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Embeddings of finite classical groups over field extensions and their geometry

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Abstract. We study some embeddings of finite classical groups defined over field extensions, focusing on their geometry. The embedded groups are subgroups of classical groups lying outside the main Aschbacher classes. We concentrate on $\mathrm{PG}(8,q)$ where the embedded groups can be seen as automorphism groups of natural geometric objects: Hermitian Veroneseans, Twisted Hermitian Veroneseans and rational curves.

1 Introduction

Let $G = G(n,q^t)$ denote a classical group with natural module V of dimension $n \geqslant 2$ over the Galois field $\mathrm{GF}(q^t)$. Let $\psi : \mathrm{GF}(q^t) \to \mathrm{GF}(q^t)$, $x \mapsto x^q$, be the Frobenius automorphism of $\mathrm{GF}(q^t)$ and let V^{ψ^i} denote the G-module V with group action given by $v \cdot g = vg^{\psi^i}$, where g^{ψ^i} denotes the matrix g with its entries raised to the q^i-th power, $i = 0, \dots, t-1$. Also let V^* denote the G-module with group action given by $v \cdot g = vg^*$, where g^* is the inverse-transpose of g.

Then one can form the tensor product module $V \otimes V^{\psi} \otimes \cdots \otimes V^{\psi^{t-1}}$, a module which can be realized over the field $\mathrm{GF}(q)$. This gives an embedding of G in a classical group, say $\overline{G}$, with an n^t-dimensional natural module over $\mathrm{GF}(q)$, yielding an absolutely irreducible representation of the group G. For t even, there is a similar module given by $V \otimes V^{*\psi} \otimes V^{\psi^2} \otimes \cdots \otimes V^{*\psi^{t-1}}$, realizable over $\mathrm{GF}(q^2)$. Such representations are given by Steinberg [15], and Seitz [14] goes so far as to describe the normalizers of such embedded subgroups as an extended Aschbacher class of subgroups.

The geometry of maximal subgroups in the Aschbacher classes is well understood (with the possible exception of the class $\mathscr{C}_6$). Our main purpose is to describe the geometry of subgroups lying outside the Aschbacher classes, little being known at present. We concentrate on classical groups of low dimension, namely with $t = 2$ and $n = 3$, and study the embeddings of $\mathrm{PGL}(3,q^2)$ in $\mathrm{PGL}(9,q)$, $\mathrm{PGL}(3,q^2)$ in $\mathrm{PU}(9,q^2)$ and $\Omega(3,q^2)$ in $\Omega(9,q)$; in the last case q is odd. We identify the normalizers of the embedded groups as (in most cases) maximal subgroups and stabilizers of geometrical configurations.

We mostly work inside Segre varieties since the geometrical configurations we shall deal with are naturally contained in such varieties.

2 The Hermitian Veronesean of $\mathrm{PG}(2,q^2)$

2.1 Tensored spaces. Let V_i, $1 \leqslant i \leqslant t$ be vector spaces of dimension n_i over the Galois field $\mathrm{GF}(q)$. Then $V = V_1 \otimes \cdots \otimes V_t$ is a vector space of dimension $\prod_{i=1}^{t} n_i = n$. Assuming that $m_i = n_i - 1 \geqslant 1$ for each i, let $\mathrm{PG}(m_1,q)$, $\mathrm{PG}(m_2,q), \ldots, \mathrm{PG}(m_t,q)$ be the projective spaces over $\mathrm{GF}(q)$ corresponding to $V_1, V_2, \ldots, V_t$. The set of all vectors in V of the form $v_1 \otimes \cdots \otimes v_t$ with $0 \neq v_i \in V_i$ corresponds to a set of points in $\mathrm{PG}(n-1,q)$ known as the *Segre variety*, $S_{m_1,\ldots,m_r}$, of $\mathrm{PG}(m_1,q), \ldots, \mathrm{PG}(m_r,q)$, [7, 25.5].

2.2 A representation of $\mathrm{GL}(3,q^2)$. Let $G = \mathrm{GL}(3,q^2)$ and let $\psi : \mathrm{GF}(q^2) \to \mathrm{GF}(q^2)$ be the Frobenius automorphism of $\mathrm{GF}(q^2)$ given by $x \mapsto x^q$; we sometimes write $\bar{x}$ for x^q. Let V_0 be the natural module for $\mathrm{GL}(3,q^2)$ over $\mathrm{GF}(q^2)$. Let V_0^{ψ} be the G-module with group action given by $v \cdot g = vg^{\psi}$, where vg^{ψ} denotes the matrix g with its entries raised to the q-th power and let $V = V_0 \otimes V_0^{\psi}$. Then we have a representation $\rho : G \to \mathrm{GL}(3^2,q^2)$ with $\rho(g) = g \otimes g^{\psi} \in \mathrm{GL}(3,q^2) \otimes \mathrm{GL}(3,q^2)$. This representation of $\mathrm{GL}(3,q^2)$ is absolutely irreducible (c.f. [15]). The two representations ρ and $\rho\psi$ are isomorphic, so this representation of G on V can be written over $\mathrm{GF}(q)$ (c.f. [1, 26.3]). Moreover if ψ_0 is the Frobenius automorphism of $\mathrm{GF}(q^2)$ given by $x \mapsto x^{q_0}$ for any $q_0 < q$, then ρ and $\rho\psi_0$ are not isomorphic (c.f. [15]) and so ρ cannot be written over $\mathrm{GF}(q_0)$.

We can give a concrete construction of a $\mathrm{GF}(q)$-subspace of V fixed by $\rho(G)$. If v_1, v_2, v_3 is a basis for V_0 and $\alpha \in \mathrm{GF}(q^2)\backslash\mathrm{GF}(q)$ is fixed, then the vectors $v_i \otimes v_i$, $v_i \otimes v_j + v_j \otimes v_i$ and $\alpha v_i \otimes v_j + \alpha^q v_j \otimes v_i$ $(i < j)$ form a basis for a 3^2-dimensional $\mathrm{GF}(q)$-subspace V_q of V fixed by G. There is an involution $\theta \in \mathrm{GL}(3^2,q^2)$ on V that takes $v_i \otimes v_j$ to $v_j \otimes v_i$ for each i, j. We see that θ fixes V_q and normalizes $\rho(G)$; it is not difficult to show that θ does not lie in $\rho(G)$. Factoring out scalars we get an embedding of $\mathrm{PGL}(3,q^2)$ in $\mathrm{PGL}(3^2,q)$. Restricting to matrices with determinant one, we find $\rho(\mathrm{SL}(3,q^2)) \leqslant \mathrm{SL}(3^2,q)$ so that $\mathrm{PSL}(3,q^2)$ is embedded in $\mathrm{PSL}(3^2,q)$. The involution $-\theta$ lies in $\mathrm{SL}(3^2,q)$ and normalizes $\rho(\mathrm{SL}(3,q^2))$.

The realization over $\mathrm{GF}(q)$ can be seen in another way. Let $\phi : V \to V$, $\lambda u_1 \otimes u_2 \to \lambda^q u_2 \otimes u_1$, with each u_i being one of v_1, v_2, v_3, extended linearly over $\mathrm{GF}(q)$. Then ϕ is a semi-linear map that commutes with $\rho(G)$. Let W be the set of all vectors in V that are fixed by ϕ. Then for all $u \in W$, $g \in G$, $\phi(g(u)) = g(\phi(u)) = g(u)$, and so $g(u) \in W$. Thus the set W is fixed by G and it is a $\mathrm{GF}(q)$-subspace of V. We observe that W contains all the vectors in V_q above. Moreover $\mathrm{GF}(q)$-linearly independent vectors in W are linearly independent over $\mathrm{GF}(q^2)$. For otherwise, consider a minimally-sized counterexample: $w_1, \ldots, w_r$ are linearly independent over $\mathrm{GF}(q)$ but not over $\mathrm{GF}(q^2)$. Then, there are scalars $\mu_1, \ldots, \mu_r \in \mathrm{GF}(q^2)$ such that $\sum_{i=1}^{r} \mu_i w_i = 0$, with not all μ_i in $\mathrm{GF}(q)$, and we may assume, without loss of generality, that $\mu_r = 1$. Now

$\sum_{i=1}^{r} \mu_i^q w_i = 0$ and so $\sum_{i=1}^{r-1} (\mu_i^q - \mu_i) w_i = 0$. We get a contradiction to the minimality of r. Given the absolute irreducibility of $\rho(G)$ we conclude that W has dimension 3^2 over $\mathrm{GF}(q)$. Thus $W = V_q$.

