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Scalar curvature of definable Alexandrov spaces

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Abstract. To any compact set definable in an o-minimal structure, we associate a signed measure, called scalar curvature measure. This generalizes the concept of scalar curvature on Riemannian manifolds. The main result states that, if the definable set is an Alexandrov space with curvature bounded below by κ, then the scalar curvature measure is bounded below by $\kappa m(m-1)\,\mathrm{vol}_m(-)$, where m is the dimension of the space and $\mathrm{vol}_m(-)$ the m-dimensional volume. This is a non-trivial generalization of a fact from differential geometry. The proof combines techniques from o-minimal theory and from Alexandrov space theory. The background of the definition of scalar curvature measure is given in the second part of the paper, where it is related to integral geometry and expressed by geometric quantities.

1 Introduction

1.1 Plan of the paper and main results. The theory of o-minimal structures is a powerful generalization of the theory of subanalytic sets. In the last few years, much progress has been made on this subject leading on the one hand to new examples of o-minimal structures and on the other hand to many interesting properties inherent to all such structures. We are mainly interested in the inner geometric properties and quantities, i.e. those that do not depend on an embedding of the set in a Euclidean space, but only on the set as a metric space. One such quantity is the volume of the set, another example is the Euler characteristic. The aim of this paper is to show that there is another inner quantity of an o-minimal set, a (signed) measure called "scalar curvature measure", which behaves in many respects like the scalar curvature of a smooth Riemannian manifold.

Metric differential geometry has seen a growing interest in the last few years. Alexandrov spaces were known for a long time, but their importance became clear in the fundamental work [6]. The set of Riemannian manifolds with given dimension, bounded diameter and sectional curvature bounded below by some fixed number κ is not compact with respect to the Gromov–Hausdorff metric. On the other hand, Alexandrov's condition is a restatement of the condition of bounded sectional curvature which makes sense for each length space and leads to a compact set of spaces, called Alexandrov spaces. The study of Alexandrov spaces yielded many new results which even in the smooth case were not known before.

The main theorem of this paper is a non-trivial generalization of the following easy fact from differential geometry: If an m-dimensional Riemannian manifold has sectional curvature bounded from below by κ, then its scalar curvature is bounded from below by $\kappa m(m-1)$. With the right interpretation, this will remain true in the setting of o-minimal structures. More precisely, we have the following (for the definition of $\mathrm{scal}(S,-)$ see Subsection 1.2 or Theorem 1.2)

Main Theorem 1.1. *Let S be a compact connected definable set of dimension m, which is an Alexandrov space with sectional curvature $\geqslant \kappa$ (with respect to the inner metric). Then* $\mathrm{scal}(S,-) \geqslant \kappa m(m-1)\,\mathrm{vol}(-)$, *i.e. for each Borel set $U \subset S$ we have* $\mathrm{scal}(S,U) \geqslant \kappa m(m-1)\,\mathrm{vol}(U)$.

Remarks. This theorem is one reason why we speak of "scalar curvature measure". A second reason is that if S happens to be a Riemannian manifold, then $\mathrm{scal}(S,-)$ is nothing else than the integral over the usual scalar curvature.

Under some minor (and necessary) topological restrictions, an analogous theorem holds true if the set has sectional curvature bounded from above by κ in the sense of CAT-spaces (see [3] for these spaces). Then $\mathrm{scal}(S,-) \leqslant \kappa m(m-1)\,\mathrm{vol}(-)$.

Furthermore, it can be shown that the measure $\mathrm{scal}(S,-)$ depends only on S as a metric space and not on the embedding of S in a Euclidean space. The proof of these last two assertions will be presented in another paper.

The plan of this paper is the following. We begin by recalling in Subsection 1.2 the basic definitions of o-minimal structures and of Alexandrov spaces. Tame stratifications are introduced for technical reasons. Lipschitz–Killing curvature measures can be defined in the setting of o-minimal structures as was seen by Bröcker–Kuppe [4] and Fu [13]. One of these Lipschitz–Killing curvature measures will be interpreted as scalar curvature measure for definable sets. The definition is given in Definition 1.9. This measure is very closely related to the inner geometry of the set and can be expressed by geometric terms. More precisely, we will show in Section 3 the following theorem:

Theorem 1.2. *Let S be a connected compact definable set of dimension m with a tame stratification $S = \bigcup_i X^i$ and $U \subset S$ a Borel subset. Then*

$$\begin{aligned}\mathrm{scal}(S,U) &= \sum_{X^m} \int_{U\cap X^m} s(x)\,d\mathrm{vol}_m(x)\\ &\quad + 2\sum_{X^{m-1}} \int_{U\cap X^{m-1}} \sum_{\alpha=1}^{k} \mathrm{tr}\, II_{w_\alpha}\, d\mathrm{vol}_{m-1}(x)\\ &\quad + 4\pi \sum_{X^{m-2}} \int_{U\cap X^{m-2}} \left(\frac{1}{2} + (-1)^m \frac{\chi_{\mathrm{lok}}(S,x)}{2} - \theta_m(S,x)\right) d\mathrm{vol}_{m-2}(x).\end{aligned}$$

The summation is to be taken over all strata of codimensions 0, 1 *and* 2 *respectively.*

The vectors $w_1, w_2, \dots, w_k$ *are the normal vectors of a codimension* 1 *stratum* X^{m-1} *in direction of the highest dimensional strata and* $\operatorname{tr} II_{w_\alpha}$ *is the trace of the second fundamental form of* X^{m-1} *in direction* w_α ($II_{w_\alpha} = -\nabla w_\alpha$). *By* $\chi_{\text{lok}}(S, x)$ *we denote the local Euler characteristic of* S *at* x *and by* $\theta_m(S, x)$ *the* m*-dimensional density of* S *at* x.

The reader mainly interested in Alexandrov spaces can take this as a definition of scalar curvature measure. Some of the terms will be explained later in the paper. The proof of the main theorem only uses the above formula. However, we want to point out that the motivation for defining scalar curvature measure the way we do comes from integral geometry.

In Section 2, we are going to prove the main theorem. This is done for each stratum dimension separately. Strata of codimension 0 can be treated as in the smooth case. Strata of codimension 2 can be easily handled using facts about Alexandrov spaces and some topological considerations. Strata of codimension 1 present more difficulties. We have to investigate the metric structure of the definable set near such a stratum. Using o-minimal theory and stratifications, we will get bounds for Alexandrov angles between certain geodesics. These geodesics can be used to construct, with the help of Alexandrov space theory, some points on the unit sphere with pairwise "big" distance. Counting volumes of sphere sections finally yields the result.

The proof of Theorem 1.2 can be found in Section 3. Again, we argue for each stratum dimension separately, the most difficult case being strata of codimension 2. We have to prove several facts about the density of definable sets in order to relate the inner geometry of the set to the exterior geometric terms. A result of independent interest is the normal section formula for densities (Theorem 3.7).

1.2 Basic definitions. For the convenience of the reader, we collect some basic definitions that will be used in the sequel.

The unique simply connected complete space form of constant curvature κ and dimension m is denoted H_κ^m. We refer to it as the "κ-plane".

Definition 1.3. An *Alexandrov space with curvature bounded below by* κ is a locally complete metric space M with the following two properties:

a) The metric of M is intrinsic, that is for any $x, y \in M, \delta > 0$ there is a finite sequence of points $z_0 = x, z_1, \dots, z_k = y$ such that $d(z_i, z_{i+1}) < \delta$ for $i = 0, \dots, k-1$ and $\sum_{i=0}^{k-1} d(z_i, z_{i+1}) < d(x, y) + \delta$.

b) For each point $x \in M$ there is a neighborhood U_x such that for any four distinct points P, A, B, C in U_x we have the inequality $\angle APB + \angle BPC + \angle CPA \leqslant 2\pi$. Here, $\angle APB$ denotes the angle at the vertex $\tilde{P}$ of a triangle $\tilde{P}\tilde{A}\tilde{B}$ in the κ-plane with side lengths $d(\tilde{P}, \tilde{A}) = d(P, A)$, $d(\tilde{P}, \tilde{B}) = d(P, B)$, $d(\tilde{A}, \tilde{B}) = d(A, B)$, and the other angles are defined in a similar way.

Since the metric spaces we will consider here are always intrinsic, we refer to the second condition as Alexandrov's condition. Alexandrov spaces are a generalization of the concept of manifolds with sectional curvature bounded below to metric spaces. More precisely, we have:

Toponogov's Theorem 1.4. *A complete Riemannian manifold* (M, g) *is an Alexandrov space with curvature* $\geqslant \kappa$ *if and only if its sectional curvature is* $\geqslant \kappa$.

Alexandrov spaces are important as limits of sequences of manifolds with bounded sectional curvature: On the set MET of metric spaces there is a distance, called Gromov–Hausdorff distance d_{G-H}. Riemannian manifolds of fixed dimension and sectional curvature $\geqslant \kappa$ form a non-compact subset. Boundary points can be spaces with difficult singularities. Even the dimension can be smaller (collapsing). On the other hand, the set of Alexandrov spaces with dimension $\leqslant n$, curvature $\geqslant \kappa$ and diameter $\leqslant D$ is a compact subset of (MET, d_{G-H}). Sequences of Riemannian manifolds satisfying these inequalities (with uniform bounds n, D, κ) have therefore subsequences converging to Alexandrov spaces. The study of the latter is therefore important even when one is mainly interested in the smooth case. Our basic reference for Alexandrov spaces is the very good survey article [6].

Next, we come to the the theory of o-minimal structures.

