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# Pseudo-parallel submanifolds of a space form

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**Abstract.** Pseudo-parallel immersions into space forms are defined as extrinsic analogues of pseudo-symmetric manifolds (in the sense of R. Deszcz) and as a direct generalization of semi-parallel immersions. In this paper we obtain a description of pseudo-parallel hypersurfaces of a space form as quasi-umbilic hypersurfaces or cyclids of Dupin. Moreover, we study pseudo-parallel immersions of surfaces and pseudo-parallel immersions with maximal first normal bundle in space forms. Finally, we give a topological classification of some complete, simply connected manifolds admitting a pseudo-parallel immersion into a space form.

**Key words.** Isometric immersion, space form, pseudo-parallel.

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## 1 Introduction

A Riemannian manifold $M^n$ is *locally symmetric* if its Riemannian curvature tensor $R$ is parallel, i.e. $\nabla R = 0$, where $\nabla$ is the Levi–Civita connection extended to act on tensors as a derivation. The integrability condition of $\nabla R = 0$ is $R \cdot R = 0$, where again $R$ is extended to act on tensors as a derivation. Manifolds which satisfy the latter condition are called *semi-symmetric* and have been classified by Szabó (see [33], [34]). Investigation of several properties of semi-symmetric manifolds gives rise to their next generalization: the *pseudo-symmetric manifolds*. For example every totally umbilic submanifold of a semi-symmetric manifold, with parallel mean curvature vector, is pseudo-symmetric (see [1]). The class of pseudo-symmetric manifolds is very large: many examples of pseudo-symmetric manifolds which are not semi-symmetric have been constructed (see e.g. [15], [16] and references therein). In the last decade, a big amount of results both intrinsic and extrinsic involving this class of manifolds have been published by several authors. Consequently many particular results are known, see, for example, [15], [16], [11], [17], [18], [19], [20], but a full classification is not yet available.

In the theory of submanifolds of a space form, conditions analogous to local symmetry and semi-symmetry have been introduced and studied quite intensively.

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Ferus and others introduced the concept of *parallel immersions*, i.e. immersions with parallel second fundamental form, and classified such immersions (see [24], [5], [35] for example). On the other hand, Deprez and others introduced the concept of semi-parallel immersions, i.e. immersions such that the curvature tensor annihilates the second fundamental form. A complete classification is not yet available, but some particular results are known, see, for example, [12], [13], [26], [27], [22], [3].

In this paper we introduce the concept of *pseudo-parallel* immersions into a space form $Q^N(c)$ of constant curvature $c$, as the extrinsic analogue of pseudo-symmetric manifolds, and as a direct generalization of semi-parallel immersions. Those immersions are defined by the condition $\bar{R}(X \wedge Y) \cdot \alpha = \phi(X \wedge Y) \cdot \alpha$, where $\alpha$ is the second fundamental form and $\phi$ is a smooth real valued function on the manifold. In Section 2 of this paper, we give the basic definitions. In Section 3 we classify the pseudo-parallel hypersurfaces: They are either quasi-umbilic hypersurfaces or cyclids of Dupin. In Section 4 we study pseudo-parallel surfaces proving, among other results, that they are isotropic or have flat normal bundle. In Section 5 we study pseudo-parallel immersions (of dimension $\geqslant 3$), with maximal first normal bundle, proving that such an immersion is a Veronese immersion into some sphere, if the manifold is complete and $\phi \leqslant 0$. Finally, in Section 6 we introduce a Jordan triple system in connection with every pseudo-parallel immersion and as a consequence we prove a topological classification for some complete simply connected manifolds that admit a pseudo-parallel immersion in $Q^N(c)$ with $c + \phi > 0$ and $\phi \geqslant 0$.

Some of those results were announced in [2] where a different proof of 6.3 was also given.

## 2 Basic definitions

We will give some basic notations since they are standard only up to sign.

Let $M^n$ be an $n$-dimensional Riemannian manifold with Levi–Civita connection $\nabla$ and curvature tensor $R(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$. Let $Q^{n+p}(c)$ be an $(n+p)$-dimensional space form with constant curvature $c$, connection $\bar{\nabla}$ and curvature tensor $\bar{R}$. Let $f : M^n \to Q^{n+p}(c)$ be an isometric immersion with normal bundle $\nu M$, and second fundamental form $\alpha : TM \otimes TM \to \nu M$. If $\xi$ is a normal vector at $x \in M$, we denote by $A_\xi : T_xM \to T_xM$, $\langle A_\xi(X), Y\rangle = \langle \alpha(X,Y), \xi\rangle$ the Weingarten operator in the $\xi$ direction and by $\nabla^\perp$, $R^\perp$ the normal connection and its curvature.

We will identify a 2-form $\omega \in \Lambda^2(T_xM)$ with the antisymmetric endomorphism $\langle \omega(X), Y\rangle = \omega(X,Y)$. Given $X, Y \in T_xM$, the bi-vector $X \wedge Y$ will be identified with the 2-form

$$X \wedge Y(Z) = \langle X, Z\rangle Y - \langle Y, Z\rangle X.$$

By the well known symmetries of $R$ the linear extension of the map $R(X \wedge Y) = R(X,Y)$ is a well defined symmetric endomorphism of $\Lambda^2(T_xM)$.

