this graph morphism is actually an isomorphism. This means f induces a bijection on points. Now the fact that all objects of the building geometry Δ_J are distinguished by their point-shadows forces the morphism $f : \Delta_J \to \Gamma$ to be an isomorphism of parapolar spaces, and completes the proof of Theorem 1 when all maximal singular subspaces possess finite projective rank.

6.4 The case of infinite singular rank. The hypotheses (A1)–(A4) show that if Γ possesses a singular subspace of infinite projective rank, then our point-residue $\mathrm{Res}_\Gamma(p)$ is the product geometry $L \times P$ where P is a polar space.

Here we shall take this structure of P of $\mathrm{Res}_\Gamma(p)$ as a hypothesis, where P is any polar space that is not a grid. Of course that means we are reproving some of the finite singular rank cases over again, but this time without resorting to the theory of locally truncated geometries. All the better!

We first require a general theorem:

Theorem 39. *Suppose Γ is a parapolar space satisfying the hypothesis.*

(A1) *If x is a point, and S is a symplecton, then $x^\perp \cap S$ is never a single point.*

Then the point-collinearity graph $G := (\mathscr{P}, \sim)$ has diameter at most three.

Proof. It suffices to show that G possesses no geodesic of length four. So by way of contradiction assume $g = (x_0, x_1, x_2, x_3, x_4)$ is a geodesic of length four and angle type $(\alpha_1, \alpha_2, \alpha_3)$. Thus each α_i is at least two.

Case 1: *One of the* $\alpha_i = 2$. Suppose first that $\alpha_1 = 2$. Then there is a symplecton R_0 on $\{x_0, x_1, x_2\}$ and $x_3^\perp \cap R_0$ contains a line N. Then, as R_0 is a polar space, there is a point $z \in x_0^\perp \cap N$, and we have a path (x_0, z, x_3, x_4) of length 3, a contradiction.

Thus $\alpha_1 \neq 2$ and by symmetry, $\alpha_3 \neq 2$.

Next suppose $\alpha_2 = 2$. Then there is a symplecton R_1 containing $\{x_1, x_2, x_3\}$, a line

$M = x_4^\perp \cap R_1$ and a point $y \in x_1^\perp \cap M$. Then (x_0, x_1, y, x_4) is a path of length three connecting x_0 and x_4, a contradiction.

Case 2. *g has angle type* $(3,3,3)$. First choose a symplecton S_1 containing the edge $\{x_1, x_2\}$. Then $x_3^\perp \cap S_1$ contains a line M. Now, since S_1 has polar rank at least three, $M^\perp \cap S_1$ is not a clique, while $x_3^\perp \cap S_1$ is. Thus there exists a point $u \in S_1 \cap M^\perp - x_3^\perp$ and so the polar pair (u, x_3) lies in a symplecton S_2 containing $\{u, x_2, x_3\}$ meeting S_1 in at least the plane $\pi = \langle u, M \rangle$. Now $x_0^\perp \cap S_1$ contains a line N_1 on x_1, and in the polar space S_1, $N_1^\perp \cap \pi$ must contain a point z. Thus (x_0, z) is a polar pair. Now also (A1) forces $x_4^\perp \cap S_2$ to contain a line N_4 which meets $z^\perp$ in at least a point q. Now $w = (x_0, x_1, z, q, x_4)$ is a path connecting x_0 and x_4, of angle type $(\beta_1, \beta_2, \beta_3)$ with $\beta_1 = 2$. But that returns us to Case 1 with w replacing g. We have seen that that case leads to a contradiction.

Thus no such geodesic of length four exists and the proof is complete.

We are now operating under this hypothesis,

(B1) *Every point-residual geometry* $\mathrm{Res}(p) = (\mathscr{L}_p, \Pi_p)$ *is one of the conclusion geometries of Theorem* 2.