Definition 1.5. An *o-minimal structure* is a sequence $\sigma = (\sigma_n)_{n=1,2,3,\ldots}$ such that

a) σ_n is a Boolean algebra of subsets of $\mathbb{R}^n$ and $\mathbb{R}^n \in \sigma_n$.

b) For $1 \leqslant i < j \leqslant n$ the set $\{x_i = x_j\}$ is contained in σ_n.

c) If $S \in \sigma_n$ then $S \times \mathbb{R} \in \sigma_{n+1}$ and $\mathbb{R} \times S \in \sigma_{n+1}$.

d) If $S \in \sigma_{n+1}$ then $\pi(S) \in \sigma_n$ where $\pi : \mathbb{R}^{n+1} \mapsto \mathbb{R}^n$ is the projection onto the first n coordinates.

e) The graphs of addition and multiplication belong to σ_3. (Equivalently: algebraic subsets belong to σ)

f) σ_1 consists exactly of all finite unions of points and intervals.

The smallest example of an o-minimal structure is the set of semi-algebraic sets. Other examples are globally subanalytic sets or sets definable over $\mathbb{R}_{\exp}$. The basic reference for o-minimal structures is [22], see also [10].

In the rest of the paper, we fix an o-minimal structure and refer to its elements as *definable sets*.

A closed definable set is embedded in a Euclidean space and has an induced length metric. We call it the *inner metric*.

Definition 1.6. We call a Whitney stratification $S = \bigcup X^i \subset \mathbb{R}^n$ of a compact definable set *tame* if there is a stratification $\bigcup N^\mu$ of the normal space $\mathrm{Nor}_e\, S \subset \mathbb{R}^n \times S^{n-1} \subset \mathbb{R}^{2n}$ satisfying the following two conditions:

a) $\{N^\mu\}$ is compatible with the sets $\mathrm{Nor}_e\, T_{X^i} X^j$, $X^i \subset \overline{X^j}$.

b) The projection $\pi : \mathrm{Nor}_e\, S \mapsto S$, $(x, v_e) \mapsto x$ is a submersion on each stratum.

Tame stratifications (and tame sets) are introduced in [16]. There the interested reader can find many facts about them. We only need here that each definable set admits a tame stratification, that the Lipschitz–Killing curvatures can be defined for such a stratification and do not depend on it.

We remark that Lipschitz–Killing curvatures are defined in various situations and under various names. Several authors introduced them in the case of convex sets, manifolds (with boundary), sets with positive reach and piecewise linear spaces. The case of definable sets presents more difficulties, as multiplicities have to be counted correctly. This is where stratified Morse theory enters. Without giving any details, we sketch the definition of Lipschitz–Killing curvatures and refer to [16] for the general theory. We remark that there is a different approach due to Fu [13] using geometric measure theory.

First we have to define an index β:

Definition 1.7. Let $S \subset \mathbb{R}^n$ be a compact definable set with a tame stratification $S = \bigcup X^i$. Let $(x, v) \in \operatorname{Nor} S$. We set

$$\beta(x, v) := 1 - \chi(O_{\delta,\theta}(x, v) \cap S), \quad 0 < \delta \ll \theta \ll 1$$

where

$$O_{\delta,\theta}(x, v) := \left\{ x + w \,\middle|\, \angle(v, w) \leqslant \frac{\pi}{2} - \theta, \left\langle w, \frac{v}{\|v\|} \right\rangle = \delta \tan \theta \right\}.$$

The index $\hat{\beta}(x, v)$ is defined analogously, where

$$\hat{O}_{\delta,\theta}(x, v) := \left\{ x + w \,\middle|\, \angle(v, w) \leqslant \frac{\pi}{2} - \theta, \|w\| = \frac{\delta}{\cos\theta} \right\}.$$

Remarks. For θ and δ sufficiently small, the above Euler characteristic is constant.

In [16], Lipschitz–Killing curvatures are defined using the index β. On a Whitney stratified space both indices are equal almost everywhere. Calculations with $\hat{\beta}$ can be easier, so we use this second index if necessary. The indices β and $\hat{\beta}$ can be interpreted as the normal Morse indices of some height or distance function respectively (see [14] for stratified Morse theory).

Using the map $E : \operatorname{Nor} S \mapsto \mathbb{R}^n$, $(x, v) \mapsto x + v$, one gets a volume form $d\operatorname{Vol}_{\operatorname{Nor} S} := E^* \, d\operatorname{Vol}_{\mathbb{R}^n}$ on $\operatorname{Nor} S$. We let $Tub_r(S, B)$ denote the set of normal vectors with base-points in some Borel set $B \subseteq S$ and lengths bounded by r.

Let b_k denote the volume of a k-dimensional Euclidean unit ball.

Proposition and Definition 1.8. *In the setting as above,*

$$\operatorname{Vol}_\beta Tub_r(S, B) := \int_{Tub_r(S,B)} \beta(x, v) \, d\operatorname{Vol}_{\operatorname{Nor} S}(x, v)$$

is a well defined polynomial in r:

$$\mathrm{Vol}_\beta\, Tub_r(S,B) =: \sum_{k=0}^{n} b_k \Lambda_{n-k}(S,B) r^k.$$

The (signed) measures $\Lambda_i(S,-)$ are called Lipschitz–Killing curvatures *and are independent of the stratification.*

If the dimension of S is m, then $\Lambda_k(S,-)$ vanishes identically for $k > m$, and $\Lambda_m(S,-)$ is the volume measure. $\Lambda_0(S,-)$ is the Gauß–Bonnet measure, in particular $\Lambda_0(S,S) = \chi(S)$, the Euler characteristic of S.

In this paper we are concerned with $\Lambda_{m-2}(S,-)$ which plays the role of scalar curvature (up to some factor). The restriction of this measure to strata of dimension smaller than $m-2$ vanishes identically, therefore our proofs consist of three parts, one for each stratum dimension. Note that it follows from the general theory that Λ_{m-2} is continuous on each stratum with respect to the corresponding volume measure.

Definition 1.9. Let S be a compact definable set of dimension m. Then the signed measure

$$\mathrm{scal}(S,-) := 4\pi\Lambda_{m-2}(S,-)$$

is called the *scalar curvature measure* of S.

Remark. The name *scalar curvature* is justified by at least two facts. If S happens to be a Riemannian manifold, this measure yields the integral over the usual scalar curvature. The second justification comes from our main theorem which is a nontrivial generalization of an easy fact from differential geometry relating sectional and scalar curvature.

2 Proof of the Main Theorem

In this section, we suppose S to be a compact connected definable set of dimension m which is an Alexandrov space with sectional curvature $\geqslant \kappa$. We have to show

$$\mathrm{scal}(S,U) \geqslant \kappa m(m-1)\,\mathrm{vol}(U)$$

for each Borel subset $U \subset S$.

From the additivity of both sides and the fact that both sides vanish on strata of dimension less than $m-2$, we see that it suffices to show that for each Borel set $U \subset X^k$, $k = m-2, m-1, m$ we have $\mathrm{scal}(S,U) \geqslant \kappa m(m-1)\,\mathrm{vol}(U)$. This will be done in Propositions 2.3, 2.6 and 2.10. First of all, we need some topological arguments showing that along strata of codimension at most 2, the topology of a definable Alexandrov space is not complicated.

2.1 Topological consequences. Recall that a boundary point of an Alexandrov space of finite dimension is defined inductively: a one-dimensional Alexandrov space is just a 1-dimensional manifold and the boundary in Alexandrov's sense is its usual boundary. Otherwise a point is a boundary point if its space of directions has non-empty boundary. The boundary of an Alexandrov space is a closed subset and is characterized by local topological properties ([6], [20]).

Lemma 2.1. *An $(m-1)$-stratum X^{m-1} lies in the boundary of exactly one or two m-strata. In the first case, each point of the stratum is a boundary point.*

Proof. Since the Hausdorff dimension of S equals m, the burst indices near each point are m (see [6], Corollary 6.5). This excludes the case that there is no m-stratum neighboring our given stratum.

If there are 3 or more such m-strata, we choose a point $P \in X^{m-1}$. The normal section $(T_P X^{m-1})^{\perp} \cap S$ consists of at least 3 one-dimensional curves γ_1, γ_2, γ_3 that can be parameterised by arc-length. For fixed positive t set $A(t) = \gamma_1(t)$, $B(t) = \gamma_2(t)$, $C(t) = \gamma_3(t)$.

Each path between two of these three points has to go through X^{m-1}. With estimates similar to those that we will use quite often in Section 2.3, we see that $\lim_{t \to 0} \angle(A(t), P, B(t)) = \pi$ and accordingly for the other angles. This contradicts Alexandrov's condition for the quadruple $(P, A(t), B(t), C(t))$ for t sufficiently small and shows the first statement.

Next, suppose there is exactly one m-stratum neighboring X^{m-1}. A neighborhood of a point $P \in X^{m-1}$ is homeomorphic to a half-space of dimension m. Therefore, P is a boundary point in Alexandrov's sense ([6], Theorem 13.3 a). □

Lemma 2.2. *For a stratum X^{m-2} of codimension 2 and a point $P \in X^{m-2}$ there are the following two possibilities*:

a) *The local Euler characteristic of the normal section at P is* 0 *and the point is a boundary point in Alexandrov's sense.*

b) *The local Euler characteristic of the normal section at P is* 1.

Proof. The normal section at P consists of a finite union of two-dimensional sets which look like two one-dimensional curves emanating from P and a two-dimensional stratum between them. Each two of them can be neighboring (this means that their boundaries contain a common one-dimensional curve) or not. It is impossible for three of them to have the same one-dimensional curve on their boundary, since this would lead to a contradiction to the above lemma. Hence, we can partition the set of these two-dimensional sets in equivalence classes for the relation "being connected via a chain of neighboring sets". Equivalence classes are sequences of the form $(A_1, A_2, \ldots, A_l)$, where two consecutive sets are neighboring and A_1 and A_l can be neighboring or not.

Suppose there are three or more of such equivalence classes. Then with similar arguments as in the proof of Lemma 2.1 we would get a contradiction to Alexandrov's condition.