Define actions of a 2-form $\omega$

$$(\omega \cdot R)(X \wedge Y) = [\omega, R(X \wedge Y)] - R(\omega(X) \wedge Y) - R(X \wedge \omega(Y))$$

$$(\omega \cdot \alpha)(X,Y) = -\alpha(\omega(X), Y) - \alpha(X, \omega(Y))$$

so that

$$R(X \wedge Y) \cdot R = ([\nabla_X, \nabla_Y] - \nabla_{[X,Y]})R.$$

It is natural to define an action of the curvature tensor of the ambient space on the second fundamental form, by

$$\bar{R}(X \wedge Y) \cdot \alpha = ([\bar{\nabla}_X, \bar{\nabla}_Y] - \bar{\nabla}_{[X,Y]})\alpha = R^{\perp}(X, Y) \circ \alpha + R(X \wedge Y) \cdot \alpha.$$

We recall that $M^n$ is said to be

- *Locally symmetric* (L.S., for short) if $\nabla R = 0$.
- *Semi-symmetric* (S.S., for short) if $R(X \wedge Y) \cdot R = 0$ for all $X, Y \in TM^n$.
- *Pseudo-symmetric* (P.S., for short) if $R(X \wedge Y) \cdot R = \phi(X \wedge Y) \cdot R$ for all $X, Y \in TM^n$ where $\phi$ is a real valued smooth function on $M^n$.

The correspondent concepts for an isometric immersion $f : M^n \to Q^{n+p}(c)$ are the following: $f$ is said to be

- *Locally parallel* (L.P., for short) if $(\nabla_X \alpha)(Y, Z) := \nabla_X^{\perp}[\alpha(Y, Z)] - \alpha(\nabla_X Y, Z) - \alpha(Y, \nabla_X Z) = 0$ for all $X, Y, Z \in TM^n$ (*extrinsically symmetric* in Ferus' terminology).
- *Semi-parallel* (S.P., for short) if $\bar{R}(X \wedge Y) \cdot \alpha = 0$ for all $X, Y \in TM^n$.
- *Pseudo-parallel* (P.P., for short) if $\bar{R}(X \wedge Y) \cdot \alpha = \phi(X \wedge Y) \cdot \alpha$ for all $X, Y \in TM^n$, where $\phi$ is a real valued smooth function on $M^n$.

The basic Gauss, Codazzi–Mainardi and Ricci equations give that the extrinsic conditions (L.P., S.P., P.P. respectively) imply the corresponding intrinsic conditions (L.S., S.S., P.S. respectively), see [24], [5], [13], [2].

## 3 Pseudo-parallel hypersurfaces

Let $f : M^n \to Q^{n+1}(c)$ be a hypersurface of a space form, $\xi$ a unit normal vector and $\{e_1, \dots, e_n\}$ an orthonormal basis of the tangent space of $M$ which diagonalizes the Weingarten operator $A_\xi$. Let $\lambda_i = \langle A_\xi e_i, e_i \rangle$ be the principal curvatures in the $\xi$ direction.

From the Gauss equations

$$R(e_i \wedge e_j) = (c + \lambda_i \lambda_j) e_j \wedge e_i,$$

we get, for $i \neq j$:

$$[\bar{R}(e_i \wedge e_j) \cdot \alpha](e_i \wedge e_j) = (\lambda_j - \lambda_i)(\lambda_i \lambda_j + c)\xi, \quad [(e_i \wedge e_j) \cdot \alpha](e_i \wedge e_j) = (\lambda_i - \lambda_j)\xi.$$

Therefore, in the above situation we have

**Proposition 3.1.** *If $f$ is P.P., then*

$$(\lambda_i\lambda_j + c + \phi)(\lambda_i - \lambda_j) = 0 \quad \text{for all } i \neq j. \tag{1}$$

*In particular, $f$ has at most two distinct principal curvatures and, if exactly two, their product is $-(c+\phi)$.*

The above condition (1) is also sufficient for P.P., up to the smoothness of $\phi$. For example, suppose that $A_\xi$ has, at every $x \in M$, at most two distinct eigenvalues $\lambda(x)$ and $\mu(x)$. Define $\phi(x) = -(c + \lambda(x)\mu(x))$. Then $\phi$ is a continuous function which is smooth outside the boundary of the set $U = \{x \in M : \lambda(x) = \mu(x)\}$, of umbilic points. If $\phi$ is smooth then the immersion is P.P.

In particular, if $n = 2$, $\phi$ is the curvature, hence is smooth. So every surface in $Q^3(c)$ is P.P.

Let $f : M^n \to Q^{n+1}$ be an hypersurface and $U \subseteq M$ be the set of umbilic points. We recall that $f$ is *quasi-umbilic* if it has a principal curvature of multiplicity at least $(n-1)$. Quasi-umbilic hypersurfaces are *conformally flat* and, if $n \geqslant 4$, conformal flatness is equivalent to quasi-umbilicity. Compact quasi-umbilic hypersurfaces are completely described in [23], for $n \geqslant 3$. Also we recall that $f$ is a *cyclid of Dupin* if it has exactly two distinct principal curvatures, each of which is constant in the direction of the corresponding eigenspace (see also [8] pp. 151–152). Such hypersurfaces have been studied intensively, see [30] for example.