2.3 The Hermitian embedding and its automorphism group. Every element $z \in \mathrm{GF}(q^2)$ has a unique representation as $x + \alpha y$ with $x, y \in \mathrm{GF}(q)$ and $\bar{z} = x + \bar{\alpha} y$. Let $\mathrm{PG}(2, q^2)$ denote the projective plane over $\mathrm{GF}(q^2)$ and consider the map φ : $\mathrm{PG}(2, q^2) \to \mathrm{PG}(8, q^2)$ defined as follows:

$$(X_0, X_1, X_2) \to (X_0^{q+1}, X_1^{q+1}, X_2^{q+1}, X_0 X_1^q, X_0^q X_1, X_0 X_2^q, X_0^q X_2, X_1 X_2^q, X_1^q X_2).$$

The map φ is well-defined and injective. φ is called the *Hermitian embedding of* $\mathrm{PG}(2, q^2)$ and we denote by $\hat{H}$ the image of such a correspondence in $\mathrm{PG}(8, q^2)$. We note that $\hat{H}$ is contained in the Segre variety $S_{2,2} \simeq \mathrm{PG}(2, q^2) \times \mathrm{PG}(2, q^2)$. In fact $\hat{H} = \{(P, \bar{P}) f : P \in \mathrm{PG}(2, q^2)\}$, where f is the Segre map sending $\mathrm{PG}(2, q^2) \times \mathrm{PG}(2, q^2)$ onto $S_{2,2}$. Indeed, the coordinate system for $\mathrm{PG}(8, q^2)$ corresponds to the basis $v_i \otimes v_j$ $(1 \leqslant i \leqslant 3, 1 \leqslant j \leqslant 3)$ for V and the points of $\hat{H}$ all lie in the Baer subgeometry of $\mathrm{PG}(8, q^2)$ determined by the subset $V_q = W$ of V. The point set $\hat{H}$ is a variety of the Baer subgeometry known as *the Hermitian Veronesean of* $\mathrm{PG}(2, q^2)$ [13], [5]. We denote the variety by $\mathscr{H}$ when regarding it as a variety in $\mathrm{PG}(8, q)$.

The variety $\mathscr{H}$ can also be described in terms of a normal line spread of $\mathrm{PG}(5, q)$ [13]. If $\tau : \mathrm{PG}(5, q^2) \to \mathrm{PG}(5, q^2)$ is the map sending the point $P(X_0, \ldots, X_5)$ to $P(\bar{X}_3, \bar{X}_4, \bar{X}_5, X_0, X_1, X_2)$, then the points fixed by τ form a subgeometry $\mathscr{G}$ of $\mathrm{PG}(5, q^2)$ isomorphic to $\mathrm{PG}(5, q)$. If π is the plane with equations $X_3 = X_4 = X_5 = 0$, then the plane $\bar{\pi}$ with equations $X_0 = X_1 = X_2$ is disjoint from π. The set of lines of $\mathrm{PG}(5, q^2)$ joining a point $P \in \pi$ with the point $\bar{P} \in \bar{\pi}$ is a normal line spread of $\mathscr{G}$ which can be represented on the Grassmannian $G_{1,5}$ of lines of $\mathrm{PG}(5, q)$ by the variety $\mathscr{H}$. The variety $\mathscr{H}$ is a $(q^4 + q^2 + 1)$-cap of $\mathrm{PG}(8, q)$ and it is not contained in any proper subspace of $\mathrm{PG}(8, q)$ [13], [5].

Let $G(\mathscr{H}) = \{\zeta \in \mathrm{PGL}(9, q) : \zeta(\mathscr{H}) = \mathscr{H}\}$. The group $G(\mathscr{H})$ is a subgroup of $\mathrm{PGL}(9, q)$ containing $\mathrm{PGL}(3, q^2)$ [13], [5]. Given a projectivity ξ of $\mathrm{PG}(2, q^2)$, the corresponding projectivity of $G(\mathscr{H}) \leqslant \mathrm{PGL}(9, q)$, denoted by $\xi^{\mathscr{H}}$, is called *the Hermitian lifting of* ξ, or briefly the *$\mathscr{H}$-lifting of* ξ [5].

Let ξ be a linear collineation of $\mathrm{PG}(2, q^2)$ with matrix representation $A = (a_{ij})$, $i, j = 0, 1, 2$. The matrix representation of the $\mathscr{H}$-lifting $\xi^{\mathscr{H}}$ of ξ is the matrix whose generic column is

$$(\bar{a}_{0i} a_{0j}, \bar{a}_{0i} a_{1j}, \bar{a}_{0i} a_{2j}, \bar{a}_{1i} a_{0j}, \bar{a}_{1i} a_{1j}, \bar{a}_{1i} a_{2j}, \bar{a}_{2i} a_{0j}, \bar{a}_{2i} a_{1j}, \bar{a}_{2i} a_{2j})$$

with $0 \leqslant i,\ j \leqslant 2$. In particular, $\xi^{\mathscr{H}}$ is the collineation induced by the Kronecker product $A \otimes A^{\psi}$. Hence, the embedding $\mathrm{PGL}(3, q^2) \leqslant \mathrm{PGL}(9, q)$ gives the representation of the group $\mathrm{PGL}(3, q^2)$ as an automorphism group of the Hermitian Veronesean $\mathscr{H}$. Notice that the involutory Frobenius automorphism of $\mathrm{GF}(q^2)$ induces a collineation of $\mathrm{PG}(8, q)$ fixing $\mathscr{H}$ (actually, it interchanges the planes π and $\bar{\pi}$).

We briefly recall Aschbacher's Theorem for classical groups over $\mathrm{GF}(q)$ [1]. Eight classes of "large" subgroups of a given classical group G are defined: $\mathscr{C}_1$, reducible subgroups; $\mathscr{C}_2$, imprimitive subgroups; $\mathscr{C}_3$, stabilizers of field extensions of $\mathrm{GF}(q)$; $\mathscr{C}_4, \mathscr{C}_7$, stabilizers of various tensor product decompositions; $\mathscr{C}_5$, classical groups over subfields of $\mathrm{GF}(q)$; $\mathscr{C}_6$, symplectic-type groups; $\mathscr{C}_8$, other classical groups over $\mathrm{GF}(q)$. Aschbacher's Theorem states that any subgroup of G, not containing the socle of G, is either contained in a member of one of $\mathscr{C}_1$–$\mathscr{C}_8$ or is almost simple and is induced by an absolutely irreducible subgroup modulo scalars. A full discussion of the theorem is given in [9]. Moreover, the same source gives tables with details of the structure of maximal subgroups in each class. In the following we make extensive use of Table 3.5.A ($\mathrm{SL}(n,q)$). We remark that a complete list of maximal subgroups of $\mathrm{SL}(9,q)$ is given by P. B. Kleidman in his Ph.D. Thesis [10]. However no proof is given there, nor have the proofs been subsequently published elsewhere. A. S. Kondratiev has results that give information on subgroups not contained in a maximal subgroup of classes $\mathscr{C}_1$–$\mathscr{C}_8$ but they do not apply to the subgroups we are interested in (c.f. [11] for a survey). An unpublished work of Aschbacher [3] is relevant to our study but does not lead to conclusions on maximality.

Proposition 2.3.1. *The full stabilizer H of the Hermitian Veronesean $\mathscr{H}$ in* $\mathrm{PSL}(9,q)$ *is almost simple and is induced by an absolutely irreducible subgroup of* $\mathrm{SL}(9,q)$ *modulo scalars.*

Proof. The stabilizer of $\mathscr{H}$ in $\mathrm{PSL}(9,q)$ contains at least $\mathrm{PSL}(3,q^2)$. We immediately see that H cannot be a member of $\mathscr{C}_1$ or $\mathscr{C}_3$ because $\rho(\mathrm{SL}(3,q^2))$ is absolutely irreducible. Moreover $\rho(\mathrm{SL}(3,q^2))$ cannot be realized over a subfield of $\mathrm{GF}(q)$ so H cannot lie inside a member of $\mathscr{C}_5$. At the same time, we can read the structure of members of $\mathscr{C}_2$, $\mathscr{C}_6$ and $\mathscr{C}_7$ from [9] and deduce that the orders of these subgroups are not divisible by the order of $\mathrm{PSL}(3,q^2)$. Thus H is not contained in a member of one of these classes. We see also that $\mathrm{PSL}(9,q)$ contains no members of $\mathscr{C}_4$, so the only possibility remaining is $\mathscr{C}_8$.

