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Minkowski versus Euclidean rank for products of metric spaces

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Abstract. We introduce a notion of the Euclidean and the Minkowski rank for arbitrary metric spaces and we study their behaviour with respect to products. We show that the Minkowski rank is additive with respect to metric products, while additivity of the Euclidean rank does not hold in general.

1 Introduction

For Riemannian manifolds there are various definitions of a rank in the literature (compare e.g. [1], [6], [3]). A notion which can easily be generalized to arbitrary metric spaces is the rank as the maximal dimension of a Euclidean subspace isometrically embedded into the manifold.

It is known that for Riemannian manifolds this *Euclidean rank* is additive with respect to products. This is not the case for more general metric spaces, even for Finsler manifolds (see Theorem 3 below).

In contrary it turns out that the *Minkowski rank* defined as the maximal dimension of an isometrically embedded normed vector space has a better functional behaviour with respect to metric products.

Definition. For an arbitrary metric space (X,d) the *Minkowski rank* is

$$\operatorname{rank}_M(X,d) := \sup_{(V,\|\cdot\|)} \{\dim V \mid \exists \text{ isometric map } i_V : (V,\|\cdot\|) \to (X,d)\}.$$

The *Euclidean rank* is defined as

$$\operatorname{rank}_E(X,d) := \sup\{n \in \mathbb{N} \mid \exists \text{ isometric map } i_{\mathbb{E}^n} : \mathbb{E}^n \to (X,d)\}.$$

In special cases, e.g. for Riemannian manifolds, these rank definitions coincide,

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since normed subspaces are forced to be Euclidean. This is well known under the hypothesis of local one sided curvature bounds in the sense of Alexandrov and follows for example from the existence of angles in such spaces (compare [5] p. 302). For the convenience of the reader we give a short proof of the following statement (see Section 2):

Theorem 1. *Let X be a locally geodesic metric space with the property that every point $x \in X$ has a neighborhood U such that the curvature in the sense of Alexandrov is bounded from below or from above in U. Then*

$$\mathrm{rank}_M(X) = \mathrm{rank}_E(X).$$

For more general metric spaces, the ranks may be different and they even have different functional behaviour with respect to metric products.

The Minkowski rank is additive, i.e., we have

Theorem 2. *Let (X_i, d_i), $i = 1, 2$, be metric spaces and denote their metric product by $(X_1 \times X_2, d)$. Then*

$$\mathrm{rank}_M(X_1, d_1) + \mathrm{rank}_M(X_2, d_2) = \mathrm{rank}_M(X_1 \times X_2, d).$$

In general the additivity of the Euclidean rank does not hold. In Section 4 we give an example of two normed vector spaces $(V_i, \|\cdot\|_i)$, $i = 1, 2$, that do not admit an isometric embedding of $\mathbb{E}^2$, although $\mathbb{E}^3$ may be embedded in their product. Thus $\mathrm{rank}_E(V_i) = 1$ for $i = 1, 2$ but $\mathrm{rank}_E(V_1 \times V_2) \geqslant 3$ and we obtain:

Theorem 3. *Let (X_i, d_i), $i = 1, 2$, be metric spaces and denote their metric product by $(X_1 \times X_2, d)$. Then it holds*

$$\mathrm{rank}_E(X_1, d_1) + \mathrm{rank}_E(X_2, d_2) \leqslant \mathrm{rank}_E(X_1 \times X_2, d),$$

but there are examples such that the inequality is strict.

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2 Minkowski rank for Alexandrov spaces

In this section we give a

Proof of Theorem 1. For the notion of (locally) geodesic metric spaces and bounds on the curvature in the sense of Alexandrov we refer to [4]. We only recall that the idea of curvature in the sense of Alexandrov is a comparison of triangles in X with triangles in the standard 2-dimensional spaces M_κ^2 of constant curvature. In our proof we need to compare the distance between a vertex of a triangle to the midpoint of the opposite side. Let therefore a, b, c be positive numbers such that there exists a triangle in M_κ^2 with corresponding side lengths. Then let $l_\kappa(a, b; c)$ be the distance in M_κ^2 of the midpoint of the side with length c to the opposite vertex. For $t > 0$ we have the scaling property $tl_\kappa(a, b; c) = l_{\kappa/t^2}(ta, tb; tc)$ and for $\kappa = 0$ we have the Euclidean formula