In the above terms we have the following description of P.P. hypersurfaces:

**Theorem 3.2.** *A P.P. hypersurface of a space form is either quasi-umbilic or a cyclid of Dupin.*

*Proof.* Consider the connected components of $M^n \backslash U$. They are open and the principal curvatures have constant multiplicity in each of them. Suppose there is such a connected component, $\mathscr{C}$, which is not quasi-umbilic. Then neither principal curvature is simple and, as a standard consequence of Codazzi's equations, $\mathscr{C}$ is a cyclid of Dupin. Suppose $\mathscr{C} \neq M^n$ and let $x_0$ be a boundary point. Then $x_0 \in U$. By a result of Pinkall (see [30]), there is a conformal transformation $A$ which sends $\mathscr{C}$ onto an open part of a compact cyclid of Dupin $\overline{M}^n$. This cyclid may have singular points, but not near $A(x_0)$ since such a point is a regular point of $\underline{A}(M^n)$. Since conformal transformations preserve umbilics, $A(x_0)$ is an umbilic of $\overline{M}^n$, which is absurd. So $\mathscr{C} = M^n$.

**Some Examples.** 1. The condition for an hypersurface of $\mathbb{R}^{n+1}$ to be S.P. is that each point is either umbilic or has two distinct principal curvatures, one being 0 (see [13]). Therefore it is easy to construct examples of P.P. hypersurfaces which are not S.P. See also [2] for examples in $Q^{n+1}(c)$ for $c \neq 0$.

2. Consider the cone over the Clifford torus $T^2 \subset S^3$. This is a minimal hypersurface of $\mathbb{R}^4$ with three distinct principal curvatures, one being 0. This hypersurface is S.S. and by Theorem 2 of [21] it is not S.P., in particular, is not P.P. Moreover, it is

a conformally flat hypersurface which is not pseudo-umbilic, a phenomenon which can occur only in dimension 3.

3. Consider $\mathbb{R}^5$ as the space of $3 \times 3$ symmetric matrices of trace zero. Then the regular orbits of the action of SO(3) by conjugation are isoparametric hypersurfaces of $S^4$ with three distinct principal curvatures, the so called *Cartan hypersurfaces*. In this family there is a minimal one, with one zero principal curvature. This one is P.S. (see [19]) but not S.S., because $c + \phi = 0$. Hence this is an example of a P.S. hypersurface which is not P.P.

**Remark 3.3.** An immediate consequence of Theorem 5.1 and Theorem 3.1(i) of [18] is the following: *If $f : M^n \to Q^{n+1}(c)$ is a P.S. hypersurface with $c + \phi \neq 0$, then $f$ is P.P.*

## 4 Pseudo-parallel surfaces

We start observing that any 2-dimensional manifold is S.S., hence P.S. with any $\phi$. In fact, in this case, $R(X \wedge Y) \cdot R = 0 = (X \wedge Y) \cdot R$.

Consider now an isometric immersion $f : M^2 \to Q^N(c)$ of a 2-dimensional manifold into an $N$-dimensional space form.

Let $\{e_1, e_2\}$ be an orthonormal basis for the tangent space and set $\alpha_{ij} = \alpha(e_i, e_j)$, where $\alpha$ is the second fundamental form of $f$. Also $K$ denotes the Gaussian curvature and $R^\perp = R^\perp(e_1, e_2)$ the normal curvature operator. With this notation the condition of pseudo-parallelism can be written as:

$$R^\perp \alpha_{11} = -2(K + \phi)\alpha_{12} = -R^\perp \alpha_{22}, \tag{2}$$

$$R^\perp \alpha_{12} = (K + \phi)(\alpha_{11} - \alpha_{22}). \tag{3}$$

In particular, if $R^\perp = 0$, then $f$ is P.P. with $\phi = -K$ (or any $\phi$ if the point is umbilic). We recall that a S.P. surface with $R^\perp = 0$ is either umbilic or flat. So any non-umbilic, non-flat surface with vanishing normal curvature is an example of a P.P. surface which is not S.P.

Next we study P.P. surfaces with non-vanishing normal curvature. We recall that an isometric immersion $f : M^n \to Q^N(c)$ is *$\lambda$-isotropic*, $\lambda : M^n \to \mathbb{R}$, if $\|\alpha(X, X)\| = \lambda(p)$ for all $p \in M$ and all $X \in T_pM$ with $\|X\| = 1$ (see [29]).