Suppose there is exactly one such equivalence class $(A_1, A_2, \dots, A_l)$. If A_1 and A_l are neighboring, then a neighborhood of P is homotopic to a Euclidean space $\mathbb{R}^2$ and the local Euler characteristic is 1. If A_1 and A_l are not neighboring, a neighborhood of P is homotopic to a two-dimensional half-space and the local Euler characteristic is 0. The curve on the boundary of A_1 which does not lie in the boundary of A_2 consists of boundary points since it corresponds to an $(m-1)$-stratum on the boundary of exactly one m-stratum. Since the boundary is closed, P must be a boundary point.

It remains the case that there are exactly two such equivalence classes. We denote them by $A = (A_1, \dots, A_l)$ and $B = (B_1, \dots, B_k)$. If A and B are closed, i.e. A_1 and A_l as well as B_1 and B_k are connected, then $\chi_{\text{lok}} = 1$ and we are in the second case. If A is open and B closed (or vice versa), $\chi_{\text{lok}} = 0$ and P is a boundary point as above.

So we can restrict to the case where A and B are both open. In this case $\chi_{\text{lok}} = -1$. Let α_1 denote the angle of A at P and α_2 the angle of B at P. If $\alpha_1 > 0$, we take its boundary curves, which are those curves which lie on the boundary of A_1 and A_l, but not on the boundary of some other A_i. We parameterise them by arc-length and call them γ_1 and γ_2. Furthermore, let γ_3 be some curve in B. Then, $\lim_{t\to 0} \angle(P, \gamma_1(t), \gamma_2(t)) = \alpha_1 > 0$, $\lim_{t\to 0} \angle(P, \gamma_{1,2}(t), \gamma_3(t)) = \pi$. This establishes a contradiction to Alexandrov's condition. Consequently, $\alpha_1 = 0$ and analogously $\alpha_2 = 0$.

It follows that the angle of the normal section vanishes and, using Proposition 3.4, we see that the density of the normal section of S at P is zero. The local Euler characteristic is constant along the stratum X^{m-2} (by Thom's Isotopy Lemma) and consequently the density of the normal section is zero along the stratum. From the normal section formula 3.7 we will be able to deduce that the density of the set S itself vanishes almost everywhere along the stratum X^{m-2}. In particular, there is at least one point $P' \in X^{m-2}$ such that $\theta(S, P') = 0$. This is a contradiction to the fact that the density at each point of an Alexandrov space is strictly positive (see Proposition 2.8). We therefore see that the case that both A and B are open cannot occur. This finishes the proof of Lemma 2.2. □

Remark. In view of Lemma 2.1 one could conjecture that a definable Alexandrov space is always a topological manifold with boundary, but this would be false. As an example, take CP^2 with a metric of positive sectional curvature ($\geqslant \kappa > 0$). Embed this space isometrically in some $\mathbb{R}^N$ (by Nash) and approximate it by a definable manifold M' with positive sectional curvature. Then embed $\mathbb{R}^N \subset \mathbb{R}^{N+1} = \mathbb{R}^N \times \mathbb{R}$ in the natural way and consider the spherical projection on the R-sphere with center $P = (0, 0, \dots, 0, R)$. By simple continuity arguments, for R big enough, the projection M'' of our M' will have sectional curvature bounded below by $\frac{\kappa}{2}$. We choose R big enough, such that $R^2 \frac{\kappa}{2} > 1$. Then the cone C over M'' with vertex P is definable and a space with curvature $\geqslant 0$ (cf. [6], Proposition 4.2.3.). On the other hand, the local Euler characteristic at P is given by $\chi_{\text{lok}}(C, C \backslash P) = 1 - \chi(M'') = 1 - \chi(CP^2) = -2$. The local Euler characteristic at each point of a topological manifold (with boundary) is -1, 0 or 1. This shows that the constructed set C is not a topological manifold.

2.2 Strata of highest dimension. From Theorem 1.2 we see that the scalar curvature measure on X^m is nothing else than the integral over the usual scalar curvature. Near smooth points, Alexandrov's condition (for some κ) is equivalent to the condition that the sectional curvature K is bounded below by κ. From $K \geqslant \kappa$ it follows that $s \geqslant \kappa m(m-1)$. Hence we have shown:

Proposition 2.3. *Let S be a connected definable Alexandrov space of dimension m and with sectional curvature bounded below by κ, then*

$$\operatorname{scal}|_{X^m}(S,-) \geqslant \kappa m(m-1) \operatorname{vol}|_{X^m}(-)$$

for each m-stratum X^m.

2.3 Strata of codimension 1. As usual, for a function $f : \mathbb{R} \mapsto \mathbb{R}$, we write $f(s) = O(s)$ if $\frac{f(s)}{s}$ stays bounded for $s \to 0$. If moreover $\lim_{s\to 0} \frac{f(s)}{s} = 0$, we write $f(s) = o(s)$.

We suppose first that we are given an $(m-1)$-stratum X^{m-1} in the boundary of exactly two m-strata. The inward normal directions are denoted by w_1, w_2.

Choose some point $P \in X^{m-1}$. On $T = T_P X^{m-1}$ we choose an orthonormal coordinate system. Coordinate lines are mapped under the exponential map of X^{m-1} to mutually orthonormal (at P) curves $x_1, \dots, x_{m-1}$ that are parameterised by arc-length. Using this map ϕ, we must show that

$$h(P) = 2(\operatorname{tr} II_{w_1} + \operatorname{tr} II_{w_2}) = 2\left\langle \sum_{i=1}^{m-1} x_i''(0), w_1 + w_2 \right\rangle$$

is non-negative on X^{m-1}, if S is an Alexandrov space with sectional curvature bounded below by κ. We do the calculation only for the case $\kappa = 0$, the general case following from easy modifications (e.g. replace $\mathbb{R}^m$ with spaces of constant curvature κ and so on). However, we will give some hints how to treat the general case at the end of the subsection.

Let $T^\perp$ denote the affine space of dimension $n - m + 1$ which is orthogonal to T and which passes through P. Then $T^\perp \cap S$ consists (locally) of a union of two definable curves that we parametrise by arc-length. Thus we have functions

$$\gamma_1, \gamma_2 : [0, \varepsilon) \mapsto T^\perp \subset \mathbb{R}^n$$

such that $\gamma_1(0) = \gamma_2(0) = P$. For $\alpha \to 0$ we get the following asymptotic behavior

$$\gamma_j(\alpha) = P + \alpha w_j + r_j(\alpha).$$

Here r_j is a function such that $h_j(\alpha) := \dfrac{\|r_j(\alpha)\|}{\alpha}$ tends to 0. Therefore

$$\gamma_j(\alpha) = P + \alpha w_j + O(\alpha h_j(\alpha)) \tag{1}$$

This relation remains true if we replace h_j by some bigger function that still tends to

0. Hence we can assume, for instance, that h_j is monotonically decreasing for $\alpha \to 0$ and that $h_j(\alpha) \geqslant \alpha^2$. This will be needed later on.

Step 1. First of all, the asymptotic behavior of the angle $\angle(x_i(s), P, x_j(s))$ is estimated for $i \neq j$.

We can write

$$x_i(s) = P + sx_i'(0) + \frac{s^2}{2} x_i''(0) + O(s^3)$$

$$x_j(s) = P + sx_j'(0) + \frac{s^2}{2} x_j''(0) + O(s^3).$$

As these curves are parameterised by arc-length, we have $\langle x'(0), x''(0) \rangle = 0$. Furthermore $d(P, x_i(s)) \leqslant s$, $d(P, x_j(s)) \leqslant s$. On the other hand,

$$d(P, x_i(s))^2 \geqslant d_E(P, x_i(s))^2 = s^2 + O(s^4)$$

hence

$$d(P, x_i(s)) \geqslant s + O(s^2).$$

Accordingly, we find $d(P, x_j(s)) \geqslant s + O(s^2)$. The other distance can be bounded below by Euclidean distance:

$$\begin{aligned} d(x_i(s), x_j(s))^2 &\geqslant d_E(x_i(s), x_j(s))^2 \\ &= \left\| s(x_i'(0) - x_j'(0)) + \frac{s^2}{2}(x_i''(0) - x_j''(0)) + O(s^3) \right\|^2 \\ &= 2s^2 - s^3(\langle x_i', x_j'' \rangle + \langle x_j', x_i'' \rangle) + O(s^4). \end{aligned}$$

From the law of cosines it follows that

$$\begin{aligned} \cos \angle(x_i(s), P, x_j(s)) &= \frac{d(x_i(s), P)^2 + d(x_j(s), P)^2 - d(x_i(s), x_j(s))^2}{2d(x_i(s), P)d(x_j(s), P)} \\ &\leqslant \frac{s^2 + s^2 - 2s^2 + s^3(\langle x_i', x_j'' \rangle + \langle x_j', x_i'' \rangle) + O(s^4)}{2(s + O(s^2))(s + O(s^2))} \\ &= \frac{s}{2}(\langle x_i', x_j'' \rangle + \langle x_j', x_i'' \rangle) + O(s^2). \end{aligned}$$

This shows that

$$\angle(x_i(s), P, x_j(s)) \geqslant \frac{\pi}{2} - \frac{s}{2}(\langle x_i', x_j'' \rangle + \langle x_j', x_i'' \rangle) \pm O(s^2) \tag{2}$$

Analogously,

$$\angle(x_i(-s), P, x_j(s)) \geqslant \frac{\pi}{2} - \frac{s}{2}(\langle -x_i', x_j'' \rangle + \langle x_j', x_i'' \rangle) \pm O(s^2).$$