For $\mathscr{C}_8$ we abuse notation to denote by ρ the representation: $\mathrm{SL}(3,q^2) \to \mathrm{SL}(9,q)$. Then ρ is equivalent to ρ^* if and only if $\rho(\mathrm{SL}(3,q^2))$ fixes a symmetric or symplectic bilinear form, while ρ^ψ is equivalent to ρ^* if and only if $\rho(\mathrm{SL}(3,q^2))$ fixes a Hermitian form. But the module for ρ is given by $V_0 \otimes V_0^\psi$, so ρ^* and ρ^ψ are given by $V_0^* \otimes V_0^{\psi *}$ and $V_0^\psi \otimes V_0$ respectively. Here ρ and ρ^ψ are known to be equivalent, so we need only show that ρ is not equivalent to ρ^*. Steinberg's Tensor Product Theorem [15] tells us that $V_0 \otimes V_0^\psi$ is equivalent to $V_0^* \otimes V_0^{\psi *}$ if and only if either V_0 is equivalent to V_0^* or V_0 is equivalent to $V_0^{\psi *}$, i.e., if and only if $\rho(\mathrm{PSL}(3,q^2))$ preserves a symmetric or symplectic bilinear form or a Hermitian form on V_0, clearly impossible. We conclude that H cannot be contained in a member of $\mathscr{C}_8$. The required result follows from Aschbacher's Theorem.

Corollary 2.3.2. *If Kleidman's list in* [10] *is correct, then H is isomorphic to* $\mathrm{PSL}(3,q^2) \cdot [(q-1,3)^2/(q-1,9)] \cdot C_2$ *and is a maximal subgroup of* $\mathrm{PSL}(9,q)$.

Proof. In Kleidman's list there are four "sporadic" maximal subgroups: M_{10}, A_7, $L_2(19)$ and $\mathrm{PSL}(3,q^2) \cdot [(q-1,3)^2/(q-1,9)] \cdot C_2$. The first three are ruled out because their orders cannot be divided by the order of $\mathrm{PSL}(3,q^2)$. It follows that H lies in the fourth maximal subgroup. To see that H is the whole of this maximal subgroup we need to recall that $\mathrm{PGL}(3,q^2)$ is embedded in $\mathrm{PGL}(9,q)$, intersecting $\mathrm{PSL}(9,q)$ in a subgroup that contains $\mathrm{PSL}(3,q^2)$ as a subgroup of index $(q-1,3)^2/(q-1,9)$, and that the involution induced by $-\theta$ lies in $\mathrm{PSL}(9,q)$ preserving $\mathscr{H}$ (see Subsection 2.2).

2.4 Generalizations. In this section we discuss two possible generalizations of the ideas above. The first concerns mappings from $\mathrm{GL}(n,q^t)$ to $\mathrm{GL}(n^t,q)$. The second concerns the possibility of infinite fields.

Remark 2.4.1. The concrete realization over $\mathrm{GF}(q)$ described above can be extended to a more general setting. Let $G=\mathrm{GL}(n,q^t)$ and let $\psi:\mathrm{GF}(q^t)\to\mathrm{GF}(q^t)$ be the Frobenius automorphism of $\mathrm{GF}(q^t)$ given by $x\mapsto x^q$. Let V_0 be the natural module for $\mathrm{GL}(n,q^t)$ over $\mathrm{GF}(q^t)$ with $V_0^{\psi^i}$ the G-module with group action given by $V\cdot g=vg^{\psi^i}$, and let $V=V_0\otimes V_0^{\psi}\otimes V_0^{\psi^2}\cdots\otimes V_0^{\psi^{t-1}}$. Then we have a representation $\rho:G\to\mathrm{GL}(n^t,q^t)$ with $\rho(g)=g\otimes g^{\psi}\otimes\cdots\otimes g^{\psi^{t-1}}$. As with the specific case above, this representation of $\mathrm{GL}(n,q^t)$ is absolutely irreducible, can be written over $\mathrm{GF}(q)$ but over no subfield of $\mathrm{GF}(q)$. This time let $\{v_1,v_2,\ldots,v_n\}$ be a basis of V_0 and let $\phi:V\to V$, $\lambda u_1\otimes u_2\otimes\cdots\otimes u_t\to\lambda^q u_t\otimes u_1\otimes\cdots\otimes u_{t-1}$, with each u_i being one of $v_1,v_2,\ldots,v_n$, extended linearly over $\mathrm{GF}(q)$. The set W of all vectors in V that are fixed by ϕ is fixed by G and is a $\mathrm{GF}(q)$-subspace of V. Moreover $\mathrm{GF}(q)$-linearly independent vectors in W are linearly independent over $\mathrm{GF}(q^t)$ and we conclude that W has dimension n^t over $\mathrm{GF}(q)$. We can write down basis vectors for W as follows. Let $\Omega=\{1,2,\ldots,t\}$ and let $c=(1234\ldots t)$, a cyclic permutation of Ω; we can consider the action of c on the set of partitions of Ω into n (possibly empty) subsets. For each orbit, Δ say, of c on these partitions, choose an element $\mathscr{P}$ of Δ (i.e., a partition of Ω into n subsets, $\mathcal{O}_1,\mathcal{O}_2,\ldots,\mathcal{O}_n$ say) and a vector $u=u_1\otimes u_2\otimes\cdots\otimes u_t$, with each u_i being one of $v_1,v_2,\ldots,v_n$ and equalling v_j if and only if $i\in\mathcal{O}_j$. Let s be the length of Δ. Then the vectors $\sum_{j=1}^{s}\phi^{j-1}(\lambda u)$, as λ ranges over $\mathrm{GF}(q^s)$, span a $\mathrm{GF}(q)$-subspace $V(\Delta)$ of V of dimension s, fixed by ϕ vector-wise. The direct sum of such subspaces gives a $\mathrm{GF}(q)$-subspace of dimension n^t. A basis for $\mathrm{GF}(q^s)$ over $\mathrm{GF}(q)$ gives rise to a basis for $V(\Delta)$.

Remark 2.4.2. Suppose that F is any field with a non-trivial involutory automorphism: $\lambda\mapsto\bar{\lambda}$ and let F_0 be the fixed subfield. Then a representation $\rho:\mathrm{GL}(n,F)\to\mathrm{GL}(n^2,F)$ can be defined as with finite fields; it is absolutely irreducible and can be realized over F_0.

When $n=3$ we also observe a connection of the Hermitian Veronesean with a notable class of algebraic varieties, the so-called *Severi varieties*, see [12], [16].

3 The Twisted Hermitian Veronesean of PG(2, q^2)

3.1 Embedding PGL(3, q^2) in PU(9, q^2). The notation here is similar to that in Section 2 with $G=\mathrm{GL}(3,q^2)$, ψ the Frobenius automorphism of $\mathrm{GF}(q^2)$ and V_0 the natural module for $GL(3, q^2)$ over $GF(q^2)$. Let V_0^* be the dual module of V_0 (with group action given by $v \cdot g = vg^* = v(g^T)^{-1}$) and let $V = V_0^* \otimes V_0^{\psi}$. Then we have an absolutely irreducible representation $\rho^* : G \to GL(3^2, q^2)$ with $\rho^*(g) = g^* \otimes g^{\psi} \in GL(3, q^2) \otimes GL(3, q^2)$ [15]. The module presented here is dual to $V_0 \otimes V_0^{\psi *}$ but is a more convenient setting from our point of view. The modules $V^* = V_0 \otimes (V_0^{\psi *})$ and $V^{\psi} = (V_0^{\psi *}) \otimes V_0$ are isomorphic and so $\rho^*(G)$ fixes a Hermitian form on V. In general such a representation cannot be realized over a subfield of $GF(q^2)$ (see [2], [9, Theorem 5.4.5]). Indeed, suppose $V_0^* \otimes V_0^{\psi}$ can be realized over a proper subfield $GF(q_0)$ of $GF(q^2)$. Then $V_0^* \otimes V_0^{\psi} \simeq V_0^{\psi_0 *} \otimes V_0^{\psi\psi_0}$, where ψ_0 is the automorphism $x \mapsto x^{q^0}$ of $GF(q^2)$. By [15] these two representations are equivalent if and only if, either $V_0^* \simeq V_0^{\psi_0 *}$ (i.e., $V_0 \simeq V_0^{\psi_0}$), which is not possible, or $V_0^* \simeq V_0^{\psi\psi_0}$ and $V_0^{\psi} \simeq V_0^{\psi_0 *}$. The latter can happen if and only if $\psi_0 = \psi$ and $V_0 \simeq V_0^*$, which in turn is possible if and only if $GL(3, q^2)$ fixes a symmetric or symplectic bilinear form on V_0. As $GL(3, q^2)$ fixes no such form on V_0, its representation on V cannot be realized over a proper subfield of $GF(q^2)$. The same applies to $SL(3, q^2)$.