**Proposition 4.1.** *An isometric immersion $f : M^2 \to Q^N(c)$ with non-vanishing normal curvature is P.P. if and only if it is $\lambda$-isotropic. Moreover, for such an immersion we have:*

1. $K + \phi > 0$,
2. $\lambda^2 = 4K + 3\phi - c > 0$,
3. $\|H\|^2 = 3K + 2\phi - c \geqslant 0$,

*where $H := \frac{1}{2}(\alpha_{11} + \alpha_{22})$ is the mean curvature vector.*

*Proof.* Suppose that $f$ is P.P. From the Ricci equations we get for $i = 1, 2$:

$$\langle \alpha_{12}, \alpha_{ii} \rangle (\alpha_{11} - \alpha_{22}) + [\langle \alpha_{22} - \alpha_{11}, \alpha_{ii} \rangle + (-1)^{i+1} 2(\phi + K)] \alpha_{12} = 0,$$
$$(\|\alpha_{12}\|^2 + (\phi + K)(\alpha_{11} - \alpha_{22}) + \langle \alpha_{22} - \alpha_{11}, \alpha_{12} \rangle \alpha_{12} = 0.$$

Since the normal curvature does not vanish, $\alpha_{12}$ and $\alpha_{11} - \alpha_{22}$ are linearly independent. It follows that

$$\langle \alpha_{12}, \alpha_{ii} \rangle = 0, \quad \langle \alpha_{22} - \alpha_{11}, \alpha_{ii} \rangle = (-1)^i 2(\phi + K), \quad i = 1, 2, \tag{4}$$
$$\|\alpha_{12}\|^2 = K + \phi > 0. \tag{5}$$

From the Gauss equation we get

$$\|\alpha_{ii}\|^2 = 4K + 3\phi - c > 0, \quad i = 1, 2,$$
$$\langle \alpha_{11}, \alpha_{22} \rangle = 2K + \phi - c,$$
$$\|\alpha_{11} - \alpha_{22}\|^2 = 4(\phi + K) > 0,$$
$$\|H\|^2 = 3K + 2\phi - c$$

and the claim follows.

Conversely, suppose that $f$ is $\lambda$-isotropic and set $X = \cos\theta e_1 + \sin\theta e_2$. Computing $\frac{d}{d\theta} \|\alpha(X, X)\|^2 = 0$ for $\theta = 0, \frac{\pi}{4}$, we get that $\langle \alpha_{ii}, \alpha_{12} \rangle = 0$. Then from the equations of Gauss and Ricci, we can see that

$$R^{\perp}\alpha_{11} = \frac{2}{3}(K - c - \lambda^2)\alpha_{12} = -R^{\perp}\alpha_{22},$$
$$R^{\perp}\alpha_{12} = \frac{1}{3}(-K + c + \lambda^2)(\alpha_{11} - \alpha_{22}).$$

Therefore $f$ is $\phi$-P.P. for $\phi = (c + \lambda^2 - 4K)/3$.

We recall that a minimal isotropic immersion is called *superminimal* (see [7]). Then we have:

**Corollary 4.2.** *Let $f : M^2 \to Q^4(c)$ be an isometric immersion with $R^{\perp} \neq 0$. Then $f$ is P.P. if and only $f$ is superminimal. Moreover, if $\phi$ is constant, then $K = \frac{c}{3} > 0$, and $f(M)$ is a piece of a Veronese surface.*

*Proof.* First observe that an isotropic immersion in codimension two is minimal. If $\phi$ is constant, then $K$ is constant and the claim follows from a theorem of Kenmotsu (see [25]).

**Remark 4.3.** By a theorem of Chern, every minimal immersion of a topological 2-sphere in $S^4(1)$ is superminimal (see [10]). So any such immersion is P.P. Moreover if the curvature is not constant, the immersion is not semi-parallel.

Finally we will study a class of P.P. surfaces in 5-dimensional space forms. We start with the following example (see [6]):

**Example.** Consider the immersion $T : \mathbb{R}^2 \to S^5(c)$:

$$T(x,y) = \frac{2}{\sqrt{6c}}\left(\cos u \cos v, \cos u \sin v, \frac{\sqrt{2}}{2}\cos 2u, \sin u \cos v, \sin u \sin v, \frac{\sqrt{2}}{2}\sin 2u\right)$$

where $u = \sqrt{\frac{c}{2}}x$, $v = \frac{\sqrt{6c}}{2}y$. A direct calculation shows that $T$ is superminimal with $\lambda^2 = \frac{c}{2}$ (see [32]). In particular, by Proposition 4.1 $T$ is P.P. with $\phi = \frac{c}{2}$.

**Theorem 4.4.** *Let $f : M^2 \to Q^5(c)$ be a P.P. immersion of a connected, complete surface, with $\phi$ constant non-negative and $c + \phi > 0$. Then we have one of the following possibilities*:

1. *$f$ is a totally umbilic immersion*;
2. *$f$ is an immersion with flat normal bundle and constant Gaussian curvature $K = -\phi \leqslant 0$*;
3. *$f(M^2)$ is a Veronese surface in some totally umbilic $S^4(\tilde{c}) \subset Q^5(c)$*;
4. *$f(M^2) \subset S^5(c)$ is congruent to the torus of the example above.*