Step 2. Next, we turn to the asymptotic behavior of the angle $\angle(x_i(s), P, \gamma_j(\alpha))$ with $j = 1, 2$. Since γ_j is parameterised by arc-length and by equation (1) we get

$$\alpha - O(\alpha h(\alpha)^{1/2}) \leqslant d(P, \gamma_j(\alpha)) \leqslant \alpha.$$

Bounding the distance between $x_i(s)$ and $\gamma_j(\alpha)$ below by Euclidean distance yields

$$\begin{aligned} d(x_i(s), \gamma_j(\alpha))^2 &\geqslant d_E(x_i(s), \gamma_j(\alpha))^2 \\ &= \left\| \gamma_j(\alpha) - P - s x_i'(0) - \frac{s^2}{2} x_i''(0) + \frac{s^3}{6} x_i'''(0) + O(s^4) \right\|^2. \end{aligned}$$

By definition, $\gamma_j(\alpha) - P \perp x_i'(0)$. Inserting the asymptotic development for $\gamma_j(\alpha)$ gives us:

$$\begin{aligned} d(x_i(s), \gamma_j(\alpha))^2 \geqslant \alpha^2 + s^2 - \alpha s^2 \langle w_1, x_i''(0) \rangle + O(\alpha^2 h_j(\alpha)) \\ + O(s^4) + O(\alpha s^3) + O(\alpha h_j(\alpha) s^2). \end{aligned}$$

We put $\alpha_j := s^2 h_j(s)^{-1/2} \leqslant s$. Then we have the following estimates:

$$\frac{\alpha_j^2 h_j(\alpha_j)}{\alpha_j s^2} = h_j(s)^{-1/2} h_j(\alpha_j) \leqslant h_j(s)^{1/2} \to 0$$

and

$$\frac{\alpha_j h_j(\alpha_j) s^2}{\alpha_j s^2} = h_j(\alpha_j) \to 0.$$

The two other O-terms behave in the same way. Thus,

$$d(x_i(s), \gamma_j(\alpha_j))^2 \geqslant \alpha_j^2 + s^2 - \alpha_j s^2 \langle w_j, x_i''(0) \rangle + o(\alpha_j s^2).$$

Again, we deduce from the law of cosines that

$$\begin{aligned} \cos \angle(x_i(s), P, \gamma_j(\alpha_i)) &= \frac{d(x_i(s), P)^2 + d(\gamma_j(\alpha), P)^2 - d(x_i(s), \gamma_j(\alpha))^2}{2 d(x_i(s), P) d(\gamma_j(\alpha), P)} \\ &\leqslant \frac{s^2 + \alpha_j^2 - \alpha_j^2 - s^2 + \alpha_j s^2 \langle w_j, x_i''(0) \rangle + o(\alpha_j s^2)}{2(s + O(s^2))(\alpha_j + O(\alpha_j h_j(\alpha_j)^{1/2}))} \\ &= \frac{s}{2} \langle w_j, x_i''(0) \rangle + o(s). \end{aligned}$$

We conclude that

$$\angle(x_i(s), P, \gamma_j(\alpha_j)) \geqslant \frac{\pi}{2} - \frac{s}{2} \langle w_j, x_i''(0) \rangle + o(s). \tag{3}$$

Analogous inequalities hold true if we replace $x_i(s)$ with $x_i(-s)$.

Step 3. For technical reasons which will become clear later on, we need some knowledge about the angles $\angle(x_i(s), P, x_i(-s))$ and $\angle(\gamma_1(\alpha_1), P, \gamma_2(\alpha_2))$. We have

$$\cos \angle(x_i(s), P, x_i(-s)) = -1 + o(1).$$

Here, $o(1)$ denotes a term that tends to 0 for $s \to 0$. Hence,

$$\angle(x_i(s), P, x_i(-s)) = \pi + o(1). \tag{4}$$

The proof is similar to the proofs in Step 1 and Step 2 using the law of cosines and bounding the distance $d(x_i(s), x_i(-s))$ by the Euclidean one.

Analogously,

$$\angle(\gamma_1(\alpha_1), P, \gamma_2(\alpha_2)) = \pi + o(1). \tag{5}$$

Here, the distance between $\gamma_1(\alpha_1)$ and $\gamma_2(\alpha_2)$ is bounded below by the sum of the two Euclidean distances $d_E(\gamma_1(\alpha_1), X^{m-1})$ and $d_E(\gamma_2(\alpha_2), X^{m-1})$. Both terms behave asymptotically as α_1 and α_2 respectively. The rest follows again from the law of cosines.

We have the following two trivial facts:

Lemma 2.4. *Given $2m$ points $(A_1, B_1, A_2, B_2, \ldots, A_m, B_m)$ on the sphere S^{m-1}, suppose that*

$$\begin{aligned} d(A_i, A_j), d(B_i, B_j) &\geqslant \frac{\pi}{2} - \varepsilon \quad \text{for } i \neq j, \\ d(A_i, B_j) &\geqslant \frac{\pi}{2} - \varepsilon \quad \text{for all } i \neq j, \quad \text{and} \\ d(A_i, B_i) &\geqslant \pi - \varepsilon. \end{aligned}$$

Then

$$\begin{aligned} d(A_i, A_j), d(B_i, B_j) &\leqslant \frac{\pi}{2} + 2\varepsilon \quad \text{for } i \neq j \quad \text{and} \\ d(A_i, B_j) &\leqslant \frac{\pi}{2} + 2\varepsilon \quad \text{for all } i \neq j. \end{aligned}$$

Lemma 2.5. *Let $V_m(d_1, \ldots, d_{\binom{m}{2}})$ be the $(m-1)$-dimensional volume of an m-simplex on the sphere S^{m-1} with side lengths $d_1, \ldots, d_{\binom{m}{2}}$. Then, for each $i = 1, \ldots, \binom{m}{2}$*

$$k_m := \frac{\partial}{\partial d_i} V_m(d_1, \ldots, d_{\binom{m}{2}}) \bigg|_{d_1 = \cdots = d_{\binom{m}{2}} = \pi/2} > 0.$$

We want to apply these lemmas to our situation. By [6] there is some map $\Phi : S \mapsto \mathbb{R}^m$ that leaves distances from P invariant and does not decrease the other distances. We denote the image points by $\tilde{P}, \tilde{A}, \ldots$ The ray emanating from $\tilde{P}$ and passing through $\tilde{A}$ intersects the sphere S^{m-1} in exactly one point $\bar{A}$. We have

$$\angle(\bar{A}, \tilde{P}, \bar{B}) = \angle(\tilde{A}, \tilde{P}, \tilde{B}) \geqslant \angle(A, P, B).$$

Next, we set $A_1 = \overline{x_1(s)}$, $B_1 = \overline{x_1(-s)}$, $A_2 = \overline{x_2(s)}$, $B_2 = \overline{x_2(-s)}, \ldots, A_{m-1} = \overline{x_{m-1}(s)}$, $B_{m-1} = \overline{x_{m-1}(-s)}$, $A_m = \overline{\gamma_1(\alpha_1)}$, $B_m = \overline{\gamma_2(\alpha_2)}$ (recall that $\alpha_j = \alpha_j(s)$).

We choose $\varepsilon > 0$ sufficiently small such that the function $(m-1)$-volume is monotonically increasing in all side lengths if these are contained in the interval $\left(\frac{\pi}{2} - 2\varepsilon, \frac{\pi}{2} + 2\varepsilon\right)$. From inequalities (2), (3), (4) and (5) we see that the assumptions of Lemma 2.4 are fulfilled for s sufficiently small. Take an m-simplex which has for each $i = 1, \ldots, m$ either A_i or B_i as a vertex. By Lemma 2.4, all its side lengths are in the interval $\left(\frac{\pi}{2} - 2\varepsilon, \frac{\pi}{2} + 2\varepsilon\right)$ where the volume function is monotonically increasing. The sum of the volumes of these 2^m simplices is exactly the volume of the $(m-1)$-unit-sphere, s_{m-1}, as they form a partition of this sphere.

Let

$$\Psi(x_1(\pm s), \ldots, x_{m-1}(\pm s), \gamma_{1,2}(\alpha_{1,2}))$$

be the volume of the simplex on S^{m-1}, whose side lengths equal the angles of the points $(x_1(\pm s), \ldots, x_{m-1}(\pm s), \gamma_{1,2}(\alpha_{1,2}))$ with P. Then

$$\sum \Psi(x_1(\pm s), \ldots, x_{m-1}(\pm s), \gamma_{1,2}(\alpha)) \leqslant s_{m-1}$$

since the angles at $\tilde{P}$ between the image points $A_1 = \overline{x_1(s)}$, $B_1 = \overline{x_1(-s)}, \ldots A_{m-1} = \overline{x_{m-1}(s)}$, $B_{m-1} = \overline{x_{m-1}(-s)}$, $A_m = \overline{\gamma_1(\alpha_1)}$, $B_m = \overline{\gamma_2(\alpha_2)}$ are not smaller than the angles between the original points and the volume function is increasing.