The representation of $GL(3, q^2)$ may be stated explicitly as follows. Assume that we have a fixed basis v_1, v_2, v_3 for V_0 as in the previous section. A non-degenerate Hermitian form is defined by $(u \otimes v, w \otimes z) = (uz^{\psi T}).(w^{\psi} v^T)$ and this is preserved by $\rho^*(g) = (g^T)^{-1} \otimes g^{\psi}$ for all $g \in G$. It follows that $PGL(3, q^2)$ can be embedded in $PU(9, q^2)$. Recall that the involution θ of $V(9, q^2)$ takes $v_i \otimes v_j$ to $v_j \otimes v_i$ for each i, j; we now observe that θ lies in $U(9, q^2)$ and normalizes (but does not lie in) $\rho^*(G)$. We find that $\rho^*(SL(3, q^2)) \leqslant SU(9, q)$ with $PSL(3, q^2)$ embedded in $PSU(9, q^2)$; $-\theta \in SU(9, q^2)$ and normalizes $\rho^*(SL(3, q^2))$. We shall shortly see that the image of $PGL(3, q^2)$ is an automorphism group of a variety that we call the *Twisted Hermitian Veronesean* of $PG(2, q^2)$ and denote by $\mathscr{H}^*$.

3.2 The Twisted Hermitian Veronesean. In considering the action of $G = GL(3, q^2)$ on $V(9, q^2)$, we see that one orbit is given by $\{(v_1 \otimes v_2)\rho^*(g) : g \in GL(3, q^2)\}$ and this orbit consists of singular vectors. The corresponding orbit in $PG(8, q^2)$ is preserved by (the image of) $PGL(3, q^2)$. Let $\mathscr{R}$ be the set of non-zero singular vectors of the form $u \otimes v$. For any $u \otimes v \in \mathscr{R}$ and any $g \in G$ we see that $(u \otimes v)g = ug^* \otimes vg^{\psi}$ is singular and so lies in $\mathscr{R}$. It is straightforward to calculate that $u \otimes v$ is singular if and only if $u.w^{\psi T} = 0$, so singular vectors of the form $v_1 \otimes w$ are precisely the vectors given by $w = \lambda v_2 + \mu v_3$ where $\lambda, \mu \in GF(q^2)$; such a singular vector is mapped to $v_1 \otimes v_2$ by the inverse of

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \lambda^q & \mu^q \\ 0 & v & \zeta \end{pmatrix},$$

where $v, \zeta \in GF(q^2)$ such that the matrix is non-singular. Thus G is transitive on $\mathscr{R}$, i.e., $\mathscr{R}$ is precisely the orbit that we initially identified. The involution $-\theta$ preserves the Hermitian form and preserves the tensor product $V_0 \otimes V_0$ so it preserves $\mathscr{R}$. Hence the stabilizer in $U(9, q^2)$ of $\mathscr{R}$ has a subgroup isomorphic to $GL(3, q^2) \cdot C_2$.

Let $\mathscr{H}^*$ be the set of points in $\mathrm{PG}(8, q^2)$ corresponding to $\mathscr{R}$. We call this the Twisted Hermitian Veronesean of $\mathrm{PG}(2, q^2)$. This set is the intersection of the Hermitian variety corresponding to the given Hermitian form and the Segre variety $S_{2,2}$. As we have seen above, the points of $\mathscr{H}^*$ corresponding to $v_1 \otimes w$ for various w are just $P(v_1 \otimes (\lambda v_2 + \mu v_3))$, i.e., are the points on a line. It follows that $\mathscr{H}^*$ consists of $q^4 + q^2 + 1$ disjoint lines of the form $u \otimes L$. At the same time $\mathscr{H}^*$ can be expressed as the disjoint union of lines of the form $L \otimes u$.

Proposition 3.2.1. *The full stabilizer H^* of the Twisted Hermitian Veronesean $\mathscr{H}^*$ in* $\mathrm{PSU}(9, q^2)$ *is almost simple and is induced by an absolutely irreducible subgroup of* $\mathrm{SU}(9, q^2)$ *modulo scalars.*

Proof. The argument here is similar to that in the proof of Proposition 2.3.1. On this occasion, the stabilizer of $\mathscr{H}^*$ in $\mathrm{PSU}(9, q^2)$ contains at least $\mathrm{PSL}(3, q^2)$ and we look at Table 3.5.B in [9]. As $\rho^*(\mathrm{SL}(3, q^2))$ is absolutely irreducible, H^* is not contained in a member of $\mathscr{C}_1$ or $\mathscr{C}_3$, and as $\rho^*(\mathrm{SL}(3, q^2))$ cannot be realized over a subfield of $\mathrm{GF}(q^2)$, H^* is not contained in a member of $\mathscr{C}_5$. The order of H^* does not divide the orders of the maximal subgroups in the classes $\mathscr{C}_2, \mathscr{C}_6$ and $\mathscr{C}_7$ and $\mathrm{PSU}(9, q^2)$ contains no subgroups in classes $\mathscr{C}_4$ and $\mathscr{C}_8$. The required result follows from Aschbacher's Theorem.

Corollary 3.2.2. *If Kleidman's list in* [10] *is correct, then H^* is isomorphic to* $\mathrm{PSL}(3, q^2)[(q+1, 3)^2/(q+1, 9)] \cdot C_2$ *and is a maximal subgroup of* $\mathrm{PSU}(9, q^2)$.

Proof. In Kleidman's list there are three "sporadic" maximal subgroups: $\mathrm{PSL}(2, q)$, J_3 and $\mathrm{PSL}(3, q^2)[(q+1, 3)^2/(q+1, 9)] \cdot C_2$. The first two are ruled out because their orders cannot be divided by the order of $\mathrm{PSL}(3, q^2)$. It follows that H^* lies in the third maximal subgroup. As in Corollary 2.3.2, we see that H^* must be the whole of this group.

3.3 Caps on the Twisted Hermitian Veronesean. Now let us suppose that S is a Singer cycle of $\mathrm{GL}(3, q^2)$. Then S is similar in $\mathrm{GL}(3, q^2)$ to the diagonal matrix

$$D = \mathrm{diag}(\omega, \omega^{q^2}, \omega^{q^4}),$$

where ω is a primitive element of $\mathrm{GF}(q^6)$ over $\mathrm{GF}(q^2)$. Consider the image S^* of S under the transpose-inverse involutory map on $\mathrm{GL}(3, q^2)$. Then $S^* = (S^T)^{-1}$ is similar in $\mathrm{GL}(3, q^6)$ to the diagonal matrix

$$D^* = \mathrm{diag}(\omega^{-1}, \omega^{-q^2}, \omega^{-q^4}).$$

Consider the Kronecker product $S^* \otimes S^\psi$. This gives in $\mathrm{GL}(9, q^6)$ the matrix

$$\mathrm{diag}(\omega^{q-1}, \omega^{q^3-1}, \omega^{q^5-1}, \omega^{q-q^2}, \omega^{q^3-q^2}, \omega^{q^5-q^2}, \omega^{q-q^4}, \omega^{q^3-q^4}, \omega^{q^5-q^4}).$$

Then $S^* \otimes S^{\psi}$ has a rational form which is the following block diagonal matrix

$$T = \mathrm{diag}(S^{q-1}, S^{q^3-1}, S^{q^5-1}).$$

Hence $\langle T \rangle$ fixes three planes, say π_1, π_2 and π_3 and all subspaces generated by them.