*Proof.* If the normal curvature is identically zero, then $f$ is totally umbilic or $\phi + K = 0$ at the non-umbilic points, by (2) and (3). But $\phi$ constant implies $K$ constant on all $M$ and therefore $K + \phi$ is constant on all $M$. Suppose now there is a point $x \in M$ such that $R^{\perp}(x) \neq 0$. Let $C$ be the connected component of the set of points where the normal curvature does not vanish, which contains $x$. We proceed similarly as in the proof of Theorem 3.7 of [3]. The calculations are quite horrible, but, after a few pages, we arrive at the conclusion that the curvature has to be constant near $x$ (which is evident if $H = 0$). In particular, by (4), $\|\alpha_{12}\|$ is a non-zero constant in $C$. If $y$ is an accumulation point of $C$, $y \notin C$, then $R^{\perp}(y) = 0$. We can therefore choose an orthonormal basis $\{e_1, e_2\}$ in $T_yM$ such that $\alpha_{12} = 0$. Extending this frame smoothly to a neighborhood of $y$, we get a contradiction. So $R^{\perp}(z) \neq 0$ for all $z \in M$, hence $f$ is $\lambda$-isotropic with $\lambda$ constant and the conclusion follows from the classification of such surfaces in [32].

## 5 Pseudo-parallel immersions with maximal first normal bundle

In this section we will study an extension of Theorem 4.4, for the case $n \geqslant 3$. Let $f : M^n \to Q^N(c)$ be a P.P. immersion. We recall that the first normal space at $x \in M^n$ is defined as

$$N_1(x) = \operatorname{span}\{\alpha(X, Y) : X, Y \in T_xM\} \subseteq \nu_x M.$$

Fix an orthonormal basis $\{e_1, \dots, e_n\}$ of $T_xM$ and let

$$\alpha_{ij} := \alpha(e_i, e_j), \quad R^{\perp}_{ij} := R^{\perp}(e_i, e_j), \quad R_{ijkl} := \langle R(e_i, e_j)e_k, e_l\rangle.$$

The condition of pseudo-parallelism can be written as

$$R^{\perp}_{ij}(\alpha_{kl}) - \sum_{m=1}^{n}(R_{ijkm}\alpha_{ml} + R_{ijlm}\alpha_{mk}) + \phi(\delta_{ik}\alpha_{jl} - \delta_{jk}\alpha_{il} + \delta_{il}\alpha_{jk} - \delta_{jl}\alpha_{ik}) = 0.$$

**Theorem 5.1.** *Let $f : M^n \to Q^N(c)$, $n \geqslant 3$, be a P.P. immersion.*

1. *If $\dim N_1(x) = \frac{1}{2}n(n+1)$ for all $x \in M^n$, then*
   (a) *$M$ has constant sectional curvature $K$ and $K + \phi > 0$;*
   (b) *$f$ is $\lambda$ isotropic with $\lambda^2 = 4K + 3\phi - c$, and pseudo-umbilic, i.e. $A_H = \|H\|^2 \operatorname{Id}$ and $\|H\|^2 = \frac{2(n+1)}{n}K + \frac{n+2}{n}\phi - c$;*
   (c) *The set $\beta = \{H, \alpha_{11} - \alpha_{22}, \dots, \alpha_{(n-1)(n-1)} - \alpha_{nn}, \alpha_{12}, \dots \alpha_{(n-1)n}\}$ is an orthogonal basis of the first normal space;*
   (d) *The normal curvature is given by*

$$R^{\perp}_{ij}\alpha_{ii} = -2(K + \phi)\alpha_{ij},$$
$$R^{\perp}_{ij}\alpha_{ij} = -(K + \phi)(\alpha_{jj} - \alpha_{ii}),$$
$$R^{\perp}_{ij}\alpha_{kk} = 0 = R^{\perp}_{ij}\alpha_{kl}.$$

2. *If $H = 0$ and $\dim N_1(x) = \frac{1}{2}n(n+1) - 1$ for all $x \in M^n$, then $\phi$ and $K$ are constant and*

$$\phi = \frac{nc - 2(n+1)K}{(n+2)}.$$

*Proof.* With the equation

$$R^{\perp}_{ij}\alpha_{kl} = \sum_{m=1}^{n}(\langle\alpha_{mj}, \alpha_{kl}\rangle\alpha_{im} - \langle\alpha_{im}, \alpha_{kl}\rangle\alpha_{mj})$$

of Ricci, the condition of pseudo-parallelism becomes

$$0 = \sum_{m=1}^{n} (\langle \alpha_{im}, \alpha_{kl} \rangle \alpha_{mj} - \langle \alpha_{mj}, \alpha_{kl} \rangle \alpha_{im} + R_{ijkm}\alpha_{ml} + R_{ijlm}\alpha_{km})$$
$$- \phi(p)(\delta_{ik}\alpha_{jl} + \delta_{il}\alpha_{jk} - \delta_{jk}\alpha_{il} - \delta_{jl}\alpha_{ik}). \quad (6)$$