$$l_0(a, b; c) = \frac{1}{2}(2a^2 + 2b^2 - c^2)^{1/2}.$$

In order to prove Theorem 1 we show that a normed vector space $(V, \|\cdot\|)$ such that the curvature in the sense of Alexandrov is bounded below or above by some constant κ in a neighborhood U of 0 is indeed a Euclidean space. To consider both possible curvature bounds let $\sim$ be either $\leqslant$ or $\geqslant$. Let x, y be arbitrary vectors in V and let $t > 0$ be large enough such that the triangle $0, \frac{x}{t}, \frac{y}{t}$ is contained in U and we can compare it with a triangle in M_κ^2. We obtain

$$\frac{1}{2}\|x + y\| = t\frac{1}{2}\left\|\frac{x}{t} + \frac{y}{t}\right\| \sim tl_\kappa\left(\left\|\frac{x}{t}\right\|, \left\|\frac{y}{t}\right\|; \left\|\frac{x}{t} - \frac{y}{t}\right\|\right) = t\frac{1}{t}l_{\kappa/t^2}(\|x\|, \|y\|; \|x - y\|).$$

Note that for $t \to \infty$ the last expression converges to

$$l_0(\|x\|, \|y\|; \|x - y\|) = \frac{1}{2}(2\|x\|^2 + 2\|y\|^2 - \|x - y\|^2)^{1/2}.$$

Thus we have

$$\|x + y\|^2 + \|x - y\|^2 \sim 2\|x\|^2 + 2\|y\|^2$$

and hence one inequality of the parallelogram equality for all $x, y \in V$. Substituting $u = x + y$ and $v = x - y$ we get that the inequality holds in the opposite direction for all $u, v \in V$ as well. Hence V is a Euclidean space.

3 Minkowski rank of products

In this section we prove that the Minkowski rank is additive for metric products. Let therefore (X_i, d_i), $i = 1, 2$, be metric spaces and consider the product $X = X_1 \times X_2$ with the standard product metric

$$d((x_1, x_2), (x_1', x_2')) = (d_1^2(x_1, x_1') + d_2^2(x_2, x_2'))^{1/2}.$$

We need an auxiliary result: Let V be a real vector space and denote by A the affine space on which V acts simply transitively. Thus for $a \in A$ and $v \in V$ the point $a + v \in A$ and for $a, b \in A$ the vector $b - a \in V$ are defined. As usual a pseudonorm on V is a function $\|\cdot\|$ which satisfies the properties of a norm with the possible exception that $\|v\| = 0$ does not necessarily imply $v = 0$. A pseudonorm $\|\cdot\|$ on V induces a pseudometric d on A via

$$d(a,b) = \|b-a\| \quad \text{for all } a,b \in A.$$

We denote the resulting pseudometric space by $(A, \|\cdot\|)$. With this notation we have:

Proposition 1. *Let (X_i, d_i), $i = 1,2$, be metric spaces and $\varphi : A \to X_1 \times X_2$, $\varphi = (\varphi_1, \varphi_2)$ be an isometric map. Then there exist pseudonorms $\|\cdot\|_i$, $i = 1,2$, on V, such that*

i) $\|v\|^2 = \|v\|_1^2 + \|v\|_2^2$ *and*

ii) $\varphi_i : (A, \|\cdot\|_i) \to (X_i, d_i)$, $i = 1,2$ *are isometric.*

For the proof of Proposition 1 we define $\alpha_i : A \times V \to [0,\infty)$, $i = 1,2$, via

$$\alpha_i(a,v) := d_i(\varphi_i(a), \varphi_i(a+v)).$$

Since φ is isometric, we have

$$\alpha_1^2(a,v) + \alpha_2^2(a,v) = d^2(\varphi(a), \varphi(a,v)) = \|v\|^2. \tag{1}$$

We will prove the following lemmata:

Lemma 1. $\alpha_i(a,v) = \alpha_i(a+v,v)$ *for* $i = 1,2$ *and* $a \in A$, $v \in V$.

Lemma 2. $\alpha_i(a,tv) = |t|\alpha_i(a,v)$ *for* $i = 1,2$ *and* $a \in A$, $v \in V$, $t \in \mathbb{R}$.

Lemma 3. $\alpha_i(a,v) = \alpha_i(b,v)$ *for* $i = 1,2$ *and* $a,b \in A$, $v \in V$.