Lemma 3.3.1. *The projective order of T is q^4+q^2+1.*

Proof. Let $r = q^4+q^2+1$. Then $\beta = \omega^r \in \mathrm{GF}(q^2)\backslash\{0\}$. Since $\omega^{(q-1)r} = \omega^{(q^3-1)r} = \omega^{(q^5-1)r} = \beta^{q-1}$, the order of T is at most r. On the other hand, if k is the order of T, then this yields, for instance, $\omega^{(q-1)k} = \omega^{(q^3-1)k}$, from which we obtain $\omega^{(q^3-q)k} = 1$. It follows that $r \mid k$ and hence $k = r$.

Lemma 3.3.2. *The action of $\langle T \rangle$ on $\mathrm{PG}(8,q^2)\backslash\{\pi_1 \cup \pi_2 \cup \pi_3\}$ is semiregular.*

Proof. Let $P = (x_0, \ldots, x_8)$ be a point in $\mathrm{PG}(8,q^2)\backslash\{\pi_1 \cup \pi_2 \cup \pi_3\}$. Assume that P is proportional to $P \cdot T^i$, with $0 \leqslant i < q^4+q^2+1$. Then there exists a non-zero element $\lambda \in \mathrm{GF}(q^2)$ such that

$$\lambda(x_0,x_1,x_2) = (x_0,x_1,x_2)S^{(q-1)i};$$
$$\lambda(x_3,x_4,x_5) = (x_3,x_4,x_5)S^{(q^3-1)i};$$

and

$$\lambda(x_6,x_7,x_8) = (x_6,x_7,x_8)S^{(q^5-1)i}.$$

This means that at least two of the linear transformations $S^{(q-1)i}$, $S^{(q^3-1)i}$ and $S^{(q^5-1)i}$ have λ as an eigenvalue. In particular, one of $S^{(q-1)i}$, $S^{(q^5-1)i}$ has λ as an eigenvalue.

Suppose that $S^{(q-1)i}$ has λ as an eigenvalue. The eigenvalues of $S^{(q-1)i}$ are the elements $\omega^{q^{2j}(q-1)i}$ with $0 \leqslant j \leqslant 2$. If one of the eigenvalues of $S^{(q-1)i}$ is in $\mathrm{GF}(q^2)$, then all of them are in $\mathrm{GF}(q^2)$ and they must be equal, so $S^{(q-1)i} = \lambda I$. But now similar arguments apply to $S^{(q^3-1)i}$ and $S^{(q^5-1)i}$: either $S^{(q^3-1)i} = \lambda I$ or $S^{(q^5-1)i} = \lambda I$. In the former case $S^{(q^3-1)i} = \lambda^{q^2+q+1}I = \lambda^{q+2}I$ (since $\lambda \in \mathrm{GF}(q^2)$) implies that $\lambda^{q+1} = 1$, so that $S^{(q^2-1)i} = I$ and $(q^6-1) \mid (q^2-1)i$. In the latter case $S^{(q^5-1)i} = \lambda^{q^4+q^3+q^2+q+1}I = \lambda^{2q+3}I$ implies that $\lambda^{2(q+1)} = 1$, so that $S^{2(q^2-1)i} = I$ and $(q^6-1) \mid 2(q^2-1)i$. In each case $(q^4+q^2+1) \mid i$.

A similar argument applies if $S^{(q^5-1)i}$ has λ as an eigenvalue with the same conclusion that $(q^4+q^2+1) \mid i$. Given that $i < q^4+q^2+1$ we conclude that $i = 0$. Hence $\langle T \rangle$ is semiregular.

Proposition 3.3.3. *Each orbit of $\langle T \rangle$ on the point set of $\mathrm{PG}(8,q^2)\backslash\{\pi_1 \cup \pi_2 \cup \pi_3\}$, not contained in any subspace generated by two of the planes π_1, π_2, π_3, is a cap.*

Proof. Let $P = (x_0, \ldots, x_8)$ be a point in $\mathrm{PG}(8,q^2)\backslash\{\pi_1 \cup \pi_2 \cup \pi_3\}$, not contained in any subspace generated by two of the planes π_1, π_2, π_3. Suppose that P, $P \cdot T^i$, $P \cdot T^j$

are distinct collinear points such that $P + \lambda P \cdot T^i + \mu P \cdot T^j$ is the zero vector, with $0 < i < j < q^4 + q^2 + 1$, $\lambda, \mu \in \mathrm{GF}(q^2)$. Thus $P \cdot (I + \lambda T^i + \mu T^j) = 0$. Expressing T as the direct sum of the three 3×3 matrices S^{q-1}, S^{q^3-1} and S^{q^5-1}, we have

$$\begin{aligned}(x_0, x_1, x_2)(I + \lambda S^{(q-1)i} + \mu S^{(q-1)j}) &= (0,0,0),\\ (x_3, x_4, x_5)(I + \lambda S^{(q^3-1)i} + \mu S^{(q^3-1)j}) &= (0,0,0),\\ (x_6, x_7, x_8)(I + \lambda S^{(q^5-1)i} + \mu S^{(q^5-1)j}) &= (0,0,0).\end{aligned}$$

It follows that the determinants of the matrices $I + \lambda S^{(q-1)i} + \mu S^{(q-1)j}$, $I + \lambda S^{(q^3-1)i} + \mu S^{(q^3-1)j}$ and $I + \lambda S^{(q^5-1)i} + \mu S^{(q^5-1)j}$ are zero.

Now the $\mathrm{GF}(q^2)$-algebra generated by S^{q-1}, say $\mathscr{A}$, is isomorphic to $\mathrm{GF}(q^6)$ and so the unique singular matrix of $\mathscr{A}$ is the null matrix. Hence $I + \lambda S^{(q-1)i} + \mu S^{(q-1)j}$ is the null matrix. Similarly for the matrices $I + \lambda S^{(q^3-1)i} + \mu S^{(q^3-1)j}$ and $I + \lambda S^{(q^5-1)i} + \mu S^{(q^5-1)j}$.

Consider the two equations

$$I + \lambda S^{(q-1)i} = -\mu S^{(q-1)j}$$

and

$$I + \lambda S^{(q^5-1)i} = -\mu S^{(q^5-1)j}.$$

Multiply each term of the first equation by the corresponding term of the second equation raised to the q-th power. Simple calculations show that $S^{(q-1)i}$ is a root of the quadratic polynomial $x^2 + ((1 + \lambda^{q+1} - \mu^{q+1})/\lambda)x + \lambda^{q-1} \in \mathrm{GF}(q^2)[x]$. This forces the eigenvalues of $S^{(q-1)i}$ to generate a subfield of $\mathrm{GF}(q^6)$ which is either $\mathrm{GF}(q^2)$ or $\mathrm{GF}(q^4)$. The latter case can never occur.

As we have seen in proving the previous proposition, if the eigenvalues of $S^{(q-1)i}$ lie in $\mathrm{GF}(q^2)$, then they are equal and $S^{(q-1)i} = \gamma I$ for some $\gamma \in \mathrm{GF}(q^2)$. Similarly $S^{(q-1)j} = \delta I$ for some $\delta \in \mathrm{GF}(q^2)$. Thus, remembering that $\gamma^{q^2} = \gamma$, $\delta^{q^2} = \delta$, we now have equations

$$\begin{aligned}1 + \lambda\gamma + \mu\delta &= 0,\\ 1 + \lambda\gamma^{q+2} + \mu\delta^{q+2} &= 0\end{aligned}$$

and

$$1 + \lambda\gamma^{2q+3} + \mu\delta^{2q+3} = 0.$$

From these we deduce that $\gamma^{q+1} = \delta^{q+1}$. But then $(S^{(q-1)i})^{q+1} = (S^{(q-1)j})^{q+1} = I$, from which we conclude that $(q^4 + q^2 + 1) \mid i$ and $(q^4 + q^2 + 1) \mid j$, a contradiction to $0 < i < j < q^4 + q^2 + 1$. Hence no three points on this orbit of $\langle T \rangle$ are collinear, i.e., the orbit is a cap.