Using the Gauss equation and the fact that the set $\gamma = \{\alpha_{11} - \alpha_{22}, \ldots, \alpha_{(n-1)(n-1)} - \alpha_{nn}, \alpha_{12}, \ldots \alpha_{(n-1)n}\}$ is linearly independent, we obtain, after some calculations, taking in account the various possibilities for the indeces $i, j, k, l$:

$$\left.\begin{aligned}
&R_{ijkl} = R_{ijjk} = 0, \quad K_{ij} = K_{12} := K,\\
&\|\alpha_{ii}\|^2 = 4K + 3\phi - c, \quad \langle \alpha_{ii}, \alpha_{jj} \rangle = 2K + \phi - c,\\
&\|H\|^2 = \frac{2(n+1)}{n}K + \frac{(n+2)}{n}\phi - c,\\
&\langle H, \alpha_{ij} \rangle = 0, \quad \langle H, \alpha_{ii} - \alpha_{jj} \rangle = 0,\\
&\langle \alpha_{ij}, \alpha_{kk} \rangle = \langle \alpha_{ij}, \alpha_{ik} \rangle = \langle \alpha_{ij}, \alpha_{ii} \rangle = \langle \alpha_{ij}, \alpha_{kl} \rangle = 0,\\
&\|\alpha_{ij}\|^2 = K + \phi,\\
&\|\alpha_{ii} - \alpha_{jj}\|^2 = 4(K + \phi),
\end{aligned}\right\} \quad (7)$$

for $i$, $j$, $k$ distinct and all $l$. From the first equation above, it follows that $\langle R(X, Y)Z, X \rangle = 0$ for all orthonormal vectors $X, Y, Z \in T_xM$ and this, together with Schur's lemma implies that $M$ has constant curvature $K$, and $K + \phi > 0$, again by the above. This proves 1(a). From the second equation above we have

$$\|\alpha(X, X)\|^2 = 4K + 3\phi - c = \lambda^2 > 0,$$

for all unit vectors $X$, and so $f$ is $\lambda$-isotropic. The second part of 1(b) and 1(c) follow directly again from the formulas above. For part 2 we observe that the formulas above are still valid and the conclusion follows easily.

We observe that if $\phi \leqslant 0$, then $M$ has constant positive curvature, so, if $M$ is complete, it is compact. For such a submanifold we have the following:

**Theorem 5.2.** *Let $f : M^n \to Q^N(c)$, $n \geqslant 3$, be a P.P. immersion of a complete manifold with $\phi \leqslant 0$ and $\dim N_1(x) = \frac{1}{2}n(n+1)$. Then $f$ is parallel. In particular it is a Veronese embedding into some totally umbilic hypersurface of a totally geodesic $Q^{(1/2)n(n+3)}(c)$.*

*Proof.* As observed above $M$ is compact. We will show that $\phi = 0$ computing the Laplacian of a suitable function and using the Hopf lemma. Let us fix some notations. Let $\{e_1, \ldots, e_n, e_{n+1}, \ldots, e_N\}$ be a local orthonormal frame field, adapted to the immersion, with $e_{n+1} = \frac{H}{\|H\|}, e_{n+2} = \frac{\alpha_{11} - \alpha_{22}}{\|\alpha_{11} - \alpha_{22}\|}, \ldots, e_{n(n+3)/2} = \frac{\alpha_{(n-1)n}}{\|\alpha_{(n-1)n}\|}$ (see the set $\gamma$ above). We write $h_{ij}^{\sigma} := \langle \alpha_{ij}, e_\sigma \rangle$ for the components of the second fundamental form,

$i, j \in \{1, \dots, n\}$, $\sigma \in \{n+1, \dots, N\}$, $h^\sigma_{ijk} := \langle (\nabla_{e_k}\alpha)(e_i, e_j), e_\sigma \rangle = \nabla_{e_k} h^\sigma_{ij}$ for the components of the covariant derivative of $\alpha$, and $h^\sigma_{ijkl} := \langle (\nabla_{e_l}\nabla_{e_k}\alpha)(e_i, e_j), e_\sigma \rangle = \nabla_{e_l} h^\sigma_{ijk} = \nabla_{e_l}\nabla_{e_k} h^\sigma_{ij}$ for the components of the second derivative of $\alpha$. Then $f$ is P.P. if and only if

$$h^\sigma_{ijkl} = h^\sigma_{ijlk} + \phi[\delta_{ki} h^\sigma_{lj} - \delta_{li} h^\sigma_{kj} + \delta_{kj} h^\sigma_{il} - \delta_{lj} h^\sigma_{ik}]. \tag{8}$$

Consider now the function $g := \|\alpha\|^2 - n\|H\|^2 = (n-1)(n+2)(K+\phi)$, where $\|\alpha\|^2 = \sum_{ij\sigma} (h^\sigma_{ij})^2$ is the length of the second fundamental form. Then, setting $H^\sigma = \langle H, e_\sigma \rangle$, $A_\sigma = A_{e_\sigma}$, we have (see [9], pp. 90–91)

$$\begin{aligned} \frac{1}{2}\Delta(g) = {} & \sum_{ij\sigma} h^\sigma_{ij} (\nabla_{e_i}\nabla_{e_j} H^\sigma) + nc(\|\alpha\|^2 - n\|H\|^2) \\ & - \sum_{\sigma\tau} [(\mathrm{trace}(A_\sigma \circ A_{tau}))^2 - \|[A_\sigma, A_\tau]\|^2 + nH^\tau \,\mathrm{trace}(A_\sigma \circ A_\tau \circ A_\sigma) \\ & + \|\nabla\alpha\|^2 - n\|H\|\Delta(\|H\|) - n\|\nabla(\|H\|)\|^2, \end{aligned} \tag{9}$$

where $\|\nabla\alpha\|^2 = \sum_{ijk\sigma} (h^\sigma_{ijk})^2$. By the pseudo-umbilicity of $f$, the first term of the sum reduces to $n\|H\|\Delta(\|H\|)$ and:

$$\|\nabla\alpha\|^2 = \sum_{ijk,\, i \neq j} \sum_{\sigma > n+1} (h^\sigma_{ijk})^2 + \sum_{ijk,\, i \neq j} (h^{n+1}_{ijk})^2 + n\|\nabla(\|H\|)\|^2. \tag{10}$$