We calculate $\Psi(x_1(s), \ldots, x_{m-1}(s), \gamma_1(\alpha_1))$:

$$\begin{aligned}
&\Psi(x_1(s), \ldots, x_{m-1}(s), \gamma_1(\alpha_1)) \\
&\quad = V_m(\angle(x_1(s), P, x_2(s)), \angle(x_1(s), P, x_3(s)), \ldots, \angle(x_1(s), P, \gamma_1(\alpha_1)), \ldots) \\
&\quad \geqslant V_m\left(\frac{\pi}{2} - \frac{s}{2}(\langle x_1', x_2'' \rangle + \langle x_2', x_1'' \rangle) \pm O(s^2), \ldots, \frac{\pi}{2} - \frac{s}{2}\langle w_1, x_1''(0) \rangle + o(s), \ldots\right) \\
&\quad = V_m\left(\frac{\pi}{2}, \ldots, \frac{\pi}{2}\right) - k_m \frac{s}{2}((\langle x_1', x_2'' \rangle + \langle x_2', x_1'' \rangle + \cdots + \langle w_1, x_1''(0) \rangle + \cdots)) + o(s).
\end{aligned}$$

Similar inequalities are true for the other combinations (i.e. $x(s)$ replaced by $x(-s)$ or $\gamma_1(\alpha_1)$ replaced by $\gamma_2(\alpha_2)$). Summing up these inequalities, all terms of type $\langle x_1', x_2'' \rangle$ vanish and we get

$$\begin{aligned} s_{m-1} &\geqslant \sum \Psi(x_1(\pm s), \dots, x_{m-1}(\pm s), \gamma_{1,2}(\alpha_{1,2})) \\ &\geqslant 2^m V_m\left(\frac{\pi}{2}, \dots, \frac{\pi}{2}\right) - \sum k_m \frac{s}{2}(\langle w_{1,2}, x_1''(0)\rangle + \cdots) + o(s) \\ &= s_{m-1} - 2^{m-2} k_m s(\langle w_1 + w_2, x_1''(0)\rangle + \cdots + \langle w_1 + w_2, x_{m-1}''(0)\rangle) + o(s). \end{aligned}$$

It follows immediately that

$$\sum_{i=1}^{m-1} \langle w_1 + w_2, x_i''(0)\rangle \geqslant 0.$$

This term is (up to a positive constant) the term we have to integrate when calculating $\mathrm{scal}(S, -)$ on X^{m-1}. Therefore this last measure is non-negative.

Remark. We have done the calculations for $\kappa = 0$. We want to indicate what has to be done if $\kappa \neq 0$. First, just apply the corresponding law of cosines to calculate angles. The asymptotic behavior of the considered angles is the same. This is not surprising since, in the small, space forms of constant curvature are almost the same.

Next, the map Φ will yield points in H_κ^m. Join them to $\tilde{P}$ by geodesics. Then the angles between these geodesics are just the angles between the corresponding points (since we are in a space form of constant curvature). In the tangent space $T_{\tilde{P}} H_\kappa^m$, which is isometric to $\mathbb{R}^m$, the tangents of these geodesics will have angles bounded below according to our formulae. Identifying such a tangent with a point on the sphere, we can apply Lemma 2.5 in exactly the same way as we did. Alternatively, instead of taking $\Phi : S \mapsto H_\kappa^m$, take $\exp_{\tilde{P}}^{-1} \circ \Phi : S \mapsto T_{\tilde{P}} H_\kappa^m = \mathbb{R}^m$. Then the same argumentation as in the case $\kappa = 0$ applies.

Thus we have shown

Proposition 2.6. *If S is an Alexandrov space with sectional curvature $\geqslant \kappa$ and X^{m-1} a stratum in the boundary of exactly two m-strata, then* $\mathrm{scal}(S, -)|_{X^{m-1}} \geqslant 0$.

Corollary 2.7. *If S is an Alexandrov space with sectional curvature $\geqslant \kappa$ and X^{m-1} a stratum in the boundary of exactly one m-stratum, then* $\mathrm{scal}(S, -)|_{X^{m-1}} \geqslant 0$.

Proof of the corollary. Here, the normal section $(T_P X^{m-1})^\perp \cap S$ consists of exactly one curve with direction w. In order to calculate $\mathrm{scal}(S, -)$ on X^{m-1}, we have to integrate over the expression $2\sum_{i=1}^{m-1} \langle w, x_i''(0)\rangle$ (see Theorem 1.2).

By Lemma 2.1, all the points of the stratum are boundary points. We can glue two copies of S along the boundary. Then, X^{m-1} becomes a stratum in the boundary of exactly two m-strata. The corresponding metric space again has sectional curvature bounded below by κ by the doubling theorem of [6]. It follows from Lemma 2.6 that $\sum_{i=1}^{m-1} \langle w, x_i''(0)\rangle \geqslant 0$. Hence, $\mathrm{scal}(S, -) \geqslant 0$ on X^{m-1}. □

2.4 Strata of codimension 2. We recall a result of Shen about the density of Alexandrov spaces [21]:

Proposition 2.8. *Let S be an Alexandrov space with sectional curvature $\geqslant k$ and dimension m, and $P \in S$. Denote the m-dimensional Hausdorff measure by* vol. *Then the density $\theta(S,P) := \lim_{r\to 0} \frac{\operatorname{vol} B_i(P,r)}{b_m r^m}$ exists, is strictly positive and bounded above by* 1. *If P is a boundary point, it is bounded above by $\frac{1}{2}$.*

Remark 2.9. In the case of a definable set of dimension m, all reasonable m-dimensional measures coincide with m-dimensional Hausdorff measure.

From the above proposition we can conclude:

Proposition 2.10. *If S is an Alexandrov space with sectional curvature $\geqslant k$ and of dimension m, and X^{m-2} a stratum of dimension $m-2$, then* $\operatorname{scal}(S,-)|_{X^{m-2}} \geqslant 0$.

Proof. Let $P \in X^{m-2}$. We use Lemma 2.2. The term to be integrated in order to calculate $\operatorname{scal}(S,-)|_{X^{m-2}}$ is by Theorem 1.2 given by the expression

$$4\pi\left(\frac{1}{2}+\frac{1}{2}\chi_{\mathrm{lok}}((T_P X^{m-2})^{\perp}\cap S,P)-\theta(S,P)\right).$$

If $\chi_{\mathrm{lok}}=1$, it equals almost everywhere $4\pi(1-\theta)\geqslant 0$. If $\chi_{\mathrm{lok}}=0$ and P a boundary point, the expression is almost everywhere $4\pi(\frac{1}{2}-\theta(S,P))\geqslant 0$. In each case we integrate over an almost everywhere non-negative function, this shows Proposition 2.10. □

3 Calculation of the scalar curvature measure

Let us first describe the content of this section. The first part (Subsections 3.1, 3.2, 3.3, 3.4) is devoted to a more geometric expression of the scalar curvature measure. One of the problems will be to calculate the scalar curvature measure of a two-dimensional definable set. This yields an expression containing the density of the set, see Subsections 3.5 and 3.6. The main problem will be to relate the density of the normal section of a point to the density of the considered set. This is the so-called normal section formula, which will be proved in Section 3.7.

Putting all these results together yields Theorem 1.2, which expresses the scalar curvature measure as

$$\begin{aligned}\operatorname{scal}(S,U) &= \sum_{X^m}\int_{U\cap X^m} s(x)\,d\operatorname{vol}_m(x)\\ &\quad+2\sum_{X^{m-1}}\int_{U\cap X^{m-1}}\sum_{\alpha=1}^{k}\operatorname{tr} II_{w_\alpha}\,d\operatorname{vol}_{m-1}(x)\\ &\quad+4\pi\sum_{X^{m-2}}\int_{U\cap X^{m-2}}\left(\frac{1}{2}+(-1)^m\frac{\chi_{\mathrm{lok}}(S,x)}{2}-\theta_m(S,x)\right)d\operatorname{vol}_{m-2}(x).\end{aligned}$$

3.1 Preliminaries. In what follows, we consider an o-minimal system and a compact definable set S of dimension m. We take a tame stratification $S = \bigcup X^i$, where X^i denotes a stratum of dimension i. Remember that this given stratification satisfies Whitney's conditions A and B. The existence of such a stratification follows from [16], Proposition 5.1.8.

We recall that b_k denotes the volume of the unit ball in k-dimensional Euclidean space. The k-volume of the k-dimensional unit sphere will be denoted by s_k. Then $s_k = (k+1)b_{k+1}$ and $b_{k+2} = \frac{2\pi}{k+2} b_k$.

In general, any Lipschitz–Killing measure on S is a stratified measure which is continuous on each stratum. That means that for each stratum X^i there are measurable functions λ_i^j such that given a Borel subset $U \subset S$, we have

$$\Lambda_i(S, U) = \sum_{X^j} \int_{U \cap X^j} \lambda_i^j \, d\,\mathrm{vol}_j.$$

According to [4], Examples 5.3., these functions are given by:

$$\lambda_i^j(x) := \frac{1}{s_{n-i-1}} \int_{S^{n-j-1}} \alpha(x, v_e) \sigma_{j-i}(II_{x, v_e}) \, dv_e, \quad j = 0, \dots, i. \tag{6}$$

Here S^{n-j-1} denotes the set of unit normal vectors at some stratum of dimension j and $\alpha(x, v_e)$ is the normal Morse index at x of the height function $h_{v_e}(y) = \langle y, v_e \rangle$.

3.2 Scalar curvature measure on X^m.

Proposition 3.1. *Let $U \subset X^m$ be a Borel measurable set. Then*

$$\mathrm{scal}(S, U) = \int_U s(x) \, d\mathrm{vol}$$

where s is the usual scalar curvature function (recall that X^m is smooth near each of its points).

Proof. Classical, see [24]. □

This proposition is the motivation for our definition of scalar curvature measure for definable sets.

3.3 Scalar curvature measure on X^{m-1}. We suppose in this subsection that we are given an $(m-1)$-stratum X^{m-1} lying on the boundary of the m-strata $X_1^m, X_2^m, \dots, X_k^m$.

Given any point x of the stratum X^{m-1}, the normal section $(T_x X^{m-1})^\perp \cap S$ is (locally) a one-dimensional set consisting of k curves. Since these curves are definable, they have well-defined directions $w_1(x), \ldots, w_k(x)$. Considered as functions of x, $w_1, \ldots, w_k$ are definable and consequently $d\,\mathrm{vol}_{m-1}$-almost everywhere differentiable.