Remark 3.3.4. We see from the previous result that many of the orbits of $\rho^*(T)$ on $\mathrm{PG}(8, q)$ are caps, three orbits are planes and the remainder is undetermined. From a different perspective we can consider the orbits of $\rho^*(T)$ on $\mathscr{H}^*$. If $0 < i < j < q^4 + q^2 + 1$, then for any non-zero vectors $u, v \in V(3, q^2)$ we have u, uS^{i*} and uS^{j*} representing distinct points of $\mathrm{PG}(2, q^2)$ and v, vS^{iq} and vS^{jq} representing distinct points of $\mathrm{PG}(2, q^2)$. This means that $(u \otimes v), (u \otimes v)\rho^*(S^i)$ and $(u \otimes v)\rho^*(S^j)$ must be non-collinear in $\mathrm{PG}(8, q)$. Thus, in particular, each orbit of $\rho^*(T)$ on $\mathscr{H}^*$ is a cap. In other words $\mathscr{H}^*$ is partitioned into caps of size $q^4 + q^2 + 1$. In fact the Segre variety is partitioned into caps of this size.

3.4 Generalizations. In an analogous manner to Subsection 2.4 we end this section with discussion of two possible generalizations of the ideas above. The first concerns mappings from $\mathrm{GL}(n, q^2)$ to $\mathrm{U}(n^2, q^2)$. The second concerns the possibility of infinite fields.

Remark 3.4.1. As with the Section 2, the situation we have described is a part of a more general picture. From [9, Lemma 2.10.15 ii, Theorem 5.4.5], there is an absolutely irreducible representation ρ^* of the group $G = \mathrm{GL}(n, q^2)$ on $V = V_0^* \otimes V_0^\psi$ over $\mathrm{GF}(q^2)$ that fixes a Hermitian form, not generally realizable over a subfield of $\mathrm{GF}(q^2)$. As argued above, ρ^* can be realized over a subfield of $\mathrm{GF}(q^2)$ if and only if $\mathrm{GL}(n, q^2)$ fixes a symmetric or symplectic bilinear form on V_0, and this can never happen. However, when we consider $\mathrm{SL}(n, q^2)$, we find that it fixes a non-degenerate symplectic bilinear form precisely when $n = 2$. In this one case, $\rho^*(\mathrm{SL}(2, q^2))$ can be realized over $\mathrm{GF}(q)$, effectively we have the well known isomorphism between $\mathrm{PSL}(2, q^2)$ and $\Omega^-(4, q)$. The non-degenerate Hermitian form defined by $(u \otimes v, w \otimes z) = (uz^{\psi T}).(w^\psi v^T)$ is preserved by $\rho^*(G)$. It now follows that $\mathrm{PGL}(n, q^2)$ can be embedded in $\mathrm{PU}(n^2, q^2)$. The involution θ lies in $\mathrm{U}(n^2, q^2)$ and normalizes (but does not lie in) $\rho^*(G)$. We find that for $n \geqslant 3$ the image of $\mathrm{PGL}(n, q^2)$ acts transitively on the intersection of a Hermitian variety and a Segre variety, the automorphism group of this intersection contains $\mathrm{PGL}(n, q^2) \cdot C_2$ and so the full automorphism group is absolutely irreducible. This intersection can be expressed as the disjoint union of subspaces of (projective) dimension $n - 2$ in two ways.

Remark 3.4.2. Suppose that F is any field with a non-trivial involutory automorphism: $\lambda \mapsto \bar{\lambda}$ and let F_0 be the fixed subfield. Then a representation $\rho^* : \mathrm{GL}(n, F) \to \mathrm{GL}(n^2, F)$ can be defined as with finite fields and is absolutely irreducible. The construction showing that $\rho^*(\mathrm{GL}(n, F)) \leqslant \mathrm{U}(n^2, F)$ applies, with the image of $\mathrm{PGL}(n, F)$ acting as a transitive automorphism group on the intersection of a Hermitian variety and a Segre variety, this intersection again being the disjoint union of subspaces of (projective) dimension $n - 2$.

4 $\mathrm{PSL}(2, q^2) \simeq \Omega(3, q^2) < \Omega(9, q), q$ odd, as the stabilizer of a rational curve

4.1 Embedding $\Omega(3, q^2)$ in $\Omega(9, q)$. Now suppose that q is odd, that $H \leqslant \mathrm{GL}(3, q^2)$ and that H fixes a non-degenerate symmetric bilinear form f_0 on V_0.

Then one can define a non-degenerate symmetric bilinear form $f = f_0 \otimes f_0$ on V by $f(u_1 \otimes u_2, w_1 \otimes w_2) = f_0(v_1, w_1).f_0(v_2, w_2)$, fixed by $\rho(H)$. Assume that the basis $\{v_1, v_2, v_3\}$ chosen for V_0 is such that $f_0(v_i, v_j) \in \mathrm{GF}(q)$ for each i, j. Recall the semi-linear map ϕ introduced in Section 2 (with W its space of fixed vectors). Then for any $u, v \in W = V_q$ we have $f(u, v) = f(\phi(u), \phi(v)) = f(u, v)^q$. Hence $f(u, v) \in \mathrm{GF}(q)$ for all $u, v \in W$. If $H = \mathrm{O}(3, q^2)$, then $\rho(H)$ is absolutely irreducible on V and therefore the restriction of f to W is non-degenerate. Thus $\rho(\mathrm{O}(3, q^2)) \leqslant \mathrm{O}(9, q)$. Indeed (considering commutator subgroups) $\rho(\Omega(3, q^2)) < \Omega(9, q)$ and the restriction of ρ to $\Omega(3, q^2)$ is injective.

4.2 A rational curve in PG(8,q^2). Let S be a Singer cycle of $\mathrm{SO}(3, q^2)$. Then S is similar in $\mathrm{SO}(3, q^4)$ to the diagonal matrix $D = \mathrm{diag}(\omega, \omega^{q^2}, 1)$, where ω has order $q^2 + 1$ as an element of $\mathrm{GF}(q^4)^*$. Consider the Kronecker product $S \otimes S^{\psi}$. Calculations show that it is similar to the matrix

$$A = \begin{pmatrix} T & 0 & 0 \\ 0 & T^{q+1} & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where $T = R^{q+1}$ with R a Singer cycle in $\mathrm{SL}(4, q)$. We observe that T has order $q^2 + 1$ so both A and its image in $\mathrm{PSL}(9, q)$ have order $q^2 + 1$. Denote by V_a, V_b and V_c the subspaces of $V(9, q)$ fixed by T, T^{q+1} and 1 respectively (having dimensions 4, 4 and 1). The collineation in $\mathrm{PSL}(4, q)$ corresponding to T has $2(q + 1)$ orbits of length $(q^2 + 1)/2$ in $\mathrm{PG}(3, q)$, each being half of an elliptic quadric. Thus each orbit spans $\mathrm{PG}(3, q)$ and T is irreducible on V_a. The same applies to T^{q+1} on V_b. With respect to the bilinear form on $V(9, q)$ preserved by A, the orthogonal complement of V_c must be $V_a \oplus V_b$ and V_a and V_b are either both totally isotropic or both non-isotropic. The first possibility is ruled out by consideration of $A^{(q^2+1)/2}$ which has block-diagonal form $(-I_4, I_4, 1)$. Hence T preserves a non-degenerate quadratic form on V_a and T^{q+1} preserves a non-degenerate quadratic form on V_b. As $\mathrm{SO}^+(4, q)$ has no element of order $(q^2 + 1)/2$, the quadratic forms on V_a and V_b are each elliptic. Finally we observe that an element of $\mathrm{SO}^-(4, q)$ of order $q^2 + 1$ is a Singer cycle of $\mathrm{SO}^-(4, q)$ and does not lie in $\Omega^-(4, q)$, while an element of order $(q^2 + 1)/2$ is the square of a Singer cycle of $\mathrm{SO}^-(4, q)$ and must therefore lie in $\Omega^-(4, q)$. It now follows that $A \in \mathrm{SO}(9, q) \backslash \Omega(9, q)$.