Lemma 4. $\alpha_i(v+w) \leqslant \alpha_i(v) + \alpha_i(w)$ *for* $i = 1,2$ *and* $v,w \in V$, *where* $\alpha_i(v) := \alpha_i(a,v)$ *with* $a \in A$ *arbitrary (compare with Lemma* 3).

From Lemmata 1–4 it follows immediately, that $\|\cdot\|_i$ defined via $\|v\|_i := \alpha_i(v)$ for all $v \in V$, $i = 1,2$, is a pseudonorm on V. Furthermore from

$$\begin{aligned} d_i(\varphi_i(a), \varphi_i(b)) &= d_i(\varphi_i(a), \varphi_i(a+(b-a))) \\ &= \alpha_i(b-a) \\ &= \|b-a\|_i \quad \text{for all } a,b \in A \end{aligned}$$

it follows that

$$\varphi_i : (A, \|\cdot\|_i) \to (X_i, \|\cdot\|_i), \quad i = 1,2,$$

are isometric mappings.

Proof of Lemma 1. The d_i's triangle inequality yields

$$\alpha_i(a, v) + \alpha_i(a + v, v) \geqslant \alpha_i(a, 2v) \tag{2}$$

and thus

$$\alpha_i^2(a, v) + 2\alpha_i(a, v)\alpha_i(a + v, v) + \alpha_i^2(a + v, v) \geqslant \alpha_i^2(a, 2v). \tag{3}$$

Using Equation (1) the sum of the Equations (3) for $i = 1$ and $i = 2$ becomes

$$\|v\|^2 + 2\left\langle \begin{pmatrix} \alpha_1(a, v) \\ \alpha_2(a, v) \end{pmatrix}, \begin{pmatrix} \alpha_1(a + v, v) \\ \alpha_2(a + v, v) \end{pmatrix} \right\rangle + \|v\|^2 \geqslant 4\|v\|^2,$$

where $\langle \cdot, \cdot \rangle$ denotes the standard scalar product on $\mathbb{R}^2$. Thus we have

$$\left\langle \begin{pmatrix} \alpha_1(a, v) \\ \alpha_2(a, v) \end{pmatrix}, \begin{pmatrix} \alpha_1(a + v, v) \\ \alpha_2(a + v, v) \end{pmatrix} \right\rangle \geqslant \|v\|^2.$$

The Euclidean norm of the vectors $(\alpha_1(a, v), \alpha_2(a, v))$ and $(\alpha_1(a + v, v), \alpha_2(a + v, v))$ equals $\|v\|$, due to Equation (1). Therefore the Cauchy–Schwarz inequality yields

$$\begin{pmatrix} \alpha_1(a, v) \\ \alpha_2(a, v) \end{pmatrix} = \begin{pmatrix} \alpha_1(a + v, v) \\ \alpha_2(a + v, v) \end{pmatrix}.$$

Proof of Lemma 2. The d_i's triangle inequality yields for all $n \in \mathbb{N}$

$$\alpha_i(a, nv) \leqslant \sum_{k=0}^{n-1} \alpha_i(a + kv, v) = n\alpha_i(a, v),$$

where the last equation follows from Lemma 1 by induction. Thus we find

$$\begin{aligned} n^2\|v\|^2 = \|nv\|^2 &= \alpha_1^2(a, nv) + \alpha_2^2(a, nv) \\ &\leqslant n^2(\alpha_1^2(a, v) + \alpha_2^2(a, v)) \\ &= n^2\|v\|^2 \quad \text{for all } n \in \mathbb{N}, v \in V, a \in A \end{aligned}$$

and therefore

$$\alpha_i(a, nv) = n\alpha_i(a, v), \ \ i = 1, 2, \quad \text{for all } n \in \mathbb{N}, v \in V, a \in A.$$

Thus for $p, q \in \mathbb{N}$, it follows

$$q\alpha_i\left(a, \frac{p}{q}v\right) = \alpha_i(a, pv) = p\alpha_i(a, v),$$

i.e.

$$\alpha_i(a, tv) = t\alpha_i(a, v) \quad \text{for all } t \in \mathbb{Q}_+$$

and by continuity even for all $t \in \mathbb{R}_+$.

Finally note that for all $t \in \mathbb{R}_+$

$$\alpha_i(a, -tv) = \alpha_i(a - tv, tv) = \alpha_i(a, tv) = t\alpha_i(a, v), \quad \text{for } i = 1, 2,$$

where the first equality is just the symmetry of the metric d_i and the second equality follows from Lemma 1.