First it is easy to see that for each normal vector v_e, $\alpha(x, v_e)$ equals 1 minus the number of vectors among $w_1, \ldots, w_k$ forming an angle bigger than $\frac{\pi}{2}$ with v_e. From this we get by an easy integration

$$\int_{S^{n-m+1}} \alpha(x, v_e) v_e \, dv_e = \frac{s_{n-m+2}}{2\pi} \sum_{j=1}^{k} w_j.$$

We recall that II_{w_j} denotes the second fundamental form of X^{m-1} in direction w_j. If we denote by II the vector valued second fundamental form of X^{m-1}, we have by equation (6):

$$\begin{aligned}
\lambda_{m-2}^{m-1}(x) &= \frac{1}{s_{n-m+2}} \int_{S^{n-m+1}} \alpha(x, v_e) \operatorname{tr}(II_{x, v_e}) \, dv_e \\
&= \frac{1}{s_{n-m+2}} \left\langle \int_{S^{n-m+1}} \alpha(x, v_e) \, dv_e, \operatorname{tr} II \right\rangle \\
&= \frac{1}{2\pi} \left\langle \sum_{j=1}^{k} w_j, \operatorname{tr} II \right\rangle \\
&= \frac{1}{2\pi} \sum_{j=1}^{k} \operatorname{tr} II_{w_j}.
\end{aligned}$$

By definition, $\mathrm{scal}(S, -)$ equals $4\pi\Lambda_{m-2}(S, -)$ which immediately yields

Proposition 3.2. *Let $U \subset X^{m-1}$ be a Borel measurable set. Then*

$$\mathrm{scal}(S, U) = 2 \int_U \sum_{j=1}^{k} \operatorname{tr} II_{w_j} \, d\mathrm{vol}_{m-1}.$$

3.4 Scalar curvature measure on X^{m-2}. Let x be a point of a codimension 2-stratum. By equation (6) we get

$$\lambda_{m-2}^{m-2}(x) = \frac{1}{s_{n-m+2}} \int_{S^{n-m+2}} \alpha(x, v_e) \, dv_e.$$

Since α only depends on the normal section, the right-hand side is nothing else than the Lipschitz–Killing curvature $\Lambda_0((T_x X^{m-2})^{\perp} \cap S, \{x\})$ of the two-dimensional normal section. We will evaluate this expression in the following subsections.

3.5 Some propositions in the two-dimensional case. Before returning to strata of dimension $m-2$, we need some propositions about two-dimensional definable sets.

Lemma 3.3. *Let $M \subset S^{n-1}$ be a one-dimensional definable subset of the $(n-1)$-dimensional unit sphere. Then*

$$\int_{S^{n-1}} \chi(B_r(x) \cap M)\, dx = a(r)\chi(M) + b(r)l(M).$$

Here, $B_r(x)$ denotes the geodesic ball of radius r around x, $a(r)$ its volume, $b(r)$ the volume of an r-tube around a big circle divided by 2π and $l(M)$ the length of M.

Proof. This follows from easy approximation arguments. □

Proposition 3.4. *Let M be a compact two-dimensional definable set, and $P \in M$. Then the limit $\alpha(M,P) := \lim_{s\to 0} \frac{l(B_s(P))}{s}$ exists and the following equation holds:*

$$\Lambda_0(M,\{P\}) = \frac{1}{2} + \frac{\chi_{\text{lok}}}{2} - \frac{\alpha}{2\pi}.$$

Here, $\chi_{\text{lok}} = \chi(M, M\backslash P)$ denotes the local Euler characteristic in P (see [2], Section 11.2), and l the length of a one-dimensional set.

Proof. Choose some Whitney stratification with P as a 0-stratum. Since both sides of the equation are Euler additive, it suffices to show the statement for each stratum separately. This is trivial for 1-strata ($\chi_{\text{lok}} = 0$, $\alpha = 0$, $\Lambda_0 = \frac{1}{2}$). So we may assume that M is a 2-stratum satisfying near P Whitney's conditions. Then $\beta(P,v) = \lim_{\theta\to 0}\lim_{\delta\to 0}\beta_{\theta,\delta}(P,v) = \lim_{\theta\to 0}\lim_{\delta\to 0}\hat{\beta}_{\theta,\delta}(P,v)$ almost everywhere. In order to calculate $\Lambda_0(M,\{P\})$ we do the same steps as in the preceding subsections and find

$$\begin{aligned}
\Lambda_0(M,\{P\}) &= \frac{1}{s_{n-1}} \int_{S^{n-1}} \beta(P,v)\, dv \\
&= \frac{1}{s_{n-1}} \int_{S^{n-1}} \lim_{\theta\to 0}\lim_{\delta\to 0} \beta_{\theta,\delta}(P,v)\, dv \\
&= \frac{1}{s_{n-1}} \int_{S^{n-1}} \lim_{\theta\to 0}\lim_{\delta\to 0} \hat{\beta}_{\theta,\delta}(P,v)\, dv \\
&\overset{(*)}{=} \frac{1}{s_{n-1}} \lim_{\theta\to 0}\lim_{\delta\to 0} \int_{S^{n-1}} \hat{\beta}_{\theta,\delta}(P,v)\, dv \\
&= 1 - \frac{1}{s_{n-1}} \lim_{\theta\to 0}\lim_{\delta\to 0} \left(a(r)\chi(S_s \cap M) + b(r)\frac{l(S_s \cap M)}{s} \right).
\end{aligned}$$

To simplify things, we have put $r := \frac{\pi}{2} - \theta$ and $s := \frac{\delta}{\cos\theta}$. Equation $(*)$ follows from

Lebesgue's Convergence Theorem (remark that β is a definable and hence integrable function). Furthermore, we have used Lemma 3.3. The existence of β almost everywhere combined with the convergence theorem yields the existence of the angle α. We therefore have

$$\begin{aligned}\Lambda_0(M,\{P\}) &= 1 - \frac{1}{s_{n-1}} \lim_{\theta\to 0}(a(r)(1-\chi_{\mathrm{lok}}(M,P)) + b(r)\alpha(M,P)) \\ &= 1 - \frac{a(\frac{\pi}{2})}{s_{n-1}}(1-\chi_{\mathrm{lok}}(M,P)) - \frac{b(\frac{\pi}{2})}{s_{n-1}}\alpha(M,P) \\ &= \frac{1}{2} + \frac{\chi_{\mathrm{lok}}(M,P)}{2} - \frac{\alpha}{2\pi}.\end{aligned}$$ □

In [17] it is proven that the following limit exists:

$$\theta(M,P) := \lim_{r\to 0} \frac{\mathrm{vol}_2(B_r(P)\cap M)}{b_2 r^2}.$$

It is called the *density* in P.

Proposition 3.5. *For a two-dimensional definable set M and $P \in M$, angle and density are equal up to some factor*:

$$\theta(M,P) = \frac{\alpha(M,P)}{2\pi}.$$

Proof. We can use a Whitney stratification of M. It suffices to show the above formula for 2-strata. This is a standard proof using Whitney's condition B and the co-area formula. □

Remark. Curvature measures for two-dimensional semi-algebraic sets are also defined in a seemingly different, but in fact equivalent way (up to a constant factor 2π) in [5]. The equivalence with the definition of [16] follows from Lemma 3.2, Subsection 3.3 and Proposition 3.5.

3.6 Comparison between inner and outer density. We have defined scalar curvature measure using integral geometry. The calculation yields terms of the outer geometry of the set. One such term is the density of the set. We are going to show that the density can already be computed using only the inner geometry of the set. This observation will be very important for the proof of our main theorem.

Proposition 3.6. *Let S be a connected definable set of dimension m and $P \in S$. Then*

$$\lim_{r\to 0} \frac{\mathrm{vol}\, B_i(P,r)}{b_m r^m} = \lim_{r\to 0} \frac{\mathrm{vol}\, B_e(P,r)}{b_m r^m}.$$

Both limits exist.

Proof. We can assume without restriction that $P = 0$. We use a Whitney stratification of S. Take the unit vector field $v(x) = -\operatorname{grad} d_e(\cdot, 0)(x)$ of $\mathbb{R}^n \setminus \{0\}$. From Whitney's condition B we deduce that for $x \in X^j \setminus \{0\}$ the angle between $v(x)$ and $T_x X^j$ will be small if x is near 0. We project v on each stratum of S and integrate with respect to this vector field. The integral curves can leave a given stratum but will then be in a stratum of smaller dimension. By the above remark about angles, the curve will be in finite time at P. The length of the curve is trivially bigger than the Euclidean distance between Q and 0. On the other hand, this length will be smaller than $(1 + \varepsilon(r))$ times the Euclidean distance between Q and 0, where $\varepsilon(r)$ is a function tending to 0 for $r \to 0$. It follows that

$$B_e\left(P, \frac{r}{1+\varepsilon(r)}\right) \subseteq B_i(P, r) \subseteq B_e(P, r).$$

We divide each term by $b_m r^m$. Then the limits of the outer terms exist for $r \to 0$ and are equal to the outer density (see [17]). Consequently the limit of the inner term exists which proves the proposition. □

3.7 The normal section formula.

Theorem 3.7. *Let S be a compact definable Whitney-stratified set and X^i a stratum of dimension i. Then at $d\,\mathrm{vol}_i$-almost each point $P \in X^i$ the density of S at P equals the density of the normal section of S at P:*

$$\theta(S, P) = \theta((T_P X^i)^\perp \cap S, P).$$

Proof. We proceed in three steps. From elementary measure theory we will verify the statement for "thick" sets. Afterwards we will see that the formula holds true if X^i is a flat stratum. The generalization to arbitrary strata is done with the help of a bi-Lipschitz map in the third step.

Step 1. We suppose that S is a thick set in $\mathbb{R}^n$ (this means S equals the closure of its interior) and that the stratum X^i is flat, i.e. $X^i \subset \mathbb{R}^i$. We write $\mathbb{R}^n$ as the orthogonal sum $\mathbb{R}^n = \mathbb{R}^i \oplus \mathbb{R}^{n-i}$.