Let us specifically choose the basis v_1, v_2, v_3 for V_0 so that the quadratic form corresponding to f_0 is given by $Q_0(\lambda_1 v_1 + \lambda_2 v_2 + \lambda_3 v_3) = \lambda_3^2 - \lambda_1 \lambda_2$. Then the points on the conic $\mathscr{C}_0$ of Q_0 can be represented by $(1, t^2, t) : t \in \mathrm{GF}(q^2)$ together with $(0, 1, 0)$. The image $\mathscr{X}$ of $\mathscr{C}_0$ in the Hermitian Veronesean $\mathscr{H}$ is then given by

$$\{P(1, t^{2q+2}, t^{q+1}, t^{2q}, t^2, t^q, t, t^{2+q}, t^{2q+1}) : t \in \mathrm{GF}(q^2)\} \cup \{P(0, 1, 0, \ldots, 0)\}.$$

Thus $\mathscr{X}$ is a rational curve, all of whose points lie in a Baer subgeometry. Put another way, $\mathscr{X}$ is just the orbit of $\rho(\mathrm{SO}(3, q^2))$ on $\mathrm{PG}(8, q^2)$ given by $\{P(v_1 g \otimes v_1 g^{\psi}) :$

$g \in \mathrm{SO}(3, q^2)\}$, or indeed an orbit of $\rho(\langle S \rangle)$ where S is the Singer cycle in $\mathrm{SO}(3, q^2)$ given above. A point $x \otimes x^\psi$ of $\mathscr{H}$ is singular precisely when x is singular. Hence if $\mathscr{Q}$ is the quadric corresponding to the bilinear form f the points of $\mathscr{Q}$ lying on $\mathscr{H}$ are precisely the points of $\mathscr{X}$, i.e., $\mathscr{X}$ is the intersection of $\mathscr{H}$ and $\mathscr{Q}$. No two points of $\mathscr{X}$ are orthogonal so $\mathscr{X}$ is a partial ovoid.

There is a further geometric description. Using the geometric setting of Subsection 2.3, take a conic C in π and $\bar{C}$ in $\bar{\pi}$. The lines joining a point on C with its conjugate on $\bar{C}$ form a set $\mathscr{Y}$ of $q^2 + 1$ lines defined over $\mathrm{GF}(q)$, and it lies in the subgeometry $\mathscr{G}$ of $\mathrm{PG}(5, q^2)$. The image of $\mathscr{Y}$ on the Grassmannian $G_{1,5}$ of lines of $\mathrm{PG}(5, q)$, under the Plücker map, is the curve $\mathscr{X}$.

Proposition 4.2.1. *Let X be the full stabilizer of the rational curve $\mathscr{X}$ in $\Omega(9, q)$ (q odd), then X contains a subgroup isomorphic to* $\mathrm{PSL}(2, q^2) \cdot C_2$.

Proof. As we have seen, $\mathscr{X}$ is an orbit of $\rho(\mathrm{SO}(3, q^2))$ so is fixed by both $\rho(\Omega(3, q^2))$ and $\rho(\mathrm{SO}(3, q^2))$. Furthermore the involution θ introduced in Section 2 (θ is induced by $v_i \otimes v_j \leftrightarrow v_j \otimes v_i$) fixes $\mathscr{X}$, preserves the bilinear form f, lies outside $\rho(\mathrm{SO}(3, q^2))$ but normalizes $\Omega(3, q^2)$ (the conjugate of $g \otimes g^\psi$ being $g^\psi \otimes g$). We already know that $-\theta$ lies in $\mathrm{SL}(9, q)$ so $-\theta \in \mathrm{SO}(9, q)$. It would be nice to think that $-\theta$ always lies in $\Omega(9, q)$ but in fact it does so precisely when 2 is square in $\mathrm{GF}(q)$ (we omit the proof). We have seen that if S is a Singer cycle of $\mathrm{SO}(3, q^2)$, then $\rho(S) \in \mathrm{SO}(9, q) \backslash \Omega(9, q)$ and clearly $\rho(S)$ normalizes $\Omega(3, q^2)$. Hence one of $-\theta, -\theta\rho(S)$ lies in $\Omega(9, q) \backslash \rho(\Omega(3, q^2))$ and normalizes $\rho(\Omega(3, q^2))$. Hence we have identified a subgroup of $\Omega(9, q)$ isomorphic to $\mathrm{PSL}(2, q^2) \cdot C_2$ that stabilizes $\mathscr{X}$.

Proposition 4.2.2. *The full stabilizer X of the rational curve $\mathscr{X}$ is almost simple and is an absolutely irreducible subgroup of* $\Omega(9, q)$.

Proof. By Proposition 4.2.1, X has a subgroup isomorphic to $\mathrm{PSL}(2, q^2) \cdot C_2$. The argument here is again similar to that in the proof of Proposition 3.1.1. This time we look at Table 3.5.D in [9]. In Section 2, we saw that $\rho(\mathrm{PSL}(2, q^2))$ is absolutely irreducible and not realizable over any proper subfield of $\mathrm{GF}(q)$ so the same must apply to X. Hence X is not contained in a member of $\mathscr{C}_1$, $\mathscr{C}_3$ or $\mathscr{C}_5$. There are no maximal subgroups of $\Omega(9, q)$ in classes $\mathscr{C}_4$, $\mathscr{C}_6$ or $\mathscr{C}_8$. The order of $\mathrm{PSL}(2, q^2)$ does not divide the orders of the maximal subgroups in the classes $\mathscr{C}_2$ and $\mathscr{C}_7$. The required result follows by Aschbacher's Theorem.

Corollary 4.2.3. *Assume that $q \neq 3$. If Kleidman's list in* [10] *is correct, then X is isomorphic to* $\mathrm{PSL}(2, q^2) \cdot C_2$ *and is maximal in* $\mathrm{P}\Omega(9, q)$.

Proof. In Kleidman's list there are seven "sporadic" maximal subgroups (existence often dependent on q): $\mathrm{PSL}(2, 8)$, $\mathrm{PSL}(2, 17)$, S_{11}, $\mathrm{PGL}(2, q)$, $A_{10}.2$, A_{10} and $\mathrm{PSL}(2, q^2) \cdot C_2$. The first five are ruled out because their orders cannot be divided by the order of $\mathrm{PSL}(3, q^2)$ (for the corresponding values of q). It follows that X lies in one of the last two maximal subgroups. The case A_{10} exists only when $q = 3$.

Hence when $q \neq 3$, X must be a subgroup of the last "sporadic" maximal subgroup, $\mathrm{PSL}(2, q^2) \cdot C_2$, and as in Corollary 2.3.2, equality is readily established.

Suppose now that $q = 3$. In this case $\mathscr{X}$ has 10 points and $\Omega(3, 9)$ can be embedded in A_{10}. We can compare the embedding of $\Omega(3, 9)$ in $\Omega(9, 3)$ given by ρ with an embedding arising from the deleted permutation module for A_{10}. Recall that $\mathscr{X} = \{P(v_1 g \otimes v_1 g^{\psi}) : g \in \Omega(3, q^2)\}$. As shown earlier, we can take as representatives for the points of $\mathscr{X}$ the vectors $y = (0, 1, 0)$ and $x_\lambda = (1, \lambda^2, \lambda)$ for $\lambda \in \mathrm{GF}(9)$. We can calculate that $(y, x_\lambda) = 1$ and $(x_\mu, x_\lambda) = (\lambda - \mu)^{2(q+1)}$; given $q = 3$ we have $(x_\mu, x_\lambda) = 1$ when $\lambda \neq \mu$.

The permutation module M for A_{10} over $\mathrm{GF}(3)$ (c.f. [9] p. 185) is given by the action of A_{10} on the coordinate vectors of $V(10, 3)$; the hyperspace $M = \{(a_1, a_2, \ldots, a_{10}) : a_1 + a_2 + \cdots + a_{10} = 0\}$ is fixed globally by A_{10}; there is a bilinear form on $V(10, 3)$ given by $((a_1, a_2, \ldots, a_{10}), (b_1, b_2, \ldots, b_{10})) = 2(a_1 b_1 + a_2 b_2 + \cdots + a_{10} b_{10})$ which is non-degenerate on restriction to M and which is preserved by A_{10}; thus A_{10} can be embedded in $\Omega(9, 3)$ (M is known as the deleted permutation module for A_{10}). We therefore have an embedding of $\Omega(3, 9)$ in $\Omega(9, 3)$. Now consider the decomposition $V = M \oplus M^{\perp}$ (with $M^{\perp}$ being $\langle (1, 1, \ldots, 1) \rangle$). The projections of the coordinate vectors for $V(10, 3)$ onto M are the vectors $z_1, z_2, \ldots, z_{10}$ with z_i having 0 in the i'th position and 1's elsewhere: these are singular vectors spanning M, permuted faithfully by A_{10}, with $(z_i, z_j) = 1$ for any $i \neq j$.