(B2) *There exists a point p for which* $\mathrm{Res}(p)$ *is a geometry* $L_p \times P_p$ *where* L_p *is a line and* P_p *is a polar space of rank* $k_p \geqslant 2$ *that is not a grid.*

In Condition (B2) the projective line L_p is thick since we are dealing with the point residue of a parapolar space with thick lines.

Now suppose $L = pq$ is a line on p. Then L lies in a unique symplecton P of Γ which has rank $k_p + 1$, which is not oriflamme of rank three. It also lies in a unique plane lying in no such symplecton. Since q is on such a line, and yields a residue in the conclusion of Theorem 2, it has a residue of shape $L_q \times P_q$, where $P_q \simeq P_p$ (being point residue geometries of distinct points of the same symplecton P) and has symplectic rank k_p, and L_q has the same cardinality as L_p. Thus we see that $\mathrm{Res}(q) \simeq \mathrm{Res}(p)$.

Lemma 40. Γ *is a parapolar space with all point-residues isomorphic to* $L \times P$ *where* L *has a constant cardinality, and* P *has a uniform polar rank* k *at least two. If* k *is greater than two, then* P *even has a constant isomorphism type.*

Remark. The last sentence follows from an unpublished theorem of Tits. We don't actually use this fact.

Lemma 41. *The following statements hold*:

1. *For any symplecton* S *and point* x *not in* S, $x^\perp \cap S$ *is empty, or is a line.*
2. *The symplecta are partitioned naturally into two sets as* $\mathscr{S} = \mathscr{D} + \mathscr{S}^+$, *where* $\mathscr{D}$ *is the collection of all oriflamme rank-three polar spaces, and* $\mathscr{S}^+$ *are the remaining symplecta* (*all of rank* k). *We have the following*:

   (a) *Any two distinct members of* $\mathscr{S}^+$ *intersect at the empty set or at a single point.*

   (b) *If* $(D, S) \in \mathscr{D} \times \mathscr{S}^+$, *then* $D \cap S$ *is the empty set or a plane.*

3. *The projective planes* Π *of* Γ *are also partitioned into two sets*:
   (a) *The* $\mathscr{S}^+$*-planes, which are the planes which lie in a (necessarily unique) member of* $\mathscr{S}^+$.
   (b) *The* $\mathscr{D}$*-planes, which are those planes which lie in no member of* $\mathscr{S}^+$ *at all. These are maximal singular subspaces of* Γ.
   (c) *Every line lies in a unique member of* $\mathscr{S}^+$ *and in a unique* $\mathscr{D}$*-plane.*
4. *If* $D \in \mathscr{D}$, *then the two oriflamme classes of planes of* D *are the* $\mathscr{D}$*-planes and the* $\mathscr{S}^+$*-planes which are contained in* D. *In particular, any two distinct* $\mathscr{S}^+$*-planes of* D *intersect at a single point.*

Proof. The symplecta in $\mathscr{D}$ can never be isomorphic to those in $\mathscr{S}^+$, so the two classes of symplecta can never fuse globally. All of the statements follow from the uniform local structure of any point-residue. For example, the $\mathscr{D}$-planes and $\mathscr{S}^+$-planes correspond to the "horizontal" and "vertical" lines respectively in the product geometry $L \times P$ representing a point-residue.

Corollary 42. *If two symplecta from* $\mathscr{S}^+$ *both intersect a common symplecton from* $\mathscr{D}$ *non-trivially, then the two symplecta either coincide or intersect at a point.*

Proof. Suppose S_1 and S_2 are distinct members of $\mathscr{S}^+$, which intersect non-trivially a symplecton $D \in \mathscr{D}$. Then by Lemma 41 2(b), the intersections $S_i \cap D$ are planes which belong to the same oriflamme class of D by Part 4 of Lemma 41. Since the planes meet at a point, so do S_1 and S_2.