It suffices to prove that

$$\int_Q \theta(S, P)\, dP = \int_Q \theta(S \cap (T_P X^i)^\perp, P)\, dP$$

for each box $Q = (a_1 - \varepsilon_1, a_1 + \varepsilon_1) \times \cdots \times (a_i - \varepsilon_i, a_i + \varepsilon_i) \subset \mathbb{R}^i$. We denote for each $0 < r < \min\{\varepsilon_1, \ldots, \varepsilon_i\}$ with $Q_-(r)$ the box

$$Q_-(r) = (a_1 - \varepsilon_1 + r, a_1 + \varepsilon_1 - r) \times \cdots \times (a_i - \varepsilon_i + r, a_i + \varepsilon_i - r)$$

and by $Q_+(r)$ the box

$$Q_+(r) = (a_1 - \varepsilon_1 - r, a_1 + \varepsilon_1 + r) \times \cdots \times (a_i - \varepsilon_i - r, a_i + \varepsilon_i + r).$$

Then $Q_-(r) \subset Q \subset Q_+(r)$.

Let $r > 0$ be a fixed real number. Then we have for each $y = (y_0, y_1) \in Q_-(r) \times \mathbb{R}^{n-i}$ with $|y_1| \leqslant r$ the following equality (use Pythagoras):

$$\int_Q 1_{B(x,r)\cap S}(y)\,dx = 1_S(y) b_i \Big(\sqrt{r^2 - |y_1|^2}\Big)^i.$$

In the following calculations, we can use dominated convergence, which follows either from the fact that the considered functions are definable or from the fact that densities of thick sets are bounded by 1. Hence we can apply the theorems of Fubini and Lebesgue. We use spherical coordinates for $\mathbb{R}^{n-i}$ and write $y_1 = \bar{r}\phi$ with $\bar{r} \in [0, \infty)$ and $\phi \in S^{n-i-1}$. Then $dy = \bar{r}^{n-i-1}\,dy_0\,d\bar{r}\,d\phi$ and we see that

$$\begin{aligned}
\frac{1}{r^n}\int_Q \operatorname{vol} B(x,r)\cap S\,dx &= \frac{1}{r^n}\int_Q \int_{\mathbb{R}^n} 1_{B(x,r)\cap S}(y)\,dy\,dx \\
&= \frac{1}{r^n}\int_{\mathbb{R}^n} \int_Q 1_{B(x,r)\cap S}(y)\,dx\,dy \\
&\geqslant \frac{1}{r^n}\int_{Q_-(r)\times B^{n-i}(r)} \int_Q 1_{B(x,r)\cap S}(y)\,dx\,dy \\
&= \frac{1}{r^n}\int_{Q_-(r)\times B^{n-i}(r)} 1_S(y) b_i\Big(\sqrt{r^2 - |y_1|^2}\Big)^i dy \\
&= \frac{1}{r^n}\int_{Q_-(r)} \int_{(0,r)} \int_{S^{n-i-1}} 1_S(y_0, \bar{r}\phi) b_i\Big(\sqrt{r^2 - \bar{r}^2}\Big)^i \\
&\quad \cdot \bar{r}^{n-i-1}\,d\phi\,d\bar{r}\,dy_0 \\
&= \int_Q \frac{1}{r^n} 1_{Q_-(r)}(y_0) \int_{(0,r)} \int_{S^{n-i-1}} 1_S(y_0, \bar{r}\phi) b_i\Big(\sqrt{r^2 - \bar{r}^2}\Big)^i \\
&\quad \cdot \bar{r}^{n-i-1}\,d\phi\,d\bar{r}\,dy_0.
\end{aligned}$$

Now, we let r tend to 0. We can write the limit on both sides under the integral sign, since we can use dominated convergence (this follows from the fact that the volume of open balls can be trivially bounded from above). By definition of the density, we have

$$\lim_{r\to 0} \frac{1}{r^n} \operatorname{vol} B(x,r) = b_n \theta(S,x).$$

We set for each $y_0 \in Q$

$$I(y_0) := \lim_{r \to 0} \frac{1}{r^n} 1_{Q_-(r)}(y_0) \int_{(0,r)} \int_{S^{n-i-1}} 1_S(y_0, \bar{r}\phi) b_i \left(\sqrt{r^2 - \bar{r}^2}\right)^i \bar{r}^{n-i-1} \, d\phi \, d\bar{r}$$

and $N(y_0) := (T_{y_0} X^i)^\perp$. With this notation we have

$$\begin{aligned} I(y_0) &= \lim_{r \to 0} \frac{1}{r^n} 1_{Q_-(r)}(y_0) \int_0^r \int_{S^{n-i-1}} 1_S(y_0, \bar{r}\phi) b_i \left(\sqrt{r^2 - \bar{r}^2}\right)^i \bar{r}^{n-i-1} \, d\phi \, d\bar{r} \\ &= \lim_{r \to 0} \frac{1}{r^n} \int_0^r \left\{ b_i \left(\sqrt{r^2 - \bar{r}^2}\right)^i \bar{r}^{n-i-1} \int_{S^{n-i-1}} 1_S(y_0, \bar{r}\phi) \, d\phi \right\} d\bar{r} \\ &= \lim_{r \to 0} \frac{1}{r^n} \int_0^r \left\{ b_i \left(\sqrt{r^2 - \bar{r}^2}\right)^i \operatorname{vol} N(y_0) \cap S \cap S(\bar{r}) \right\} d\bar{r}. \end{aligned}$$

Set

$$\begin{aligned} a(\bar{r}) &:= b_i \left(\sqrt{r^2 - \bar{r}^2}\right)^i \\ b(\bar{r}) &:= \operatorname{vol} N(y_0) \cap S \cap S(\bar{r}) \\ B(\bar{r}) &:= \int_0^{\bar{r}} b(s) \, ds. \end{aligned}$$

From the co-area formula we get $B(\bar{r}) = \operatorname{vol} N(y_0) \cap S \cap B(\bar{r})$ and consequently

$$\lim_{r \to 0} \frac{B(\bar{r})}{b_{n-i} \bar{r}^{n-i}} = \theta(N(y_0) \cap S, y_0).$$

It is easy to see that

$$\int_0^r a'(\bar{r}) \bar{r}^{n-i} \, d\bar{r} = -r^n \int_0^{\pi/2} b_i i (\cos \alpha)^{i-1} (\sin \alpha)^{n-i+1} \, d\alpha = -C_{n,i} r^n$$

with some positive constant $C_{n,i}$.

From partial integration and from the inequality $a'(\bar{r}) < 0$ we conclude that for given $\varepsilon > 0$ and sufficiently small $r > 0$ we have:

$$\begin{aligned} \int_0^r a(\bar{r}) b(\bar{r}) \, d\bar{r} &= -\int_0^r a'(\bar{r}) B(\bar{r}) \, d\bar{r} \\ &\geqslant -\int_0^r a'(\bar{r}) b_{n-i} \bar{r}^{n-i} (\theta(N(y_0) \cap S, y_0) - \varepsilon) \, d\bar{r} \\ &= C_{n,i} b_{n-i} (\theta(N(y_0) \cap S, y_0) - \varepsilon) r^n. \end{aligned}$$

It follows $I(y_0) \geqslant C_{n,i} b_{n-i} \theta(N(y_0) \cap S, y_0) - \varepsilon$. Since this is true for arbitrary $\varepsilon > 0$, we see that $I(y_0) \geqslant C_{n,i} b_{n-i} \theta(N(y_0) \cap S, y_0)$. By Lebesgue's Theorem we conclude that

$$\int_Q b_n \theta(S, x)\, dx \geqslant \int_Q C_{n,i} b_{n-i} \theta(N(y_0) \cap S, y_0)\, dy_0.$$

With similar computations we find

$$\int_Q b_n \theta(S, x)\, dx \leqslant \int_Q C_{n,i} b_{n-i} \theta(N(y_0) \cap S, y_0)\, dy_0$$

with the same constant $C_{n,i}$. Therefore,

$$\int_Q \theta(S, x)\, dx = \frac{C_{n,i} b_{n-i}}{b_n} \int_Q \theta(N(y_0) \cap S, y_0)\, dy_0.$$

With the help of an example, e.g. $S = \mathbb{R}^n$, we calculate $C_{n,i} = \frac{b_n}{b_{n-i}}$. This finishes Step 1 of the proof.

Step 2. Let us suppose now that $S \subset \mathbb{R}^n$ is a compact definable set of dimension m and X^i a flat stratum. This means that there is a Euclidean space $\mathbb{R}^i$ embedded in $\mathbb{R}^n$ such that $X^i \subset \mathbb{R}^i$. Let $P \in X^i$ be a point in the stratum and $\varepsilon > 0$ be a fixed real number.

Assertion: There is a neighborhood $U_P \subset \mathbb{R}^n$ of P and a decomposition of $S \cap U_P$ in disjoint ε-analytic pieces $\Gamma_1, \ldots, \Gamma_q$ such that the following conditions are satisfied for each $j = 1, \ldots, q$:

- There is an m-dimensional affine subspace E_j of $\mathbb{R}^n$ that contains $\mathbb{R}^i$, an open definable subset $U_j \subset E_j$ and an analytic map $\phi_j : U_j \mapsto E_j^\perp$ such that Γ_j can be written as the graph of ϕ_j in $E_j \oplus E_j^\perp = \mathbb{R}^n$ and such that $\|D_u \phi_j\| \leqslant \varepsilon$ for each $u \in U_j$.
- $S \cap U_P$ and $\bigcup_j \Gamma_j$ differ by a set of positive codimension.

Note that the important condition is the inclusion $\mathbb{R}^i \subset E_j$, the rest of the above conditions is simply the fact that $\bigcup_j \Gamma_j$ is a decomposition of S in ε-analytic pieces.