Proof of Lemma 3. For $n \in \mathbb{N}$ we have

$$\begin{aligned} |\alpha_i(a, nv) - \alpha_i(b, nv)| &= |d_i(\varphi_i(a), \varphi(a + nv)) - d_i(\varphi_i(b), \varphi(b + nv))| \\ &\leqslant d_i(\varphi_i(a), \varphi_i(b)) + d_i(\varphi_i(a + nv), \varphi_i(b + nv)) \\ &\leqslant d(\varphi(a), \varphi(b)) + d(\varphi(a + nv), \varphi(b + nv)) \\ &= 2\|b - a\|, \quad \text{for } i = 1, 2, \end{aligned}$$

and therefore

$$\alpha_i(a, v) = \lim_{n\to\infty} \frac{1}{n}\alpha_i(a, nv) = \lim_{n\to\infty} \frac{1}{n}\alpha_i(b, nv) = \alpha_i(b, v), \quad i = 1, 2.$$

Proof of Lemma 4. The claim simply follows from

$$\alpha_i(v + w) = \alpha_i(a, v + w) \leqslant \alpha_i(a, v) + \alpha_i(a + v, w) = \alpha_i(v) + \alpha_i(w),$$

where the inequality follows from the d_i's triangle inequality and the last equation is due to Lemma 3.

With that we are now ready for the

Proof of Theorem 2. i) Superadditivity follows as usual: Let $i_j : (V_j, \|\cdot\|_j) \to (X_j, d_j)$ be isometries of the normed vector spaces $(V_j, \|\cdot\|)$ into the metric spaces (X_j, d_j). Then, with $\|\cdot\| : (V_1 \times V_2) \to \mathbb{R}$ defined via

$$\|(v, w)\| := \sqrt{\|v\|_1^2 + \|w\|_2^2}, \quad \text{for all } v \in V_1, w \in V_2,$$

the map $i := i_1 \times i_2 : (V_1 \times V_2, \|\cdot\|) \to (X, d) := \left(X_1 \times X_2, \sqrt{d_1^2 + d_2^2}\right)$ is an isometry. Thus $\mathrm{rank}_M(X_1, X_2) \geqslant \mathrm{rank}_M X_1 + \mathrm{rank}_M X_2$.

ii) Let $\mathrm{rank}_M(X, d) = n$ and let $\varphi : A \to X$ be an isometric map, where A is the affine space for some n-dimensional normed vector space $(V, \|\cdot\|)$. By Proposition 1 there are two pseudonorms $\|\cdot\|_i$, $i = 1, 2$, on V such that $\|\cdot\|_1^2 + \|\cdot\|_2^2 = \|\cdot\|^2$ and such that $\varphi_i : (A, \|\cdot\|_i) \to (X_i, d_i)$ are isometric. Let V_i be vector subspaces transversal to $\mathrm{kern}\|\cdot\|_i$. Then $\dim V_1 + \dim V_2 \geqslant n$ and $\varphi_i : (V_i, \|\cdot\|_i) \to X$ are isometric maps. Thus $\mathrm{rank}_M(X_i, d_i) \geqslant \dim V_i$.

4 Euclidean rank of products

In this section we prove Theorem 3. The superadditivity of the Euclidean rank is obvious. Thus it remains to construct an example such that the equality does not hold. Therefore we construct two norms $\|\cdot\|_i$, $i = 1, 2$, on $\mathbb{R}^3$, such that

i) there does not exist an isometric embedding of $\mathbb{E}^2$ in $(\mathbb{R}^3, \|\cdot\|_i)$, $i = 1, 2$, i.e.,

$$\operatorname{rank}_E(\mathbb{R}^3, \|\cdot\|_i) = 1, \quad \text{for } i = 1, 2, \text{ and}$$

ii) the diagonal of $(\mathbb{R}^3, \|\cdot\|_1) \times (\mathbb{R}^3, \|\cdot\|_2)$ is isometric to the Euclidean space $\mathbb{E}^3 = (\mathbb{R}^3, \|\cdot\|_e)$, i.e.,

$$\operatorname{rank}_E((\mathbb{R}^3, \|\cdot\|_1) \times (\mathbb{R}^3, \|\cdot\|_2)) \geqslant 3.$$