Lemma 43. *Suppose* S_1, S_2 *and* S_3 *are pairwise distinct members of* $\mathscr{S}^+$ *on a common point* p. *Suppose* R *is a member of* $\mathscr{S}^+$ *which does not contain* p. *If* R *intersects* S_1 *and* S_2 *non-trivially, then it intersects* S_3 *non-trivially.*

Proof. Let $\{x_i\} := S_i \cap R$, $i = 1, 2$. Let's get rid of an easy case first. Suppose p were collinear with one of the x_i, say x_1. Then $p^\perp \cap R$ is a line L on x_1. Since L is not in S_1 (for $S_1 \cap R = \{x_1\}$), the plane $\langle p, L\rangle$ has to be the unique $\mathscr{D}$-plane on px_1. Then $\langle p, L\rangle$ intersects every symplecton of $\mathscr{S}^+$ on p at a line. Therefore L intersects every symplecton of $\mathscr{S}^+$ on p at a point. Thus there is a point in $L \cap S_3 \subseteq R \cap S_3$. Thus $R \cap S_3$ is non-empty.

So we may assume that $d(p, x_1) = d(p, x_2) = 2$ in the point-collinearity graph. If x_1 were collinear with x_2 then $x_1^\perp \cap S_2$ would be a line on x_2 carrying a point u of $p^\perp \cap S_2$. Then $u \in p^\perp \cap x_2^\perp \subseteq S_2$, while $u \in p^\perp \cap x_1^\perp \subseteq S_1$. This is impossible as $S_1 \cap S_2 = \{p\}$.

Thus x_1 is not collinear with x_2. Select a point $t \in x_1^\perp \cap x_2^\perp \subseteq R$. Then $t^\perp \cap S_i$ carries a point s_i of $p^\perp$, and $s_1 \neq s_2$. Then there is a symplecton D on p and t, and $D \in \mathscr{D}$ since it meets S_1 and S_2 at lines, at least. Once again, $D \cap R$ and $D \cap S_3$ are non-empty, so Corollary 42 can be invoked to yield $R \cap S_3 \neq \varnothing$ in this case as well.

Lemma 44. *If p is a point and $R \in \mathcal{S}^+$ is a symplecton not on p, then there is a symplecton $S \in \mathcal{S}^+$ containing p and meeting R non-trivially.*

Proof. First suppose p is collinear with a point x of R. By Lemma 41 (3)(c) there is an element $S \in \mathcal{S}^+$ on the line px. Then $S \cap R \neq \emptyset$ and we are done.

Next suppose $d(p,r) = 2$ for some point $r \in R$. Choose $z \in p^\perp \cap r^\perp$. Then $z^\perp \cap R$ is a line L of R. Choose a point $y \in R \cap L^\perp - L$ (this is possible since R has polar rank at least three). Then there is a symplecton $D := \ll z, y \gg$ in $\mathcal{D}$. Now by Lemma 41, Part (3)(c), the line pz lies in a symplecton S of $\mathcal{S}^+$. But now $S \cap D$ and $R \cap D$ are both non-trivial, so $S \cap R \neq \emptyset$ by Corollary 42.

So we must assume $d(p,r) \geqslant 3$ for all points r in R. But by Theorem 39, we must assume

$$R \subseteq \Delta_3(p). \tag{6}$$

Let (p,u,v,r) be a geodesic from p to a point r of R. Then there is a symplecton D of $\mathcal{D}$ on line vr and meeting R at a plane (Lemma 41 3(c) and 2(b)). Since Equation (6) implies $u^\perp \cap R = \emptyset$, one has $u \notin D$. Then there is an element S of $\mathcal{S}^+$ on the line uv (Lemma 41 3(c) again). Now as S and R are elements of $\mathcal{S}^+$ meeting D non-trivially, there exists a point $s \in S \cap R$, by Corollary 42. But $p^\perp \cap S$ is a line on u carrying a point of $s^\perp$. Thus $d(p,s) = 2$ against $R \subseteq \Delta_3(p)$. The proof is complete.

Lemma 45. *There is no symplecton in $\mathcal{S}^+$ which intersects non-trivially all other symplecta from $\mathcal{S}^+$.*

Proof. Suppose $R \in \mathcal{S}^+$ has the "radical" property—that $R \cap S \neq \emptyset$ for all $S \in \mathcal{S}^+$.