A direct comparison between the two sets of 10 vectors shows that the two embeddings of $\Omega(3, 9)$ in $\Omega(9, 3)$ are equivalent and it follows that X must contain a subgroup isomorphic to A_{10}. Hence if A_{10} is maximal we must conclude that in this case X is isomorphic to A_{10}. We have established:

Corollary 4.2.4. *If $q = 3$ and Kleidman's list in* [10] *is correct, then X is isomorphic to A_{10} and is maximal in* $\mathrm{P}\Omega(9, 3)$.

4.3 Generalizations. Once again we finish the section with discussion of possible generalizations of the ideas above. On this occasion we consider different forms as well as mappings from subgroups of $\mathrm{GL}(n, q^t)$ to $\mathrm{GL}(n^t, q)$, and we consider possible embeddings of alternating groups.

Remark 4.3.1. If $\mathrm{O}(n, q^t)$ is the orthogonal group of a non-degenerate symmetric bilinear form f_0 on $V(n, q^t)$ (with q odd) and if ρ is the representation of $\mathrm{GL}(n, q^t) \to \mathrm{GL}(n^t, q^t)$ described in Subsection 2.4, then $\rho(\mathrm{O}(n, q^t))$ preserves a non-degenerate symmetric bilinear form $f = f_0 \otimes \cdots \otimes f_0$ (t copies of f_0). If an appropriate basis is chosen for V_0, then f is defined on $V_q = W$ over $\mathrm{GF}(q)$ and $\rho(\mathrm{O}(n, q^t)) \leqslant \mathrm{O}(n^t, q)$. If we assume $n \geqslant 3$ and exclude the case $\mathrm{O}^+(4, q^t)$, the subgroup $\rho(\Omega(n, q^t))$ is absolutely irreducible and cannot be written over a subfield of $\mathrm{GF}(q)$.

If $\mathrm{Sp}(n, q^t)$ is the symplectic group of a non-degenerate alternating form f_0 on $V(n, q^t)$ (with n even but q odd or even), then $\rho(\mathrm{Sp}(n, q^t))$ preserves the tensor product form f. If t is odd, then f is an alternating form and we find that $\rho(\mathrm{Sp}(n, q^t))$ is a subgroup of $\mathrm{Sp}(n^t, q)$. If t is even and q is odd, then f is a symmetric bilinear form and $\rho(\mathrm{Sp}(n, q^t))$ is a subgroup of $\mathrm{O}(n^t, q)$. If q is even (and n must then be even), then

$\mathrm{O}(n, q^t)$ maybe regarded as a subgroup of $\mathrm{Sp}(n, q^t)$ so $\rho(\mathrm{O}(n, q^t)) \leqslant \mathrm{Sp}(n^t, q)$, but more than this $\rho(\mathrm{Sp}(n, q^t))$ preserves a quadratic form on $V_q = W$ so $\rho(\mathrm{O}(n, q^t)) \leqslant \rho(\mathrm{Sp}(n, q^t)) \leqslant \mathrm{O}(n^t, q)$. If $\mathrm{U}(n, q^t)$ is the unitary group of a non-degenerate Hermitian form f_0 on $V(n, q^t)$ (with q square and t odd), then the tensor product form f is an Hermitian form preserved by $\rho(\mathrm{U}(n, q^t))$ and $\rho(\mathrm{U}(n, q^t)) \leqslant \mathrm{U}(n^t, q)$. [Except in the case of $\mathrm{O}^+(4, q^t)$, the image under ρ is absolutely irreducible and cannot be written over a subfield of $\mathrm{GF}(q)$.]

It is worth noting that the restrictions on n mean that there is no irreducible subgroup $\rho(\mathrm{Sp}(3, q^2))$ of $\mathrm{SL}(9, q)$ and thus, for q even, no irreducible subgroup $\rho(\mathrm{O}(3, q^2))$ of $\mathrm{SL}(9, q)$. The restriction on t for $\mathrm{U}(n, q^t)$ is more subtle. Steinberg's Tensor Product Theorem leads us to believe that for t even $\rho(\mathrm{U}(n, q^t))$ is not absolutely irreducible. Indeed for the case $t = 2$ it is known that $\rho(\mathrm{U}(n, q^2))$ is reducible, for it follows from [6, Theorem 43.14] that $\rho(\mathrm{U}(n, q^2))$ fixes all vectors in a 1-dimensional subspace of $V(n^2, q^2)$; moreover the restriction of the Hermitian form f to $V_q = W$ is actually a symmetric bilinear form so $\rho(\mathrm{U}(n, q^2))$ is a subgroup of $\mathrm{O}(n^2, q)$ (for q odd) or $\mathrm{Sp}(n^2, q)$ (for q even).

Remark 4.3.2. In Remark 4.2.4 we have seen that $\Omega(3, 9) \simeq A_6 < A_{10} < \Omega(9, 3)$. More generally, the embedding of $\Omega(3, 3^t)$ in $\Omega(3^t, 3)$ given by ρ is equivalent to an embedding arising via the deleted permutation module and leads to an intermediary alternating group. We can start with the curve $\mathscr{X} = \{P(v_1 g \otimes v_1 g^{\psi} \otimes \cdots \otimes v_1 g^{\psi^{t-1}}) : g \in \Omega(3, 3^t)\}$ in $\mathrm{PG}(3^t - 1, 3)$; again this is a rational curve and a partial ovoid. The points in $\mathscr{X}$ have representatives $y = v_2 \otimes v_2 \otimes \cdots \otimes v_2$ and $x_\lambda = u_\lambda \otimes u_\lambda^{\psi} \otimes \cdots \otimes u_\lambda^{\psi^{t-1}}$ where $u_\lambda = v_1 - \lambda^2 v_2 + \lambda v_3 \in V_0$ and λ ranges over $\mathrm{GF}(q^t)$. As $\mathscr{X}$ has $3^t + 1$ points and is fixed globally by $\rho(\Omega(3, q^t))$, we have an embedding of $\Omega(3, 3^t)$ in A_{3^t+1}. We see further that $(y, x_\lambda) = 1, (x_\lambda, x_\mu) = (-1)^t$, leading to a direct comparison with the deleted permutation module for A_{3^t+1} (for t odd we need to replace y by $-y$ and consider the bilinear form on $V(3^t + 1, 3)$ given by $((a_1, a_2, \ldots, a_{3^t+1}), (b_1, b_2, \ldots, b_{3^t+1})) = a_1 b_1 + a_2 b_2 + \cdots + a_{3^t+1} b_{3^t+1})$. We deduce that the embedding of $\Omega(3, 3^t)$ in $\Omega(3^t, 3)$ given by ρ is indeed equivalent to that arising from the deleted permutation module and thus $\rho(\Omega(3, 3^t)) < A_{3^t+1} < \Omega(3^t, 3)$.

Let us consider briefly whether a generic embedding of $\mathrm{P}\Omega(n, q^t)$ in $\mathrm{P}\Omega(n^t, q)$ is likely to lead to an intermediary alternating group. For example we can consider the possibility that $\Omega(3, q^t) < A_r < \Omega(3^t, q)$ for some r, some odd q and some $t \geqslant 2$. Let d be the minimal degree for a permutation representation of the group $\Omega(3, q^t)$. From [4, Table 1] (reproduced in [9, Table 5.2.A]) we see that $d = q^t + 1$ if $q^t \neq 9$ and 6 if $q^t = 9$. Furthermore, from [9, Proposition 5.3.7] we see that $r \leqslant 3^t + 2$. Hence, with a single exception, we have $q^t + 1 \leqslant r \leqslant 3^t + 2$. It follows immediately that $q = 3$ and $r = 3^t + 1$ or $3^t + 2$. By [9, Proposition 5.3.5], V (the module for $\Omega(3^t, q)$) is isomorphic to the fully deleted permutation module for A_r, with $r = 3^t + 1$. A similar consideration of $\Omega(5, q^t) < A_r < \Omega(5^t, q)$ yields $q^{3t} + q^{2t} + q^t + 1 \leqslant r \leqslant 5^t + 2$, which has no solutions for q odd and $t \geqslant 2$, and indeed there are no other instances of intermediary alternating groups for simple groups $\mathrm{P}\Omega(n, q^t)$ embedded in $\mathrm{P}\Omega(n^t, q)$ when q is odd. Hence the only possibility for an intermediary alternating group is the one we have already seen.

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