By the choice of the frame and equations (7), the Weingarten operators of the immersion are given by:

$$\begin{aligned} A_{(1,1)} &= \|H\| \,\mathrm{Id}, \\ A_{(i,i)} &= (K+\phi)^{1/2}(E_{11} - E_{ii}), \quad i = 2, \dots, n, \\ A_{(i,j)} &= (K+\phi)^{1/2}(E_{ij} + E_{ji}), \quad 1 \leqslant i < j \leqslant n, \\ A_\sigma &= 0, \qquad \frac{1}{2}(n+1)(n+2) < \sigma \leqslant N, \end{aligned}$$

where $(i, j) = \min\{i, j\} + \frac{1}{2}|i - j|(2n + 1 - |i - j|) + n$, and $E_{ij}$ is the matrix with 1 in the entry $i, j$ and zero elsewhere. From this we get

$$\sum_{\sigma,\tau} [\mathrm{trace}(A_\sigma \circ A_\tau)]^2 = n^2\|H\|^4 + 2(n-1)(n+2)(K+\phi)^2,$$

$$\sum_{\sigma,\tau} \|[A_\sigma, A_\tau]\|^2 = 2n(n-1)(n+2)(K+\phi)^2,$$

$$\sum_{\sigma,\tau} H^\tau \,\mathrm{trace}(A_\sigma \circ A_\tau \circ A_\sigma) = n^2\|H\|^4 + n(n-1)(n+2)(K+\phi)\|H\|^2.$$

Substituting in (5), and using (6), we get:

$$\frac{1}{2}\Delta(g) = -n(n-1)(n+2)(K+\phi)\phi + \sum_{i,j,k}\sum_{\sigma>n+1}(h^{\sigma}_{ijk})^2 + \sum_{i,j,k}(h^{n+1}_{ijk})^2. \quad (11)$$

Since $\phi \leqslant 0$, we have $\Delta(g) \geqslant 0$ and hence, since $M$ is compact, $\Delta(g) = 0$. In particular, from (8), $\phi = 0$ and $\nabla\alpha = 0$.

**Remark 5.3.** It is natural to ask if the compactness condition is essential in the theorem above, at least to conclude that $\phi$ is constant. In fact we can prove that this is the case if $n = 3, N = 9$. However our proof involves such lengthy computations that is practically impossible to generalize it to higher dimensions even using computer programming.

## 6 Pseudo-parallel immersions and Jordan triple systems

Let $\mathbb{H}$ be a real $N$-dimensional inner product space, $\mathbb{E} \subset \mathbb{H}$ an $n$-dimensional subspace and $a : \mathbb{E} \oplus \mathbb{E} \to \mathbb{E}^{\perp}$ be a symmetric bilinear map. For $\xi \in \mathbb{E}^{\perp}, X, Y \in \mathbb{E}$, we define:

1. $a_{\xi} : \mathbb{E} \to \mathbb{E}$, $\langle a_{\xi}X, Y\rangle = \langle a(X, Y), \xi\rangle$,
2. $r^{\perp}(X, Y) : \mathbb{E}^{\perp} \to \mathbb{E}^{\perp}$, $r^{\perp}(X, Y)\xi = a(X, a_{\xi}Y) - a(Y, a_{\xi}X)$.

Let $c \in \mathbb{R}$. The *triple system for* $(\mathbb{H}, c)$ *with initial data* $(\mathbb{E}, a)$ is the bilinear map

$$l : \mathbb{E} \oplus \mathbb{E} \to \mathrm{End}(\mathbb{E}), \quad l(X, Y) = s(X, Y) + r(X, Y)$$

where

$$s(X, Y)Z = c\langle X, Y\rangle Z + a_{a(X,Y)}Z, \quad r(X, Y)Z = s(Y, Z)X - s(X, Z)Y.$$

We recall that the above triple system is called a *Jordan triple system*, J.T.S. for short, if

$$r^{\perp}(X, Y)a(Z, W) = a(r(X, Y)Z, W) + a(Z, a(X, Y)W).$$

Let $f : M^n \to Q^N(c)$ be an isometric immersion with second fundamental form $\alpha$. For $x \in M$ we consider the triple system for $(\mathbb{H} = T_xQ^N(c), c)$ with initial data $(\mathbb{E} = T_xM, \alpha)$. Using the basic equations, the above triple system may be written as

$$l(X, Y) = c\langle X, Y\rangle \mathrm{Id} + A_{\alpha(X,Y)} + R(X, Y).$$

It follows easily that the above system is a J.T.S. if and only if the immersion is S.P. For P.P. immersions we have a similar characterization. Consider the triple system $l_\phi$ for $(\mathbb{H} = T_x Q^N(c), c + \phi)$ with initial data $(\mathbb{E} = T_x M, \alpha)$. This triple system may be written as

$$l_\phi(X, Y) = (c + \phi)\langle X, Y\rangle \operatorname{Id} + A_{\alpha(X, Y)} + R(X, Y) - \phi(X \wedge Y).$$

A simple calculation gives

**Lemma 6.1.** *The immersion is P.P. if and only if the triple system $l_\phi$ is a J.T.S.*

The above fact has several interesting consequences and we will discuss two of them. The first, punctual in nature, is similar to the semi-parallel case. The second, global in nature, seems new even for the semi-parallel case.