The norms will be obtained by perturbations of the Euclidean norm $\|\cdot\|_e$ in the following way:

$$\|v\|_i = \varphi_i\left(\frac{v}{\|v\|_e}\right)\|v\|_e, \quad \text{for all } v \in \mathbb{R}^3,$$

where the φ_i are appropriate functions on S^2 that satisfy $\varphi_i\left(\frac{v}{\|v\|_e}\right) = \varphi_i\left(-\frac{v}{\|v\|_e}\right)$, $i = 1, 2$, and $\varphi_2 = \sqrt{2 - \varphi_1^2}$. Thus their product norm $\|\cdot\|_{1,2}$ satisfies

$$\|(v, v)\|_{1,2}^2 = \|v\|_1^2 + \|v\|_2^2 = \varphi_1^2\|v\|_e^2 + (2 - \varphi_1^2)\|v\|_e^2 = 2\|v\|_e^2$$

and the diagonal in $(\mathbb{R}^6, \|\cdot\|_{1,2})$ is isometric to $\mathbb{E}^3$ and thus ii) is satisfied. It remains to show that for φ_i suitable i) holds.

Note that for $\varphi_i\left(\frac{v}{\|v\|_e}\right) = 1 + \varepsilon_i\left(\frac{v}{\|v\|_e}\right)$, $i = 1, 2$, with ε_i, $\mathrm{D}\varepsilon_i$ and $\mathrm{DD}\varepsilon_i$ sufficiently bounded, the strict convexity of the Euclidean unit ball implies strict convexity of the $\|\cdot\|_i$-unit balls. Since $\|\cdot\|_i$ is homogeneous by definition it follows that $\|\cdot\|_i$, $i = 1, 2$, are norms.

In order to show that $\operatorname{rank}_E(\mathbb{R}^3, \|\cdot\|_i) = 1$ for suitable functions $\varphi_i = 1 + \varepsilon_i$ we use the following result:

Lemma 5. *Let $(V, \|\cdot\|)$ be a normed vector space with strictly convex norm ball and let $i : \mathbb{E}^2 \to (V, \|\cdot\|)$ be an isometric embedding. Then i is an affine map and the image of the unit circle in $\mathbb{E}^2$ is an ellipse in the affine space $i(\mathbb{E}^2)$.*

Remark. We recall that the notion of an ellipse in a 2-dimensional vector space is a notion of affine geometry. It does not depend on a particular norm. Let A be a 2-dimensional affine space on which V acts simply transitively. A subset $W \subset V$ is called an *ellipse*, if there are linearly independent vectors $v_1, v_2 \in V$ and a point $a \in A$ such that

$$W = \{a + (\cos\alpha\, v_1 + \sin\alpha\, v_2) \mid \alpha \in [0, 2\pi]\}.$$

Proof of Lemma 5. In a normed vector space $(V, \|\cdot\|)$ the straight lines are geodesics. If the norm ball is strictly convex, then these are the unique geodesics.

The isometry i maps geodesics onto geodesics and hence straight lines in $\mathbb{E}^2$ onto straight lines in V. Note that the composition of i with an appropriate translation of V yields an isometry that maps the origin of $\mathbb{E}^2$ to the origin of V. Let us therefore assume that i maps 0 to 0. It follows that i is homogeneous. Furthermore it is easy to see that parallels are mapped to parallels and this finally yields the additivity of i and thus the claim.

Now we define functions $\varphi_i = 1 + \varepsilon_i$ on S^2 in a way such that the intersection of the unit ball in $(\mathbb{R}^3, \|\cdot\|_i)$ with a 2-dimensional linear subspace is never an ellipse. Therefore we will define the ε_i's such that their null sets are 8 circles, 4 of which are parallel to the equator γ, the other 4 parallel to a great circle δ that intersects the equator orthogonally; these null sets being sufficiently close to γ and δ such that each great circle of S^2 intersects those circles in at least 8 points.

Furthermore no great circle of S^2 is completely contained in the null set.

Using spherical coordinates $\Theta \in [0, \pi]$, $\Phi \in [0, 2\pi]$, $r \in \mathbb{R}^+$, we define

$$\tilde{\varepsilon}_1(\Theta, \Phi, r) := \frac{1}{n} \prod_{k=2}^{3} \sin\left(\Theta + \frac{k\pi}{8}\right) \sin\left(\Theta + \frac{(8-k)\pi}{8}\right),$$

with $n \in \mathbb{N}$ sufficiently large, such that the norm $\|\cdot\|_1$ we will obtain admits a strictly convex unit ball.