We omit the proof, which is almost the same as the proof of the existence of the decomposition in ε-analytic pieces (see [17]). The only new information to use is Whitney's condition A which guarantees that all limit tangent spaces on the stratum X^i contain $\mathbb{R}^i$.

Let $\bigcup_k Y^k$ be a Whitney stratification of $\mathbb{R}^n$ that is compatible with $\mathbb{R}^i$, E_j, $j = 1, \ldots, q$ as well as U_j, $j = 1, \ldots, q$. It induces a stratification of the set $\overline{U_j}$; this set is thick in the m-dimensional space E_j (since it is the closure of the open set U_j). Consider an i-dimensional stratum Y^i which is contained in $\mathbb{R}^i$ and lies on the boundary of U_j. By Step 1 (applied to $\overline{U_j} \subset E_j$) we have for almost each point $Q \in Y^i$

$$\theta(U_j, Q) = \theta(U_j \cap (\mathbb{R}^i_Q)^\perp, Q).$$

If Y^i is a stratum in $\mathbb{R}^i$ which does not lie on the boundary of U_j, the above equation trivially holds, since then Q is not on the boundary of U_j and both the density of U_j at Q as well as the density of the normal section at Q are 0.

As the finite union of zero sets is a zero set, we have for almost each point $Q \in X^i \cap U_P$ the equations $(j = 1, \ldots, q)$:

$$\theta(U_j, Q) = \theta(U_j \cap (\mathbb{R}^i_Q)^\perp, Q).$$

In the following, we write $A \sim B$ if there is a function $\psi : (0, \infty) \mapsto (1, \infty)$ such that $\lim_{\varepsilon \to 0} \psi(\varepsilon) = 1$ and $\psi(\varepsilon)^{-1} A \leqslant B \leqslant \psi(\varepsilon) A$. By [17] each ε-analytic piece Γ_j satisfies the relation

$$\theta(U_j, Q) \sim \theta(\Gamma_j, Q).$$

For each of the Γ_j the set $(\mathbb{R}^i_Q)^\perp \cap \Gamma_j$ is trivially an ε-analytic piece in $(\mathbb{R}^i_Q)^\perp$. Therefore we also have

$$\theta((\mathbb{R}^i_Q)^\perp \cap U_j, Q) \sim \theta((\mathbb{R}^i_Q)^\perp \cap \Gamma_j, Q).$$

We know that $S \backslash \bigcup_j \Gamma_j$ has dimension at most $m - 1$ (in particular, it is a zero set with respect to m-volume). For almost each point $Q \in X^i$ we therefore get that $(S \cap (\mathbb{R}^i_Q)^\perp) \backslash (\bigcup_j \Gamma_j \cap (\mathbb{R}^i_Q)^\perp)$ is a set of positive codimension, in particular a zero set for $(m - i)$-volume.

It follows for almost every point $Q \in X^i \cap U_P$ that

$$\theta(S, Q) \sim \theta\Big(\bigcup_j \Gamma_j, Q\Big) = \sum_j \theta(\Gamma_j, Q) \sim \sum_j \theta(U_j, Q) = \sum_j \theta(U_j \cap (\mathbb{R}^i_Q)^\perp, Q)$$

$$\sim \sum_j \theta((\mathbb{R}^i_Q)^\perp \cap \Gamma_j, (Q)) = \theta\Big(\bigcup_j \Gamma_j \cap (\mathbb{R}^i_Q)^\perp, Q\Big) = \theta(S \cap (\mathbb{R}^i_Q)^\perp, Q).$$

We rewrite this explicitly as

$$\theta(S, Q) \sim \theta(S \cap (\mathbb{R}^i_Q)^\perp, Q)$$

for almost every point $Q \in X^i \cap U_P$ (with respect to i-dimensional volume).

To finish Step 2, we apply simple measure theoretic arguments to show that from the last relation it follows that for almost every point we already have the equality $\theta(S, Q) = \theta(S \cap (\mathbb{R}^i_Q)^\perp, Q)$. Essential is that a countable union of zero sets is a zero set and that X^i is σ-compact. The details are easy but technical and will not be given here.

This finishes Step 2.

Step 3. Finally, let S be an arbitrary compact definable set and X^i an arbitrary one of its i-strata.

Let $\varepsilon > 0$ and $P \in X^i$ be given. Let $\phi : \mathbb{R}^i \supset V \mapsto U \subset X^i$, $(\xi_1, \dots, \xi_i) \mapsto \phi(\xi_1, \dots, \xi_i)$ be a coordinate system for a neighborhood of P in X^i such that the tangent vectors $\frac{\partial\phi}{\partial\xi_j}$ form an orthonormal base in P. We choose a definable family of differentiably varying normal vector fields $v_1, \dots, v_{n-i}$ that form in each point Q of a neighborhood of P an orthonormal base of the normal space $(T_Q X^i)^\perp$.

We define a map $H : V \times \mathbb{R}^{n-i} \mapsto \mathbb{R}^n$ by

$$H(\xi_1, \dots, \xi_{n-i}, y_1, \dots, y_i) = \phi(\xi_1, \dots, \xi_{n-i}) + y_1 v_1 + y_2 v_2 + \cdots + y_i v_i.$$

The differential of H in $(0, \dots, 0)$ with respect to the basis $\left\{\frac{\partial}{\partial\xi_1}, \dots, \frac{\partial}{\partial\xi_{n-i}}, \frac{\partial}{\partial y_1}, \dots, \frac{\partial}{\partial y_i}\right\}$ and $\left\{\frac{\partial\phi}{\partial\xi_1}, \dots, \frac{\partial\phi}{\partial\xi_{n-i}}, v_1, \dots, v_i\right\}$ is given by the identity.

Consequently there are sufficiently small neighborhoods U_1 of $(0, \dots, 0)$ and U_2 of P in $\mathbb{R}^n$ respectively, such that H is a bi-Lipschitz map between U_1 and U_2 with factor $1 + \varepsilon$. The pre-image S' of $U_2 \cap S$ under H is a definable set. The stratum X^i transforms under H in a flat stratum $\mathbb{R}^i \cap U_1$ and we can apply Step 2. Therefore, for almost each point $Q \in \mathbb{R}^i \cap U_1$ the density of the normal section $(T_Q\mathbb{R}^i)^\perp \cap S'$ at Q equals the density of the set S' at Q:

$$\theta((T_Q\mathbb{R}^i)^\perp \cap S', Q) = \theta(S', Q).$$

Under H, the normal sections of S' along the stratum $\mathbb{R}^i \cap U_1$ transform in the normal sections of S along X^i. Furthermore, we know that H is bi-Lipschitz with factor $1 + \varepsilon$. It follows that

$$\theta(S', Q') \sim \theta(S, H(Q'))$$

and

$$\theta((T_{Q'}\mathbb{R}^i)^\perp \cap S', Q') \sim \theta((T_{H(Q')}X^i)^\perp \cap S, H(Q')).$$

For almost each point Q in $X^i \cap U_2$ we thus have

$$\theta(S, Q) \sim \theta((T_Q X^i)^\perp \cap S, Q).$$

To sum up what we have proven so far: For given $\varepsilon > 0$ there is for each point $P \in X^i$ a neighborhood U_P, where the density of S and the density of the normal section of S differ almost everywhere by at most a factor $\psi(\varepsilon)$, where $\psi(\varepsilon) \to 1$ for $\varepsilon \to 0$.

From simple measure-theoretic arguments we conclude that the density of S and the density of the normal section of S agree in almost each point of X^i. This finishes Step 3 and the proof of Theorem 3.7. □

Remark. For Lipschitz stratified sets (see [19]), the statement of Theorem 3.7 holds not only almost everywhere, but for each point. This follows from simple arguments (integration of vector fields and a product formula for densities). In the category of globally subanalytic sets, the existence of Lipschitz stratifications was shown by Parusiński. In general, the question of existence of Lipschitz stratifications is not solved. For our Main Theorem, it suffices to have this weaker statement, since we are only interested in measures. However, we conjecture that for each Verdier stratified set the normal section formula holds true for each point.

Summing up what we have shown in this section, we get the following

Proof of Theorem 1.2. We have to show the following formula:

$$\begin{aligned}\operatorname{scal}(S,U) = {} & \sum_{X^m} \int_{U\cap X^m} s(x)\, d\mathrm{vol}_m(x) \\ & + 2\sum_{X^{m-1}} \int_{U\cap X^{m-1}} \left(\sum_{i=1}^k \operatorname{tr} II_{w_i}\right) d\mathrm{vol}_{m-1}(x) \\ & + 4\pi \sum_{X^{m-2}} \int_{U\cap X^{m-2}} \left(\frac{1}{2} + (-1)^m \frac{\chi_{\mathrm{lok}}(S,x)}{2} - \theta_m(S,x)\right) d\mathrm{vol}_{m-2}(x).\end{aligned}$$

The third term follows from Subsections 3.4 and 3.5 and Theorem 3.7. Note that this formula remains true even if X^{m-2} is a stratum which does not lie on the boundary of an m-stratum. If, in this case, there are exactly i strata of dimension $m-1$ with X^{m-2} on their boundaries, then we get from Euler additivity

$$\chi_{\mathrm{lok}} = (-1)^m(1-i), \quad \Lambda_{m-2}(S,-)|_{X^{m-2}} = \left(1-\frac{i}{2}\right).\mathrm{vol}(S,-)|_{X^{m-2}}$$

The second term follows from the calculations in Subsection 3.3, it also remains the same for strata not lying on the boundary of m-strata, because in this case $k=0$ and $\operatorname{scal}|_{X^{m-1}} \equiv 0$.

The first term follows from Subsection 3.2. There are no other terms, since Λ_{m-2} and therefore scal vanish identically on strata of dimension $< m-2$. □

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