Fix a point p in R, and a point $r \in R - p^\perp$. Let T be any symplecton of $\mathcal{S}^+$ on r with $T \neq R$. Then $T \cap R = \{r\}$. Choose $z \in T - r^\perp$. We claim that z is not collinear with any point of R. For if there were such a point, then $z^\perp \cap R \cap r^\perp$ would contain a point in $R \cap T - \{r\}$, an absurdity. Thus

$$z^\perp \cap R = \emptyset.$$

By assumption all symplecta of $\mathcal{S}^+$ which lie on z must meet R. Let S_1 and S_2 be two of these—that is $S_1 \neq S_2$ not in name only. Set $\{x_i\} := S_i \cap R$. Since $z^\perp \cap R = \emptyset$, z is distinct from both x_1 and x_2. If x_1 were collinear with x_2, then, by Lemma 41 (1), $x_1 \cap S_2$ contains a line carrying a point w of $z^\perp$. Then by convexity of S_1, the line zw is in S_1 as well as S_2, against Lemma 41 (2a). Thus we may assume $d(x_1, x_2) = 2$.

Now select a point $t \in x_1^\perp \cap x_2^\perp \subseteq R$. Set $t^\perp \cap S_i := N_i$, and set $\{y_i\} := z^\perp \cap N_i$, $i = 1, 2$. Then $t^\perp \cap z^\perp$ contains two distinct points y_1 and y_2, so the convex closure $D := \ll t, z \gg$ is a symplecton whose intersection with each S_i contains a line zy_i. Thus $D \in \mathcal{D}$. Then $D \cap R$ is a plane π (Lemma 41 2(b)), and so $z^\perp \cap \pi$ is a line in $z^\perp \cap R$. But by construction the latter set is empty. Thus no such R exists and the proof is complete.

Theorem 46. *Under the hypotheses of Lemma* 40, *the rank-two geometry* $P^* := (\mathscr{S}^+, \mathscr{P})$ *is a nondegenerate polar space.*

Proof. By Lemmas 43 and 44 P^* is a polar space. It is nondegenerate by Lemma 45.

Corollary 47. *Under the hypotheses* (B1) *and* (B2) *which headed this subsection*, Γ *is the polar Grassmannian of lines of a non-degenerate polar space* P^* *of rank at least four. Otherwise the nature of* P^* *is arbitrary.*

Theorem 1 has now been proved in the two cases in which all singular subspaces have finite projective rank and otherwise.

Remark. By now the reader has noticed how the phantom hypotheses of finite singular rank weave in and out of the two theorems. How they weave in is as dramatic as how they weave out. The authors do not have a good explanation for this. To a large extent the assumption is in when the proof wishes to invoke one of the following: (1) the Cohen–Cooperstein theorem in some form (for example, Theorem 15, though free of the assumption of constant symplectic rank, uses the Cohen–Cooperstein Theorem) or (2) the theory of locally truncated geometries—while still struggling to maturity—requires Tits' "Local Approach Theorem" in order to say anything useful. Indeed, it would not be unfair to say that this theory is the *major application* of that beautiful theorem. But its application requires the covering chamber system C of $C(\mathscr{F})$ to be residually connected so that the functor $\mathbf{\Gamma}$ can be applied. That in turn requires a finite Coxeter matrix, and that means finite rank.

Theorem 2 uses the Cohen–Cooperstein theory only under a very special low-diameter circumstance in which $x^\perp \cap S$ is always non-empty. If it were possible to prove finite singular rank in such a special case, one could dispense with the finite singular rank assumptions altogether in both Theorems 1 and 2. It is worth mentioning.

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A. Kasikova, Department of Mathematics, University of California at Santa Cruz, Santa Cruz, CA, 95064, U.S.A.
Email: kasikova@math.ksu.edu

E. Shult, Department of Mathematics, Kansas State University, Manhattan, KS, 66502, U.S.A.
Email: shult@math.ksu.edu