**Lemma 6.2.** *Let $f : M^n \to Q^N(c)$ be a P.P. immersion. If $f$ is pseudo-umbilic, then for every $p$ in $M$ and $X, Y \in T_p M$ we have*

1. $(c + \phi(p) + \|H\|^2)((c + \phi(p))\langle X, Y\rangle^2 + \|\alpha(X, Y)\|^2) \geqslant 0,$
2. $(c + \phi(p) + \|H\|^2)\langle R(X, Y)X - \phi(p)(X \wedge Y)X, Y\rangle \leqslant 0.$

*Moreover equality holds in the first (respectively in the second) inequality if and only if $A_{\alpha(X, Y)} = -(c + \phi(p))\langle X, Y\rangle \operatorname{Id}_{T_pM}$ (respectively $R(X, Y) = \phi(p)(X \wedge Y)$).*

*Proof.* Recall that $f$ is *pseudo-umbilic* if

$$\langle \alpha(X, Y), H\rangle = \langle X, Y\rangle\langle H, H\rangle,$$

for all tangent vectors $X, Y$. Since $l_\phi = s + r$ is a J.T.S., it follows from Theorem 1, pp. 268 of [4] that

$$\operatorname{trace}(s(s(X, Y)X, Y)) \geqslant 0 \quad \text{and} \quad \operatorname{trace}(s(r(X, Y)X, Y)) \leqslant 0 \tag{12}$$

for every $p \in M$ and $X, Y \in T_p M$ and equalities hold if and only if $s(X, Y) = 0$ and $r(X, Y) = 0$, respectively. Thus if $\{e_1, \ldots, e_n\}$ is an orthonormal basis of $T_p M$ we have

$$\begin{aligned}
\frac{1}{n}\operatorname{trace}(s(s(X, Y)X, Y)) &= \frac{1}{n}\sum_{i=1}^{n}\langle s(s(X, Y)X, Y)e_i, e_i\rangle \\
&= (c + \phi(p))\langle s(X, Y)X, Y\rangle + \langle \alpha(s(X, Y)X, Y), H\rangle \\
&= (c + \phi(p))\langle X, Y\rangle^2 + (c + \phi(p))\|\alpha(X, Y)\|^2 \\
&\quad + (c + \phi(p))\langle X, Y\rangle\langle \alpha(X, Y), H\rangle + \langle A_{\alpha(X, Y)}X, A_H Y\rangle \\
&= (c + \phi(p) + \|H\|^2)((c + \phi(p))\langle X, Y\rangle^2 + \|\alpha(X, Y)\|^2),
\end{aligned}$$

which gives the first inequality. The second one follows from analogous computations starting from $\operatorname{trace}(s(r(X, Y)X, Y)) \leqslant 0$.

**Corollary 6.3.** *Let* $f : M^n \to Q^N(c)$ *be a P.P. immersion. If* $f$ *is minimal and* $c + \phi \leqslant 0$, *then* $f$ *is totally geodesic.*

**Theorem 6.4.** *Let* $f : M^n \to Q^N(c)$ *be a P.P. immersion,* $M^n$ *complete and simply connected, with* $c + \phi > 0$, $\phi \geqslant 0$. *Then* $M^n$ *is a Riemannian product of manifolds of the following type*:

1. *Manifolds homeomorphic to spheres*,
2. *Manifolds diffeomorphic to Euclidean spaces*,
3. *Manifolds biholomorphic to complex projective planes*,
4. *Symmetric spaces of compact type.*

*Moreover, if* $\phi$ *is not identically zero, then* $M^n$ *is of the first or second type.*

*Proof.* It is known that for all J.T.S. $l$ with positive constant $c$, there is a standard embedding of a symmetric R-space $\bar{f} : \overline{M}^n \to Q^N(c)$ and a point $x \in \overline{M}^n$ such that the associated J.T.S. at that point is the given one (see [24]). Applying this fact to our situations we get that, for $p \in M^n$, the system $l_{\phi(p)}$ is realizable as the J.T.S. of a symmetric R-space in $S^N(c + \phi(p))$. In particular the operator $\bar{R} = R - \phi\,\mathrm{Id} : \Lambda^2 T_p M \to \Lambda^2 T_p M$ is the curvature operator of a symmetric space of compact type, hence a non-negative operator. In particular $R = \bar{R} + \phi(p)$ is non-negative, and positive if $\phi(p)$ is positive. The conclusion follows from the classification of complete simply-connected manifolds with non-negative curvature operator (see [MN] for a survey on the subject).

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