One can easily check that $\tilde{\varepsilon}_1^k(\Theta, \Phi, r) = \sin\left(\Theta + \frac{k\pi}{8}\right) \sin\left(\Theta + \frac{(8-k)\pi}{8}\right)$, $k \in \mathbb{N}$, satisfies $\tilde{\varepsilon}_1^k(\Theta, \Phi, r) = \tilde{\varepsilon}_1^k(-\Theta, \Phi, r)$ and so does $\tilde{\varepsilon}_1$. Since $\tilde{\varepsilon}_1$ is independent of Φ it satisfies $\tilde{\varepsilon}_1\left(\frac{v}{\|v\|}\right) = \tilde{\varepsilon}_1\left(-\frac{v}{\|v\|}\right)$. Its null set is the union of the circles parallel to the equator γ at $\Theta = \left\{\frac{1}{4}\pi, \frac{3}{8}\pi, \frac{5}{8}\pi, \frac{3}{4}\pi\right\}$.

Define $\hat{\varepsilon}_1$ analogous to $\tilde{\varepsilon}_1$ but with the null set consisting of circles parallel to δ instead of the equator γ.

With that we set $\varphi_1 = 1 + \tilde{\varepsilon}_1 \hat{\varepsilon}_1$ and $\|\cdot\|_1$ defined via

$$\|v\|_1 := \varphi_1\left(\frac{v}{\|v\|_e}\right)\|v\|_e$$

is a norm on $\mathbb{R}^3$ whose unit ball coincides with the $\|\cdot\|_e$-unit ball exactly on the null set of $\tilde{\varepsilon}_1 \hat{\varepsilon}_1$. Obviously

$$\|\cdot\|_2 := \sqrt{2 - \varphi_1^2}\|\cdot\|_e$$

is another norm on $\mathbb{R}^3$ whose unit ball also intersects the $\|\cdot\|_e$-unit ball on the null set of $\tilde{\varepsilon}_1 \hat{\varepsilon}_1$.

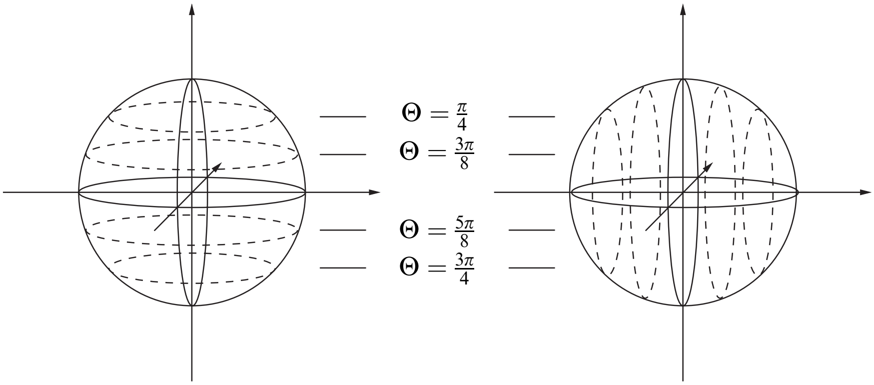

Figure 1. The dashed circles in this figure are the sections of the $\|\cdot\|_1$-, $\|\cdot\|_2$- and $\|\cdot\|_e$-unit balls.

We finally conclude that $\mathrm{rank}_E(\mathbb{R}^3, \|\cdot\|_j) = 1$, $j = 1, 2$.

Assume to the contrary, that there exists an isometric embedding $i : \mathbb{E}^2 \to (\mathbb{R}^3, \|\cdot\|_j)$. By Lemma 5 we can assume (after a translation) that i is a linear isometry and that the image of the unit circle $S \subset \mathbb{E}^2$ is an ellipse in the linear subspace $i(\mathbb{E}^2)$ which is in addition contained in the unit ball B_j of $\|\cdot\|_j$. Note that $i(\mathbb{E}^2) \cap B_j$ and $i(\mathbb{E}^2) \cap S^2$ are ellipses which coincide by construction in at least 8 points. Since two ellipses with more than 4 common points coincide, we have $i(\mathbb{E}^2) \cap B_j = i(\mathbb{E}^2) \cap S^2$. This contradicts the fact that by construction $i(\mathbb{E}^2) \cap B_j \cap S^2$ is a discrete set.

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