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Generalizing flocks of $Q^+(3,q)$

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Abstract. We define flocks of Segre varieties $S_{n,n}$ as a generalization of flocks of $Q^+(3,q)$, studying the connections with translation planes.

Key words: Flock, spread, nearfield, Segre variety, Veronese variety, Singer cycle.

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1 Introduction

Let $Q^+(3,q)$ denote the hyperbolic quadric of $\mathrm{PG}(3,q)$, q any prime power. A *flock* of $Q^+(3,q)$ is a partition of the quadric in $q+1$ irreducible conics. A flock is *linear* if all the planes of the conics of the flock contain a common line. Flocks of $Q^+(3,q)$ are related to maximal exterior sets of hyperbolic quadrics ([8]) and to inversive planes ([14]). Also, they are equivalent to certain translation planes of order q^2 whose kernels contains $\mathrm{GF}(q)$, as we explain now. Embed $Q^+(3,q)$ in the Klein quadric $Q^+(5,q)$ as a section with a 3-space L, and let l be the polar line of L with respect to $Q^+(5,q)$. Then, $l \cap Q^+(5,q) = \{a,b\}$ for certain points a and b. The polar plane of each plane of the flock intersects $Q^+(5,q)$ in an irreducible conic containing the points a and b, the union of these conics is an ovoid $\mathcal{O}$, and the Klein correspondence ϕ maps $\mathcal{O}$ to a line spread of $\mathrm{PG}(3,q)$ consisting of reguli sharing the lines $A = a^\phi$ and $B = b^\phi$, hence it is an (A,B)-regular spread. Conversely, any (A,B)-regular line spread gives a flock of $Q^+(3,q)$ by reversing the above construction ([13], [16]).

Flocks of $Q^+(3,q)$ have been classified for q even, and it was proven that they are necessarily linear ([12]). For q odd, the study of conic configurations allowed to prove that the translation plane associated with a flock of $Q^+(3,q)$ is coordinatized by a nearfield ([14], [2]), obtaining a complete classification of the translation planes defined by (A,B)-regular spreads and of the flocks of $Q^+(3,q)$, which are either

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linear, or of Thas type (obtained by taking two halves of suitable linear flocks [12]), or exceptional (existing for $q = 11, 23, 59$ [1]). See also [3] for related results.

As $Q^+(3,q) = S_{1,1}$ is the smallest Segre variety and the Klein quadric is the Grassmannian of the lines of $\mathrm{PG}(3,q)$, our aim is to extend the notion of flock to the Segre variety $S_{n,n}$, studying it via the Grassmannian $\mathscr{G}_{1,2n+1}$. We first prove that any (A,B)-regular spread of $\mathrm{PG}(2n+1,q)$ is equivalent to a partition of $S_{n,n}$ into Veronese varieties canonically embedded in the Segre variety; such a partition we call a *flock*, so that flocks of $S_{n,n}$ are equivalent to a class of translation planes. Further, we define linear flocks and we show that they always exist.

In Section 3 we study the families of translation planes associated with flocks. In particular, in Section 3.1, for any $n > 1$, starting with the Dickson nearfield $N(n+1,q)$, a flock of $S_{n,n}$ is constructed, both for q even and q odd, which is the union of equivalent "blocks" of partial linear flocks: this is, for $n = 1$, the original construction of the Thas flock of $Q^+(3,q)$. Furthermore, as proper semifields of dimension at least three over the center $\mathrm{GF}(q)$ do exist, the class of flocks of $S_{n,n}$, $n > 1$, associated with (proper) A-regular spreads, is not empty; a geometric characterization of these flocks is given in Section 3.2. The connections between linear flocks and desarguesian spreads are discussed in Section 3.3.

For q odd, the classification of the flocks of $S_{1,1}$ was obtained using Thas' Lemma ([14]), which states that any involutorial collineation of $Q^+(3,q)$, with axis a plane of the flock, fixes the flock. This Lemma allowed to prove that the translation plane associated with any flock of $Q^+(3,q)$ is coordinatized by a nearfield. In Section 4 we observe that, for $n > 1$, Thas' Lemma does not hold, even with a weaker statement, hence one would expect a number of non-isomorphic families of flocks of $S_{n,n}$.

Finally, in Section 5 we remark that it is impossible to extend the construction of the ovoid consisting of conics with two common points.

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2 Flocks of Segre varieties

Let $\mathrm{PG}(n,K)$ be the projective space of dimension n over the field K, with $n \geqslant 1$. Set $N = n^2 + 2n$. The *Segre variety* $S_{n,n}$ of $\mathrm{PG}(N,K)$ consists of all points represented by the vectors $u \otimes v$, as u and v vary over all points of $\mathrm{PG}(n,K)$. Denote by $\mathscr{G}_{1,n}$ the Grassmannian of lines of $\mathrm{PG}(n,K)$, i.e. the variety of $\mathrm{PG}(m,K)$, $m = \binom{n+1}{2}$, representing, under the Plücker map, the 1-dimensional subspaces of $\mathrm{PG}(n,K)$. Recall that $Q^+(3,K)$ is the Segre variety $S_{1,1}$, and the Grassmannian $\mathscr{G}_{1,3}$ of the lines of $\mathrm{PG}(3,K)$ is the Klein quadric. For more details, see e.g. [7, Sections 24 and 25].

A **flock** of $S_{n,n}$ is a partition of the point set of $S_{n,n}$ into Veronese varieties, obtained as sections of $S_{n,n}$ by subspaces of $\mathrm{PG}(N,K)$ of dimension $n(n+3)/2$.

Note that one might also construct different partitions of $S_{n,n}$, e.g. into caps of the same size as the Veronese varieties, with different geometric properties, but our definition is motivated by the connection with translation planes in Theorem 2.

An *n-spread* of $\mathrm{PG}(2n+1,K)$ is a set of n-dimensional subspaces such that every point is contained in exactly one subspace. An *n-regulus* in $\mathrm{PG}(2n+1,K)$ is a set of mutually skew n-dimensional subspaces, such that every line ℓ meeting any three of them meets all of them, and any point of ℓ is on (exactly) one element of the n-regulus. Such a line is called a *transversal* to the regulus. We simply say *spread* and *regulus* whenever the dimension is clear from the context. A 1-spread is sometimes called a line spread.

A spread $\mathscr{S}$ of $\mathrm{PG}(2n+1,K)$ is said to be (A,B)*-regular* if there exist $A,B\in\mathscr{S}$ such that, for any $C\in\mathscr{S}$, $C\neq A,B$, the regulus containing A,B,C consists of elements of $\mathscr{S}$. If $\mathscr{S}$ is (A,B)-regular for all B in $\mathscr{S}$ different from A, then $\mathscr{S}$ is *A-regular*. A spread is called *regular* if the regulus containing any three elements of the spread completely consists of elements of the spread. Note that for $q=2$ all spreads are regular.

Theorem 1. *Each Veronese variety on $S_{n,n}$ represents the set of the transversal lines to a regulus of an n-spread $\mathscr{S}$ of a $(2n+1)$-dimensional projective space.*

Proof. The Segre variety $S_{n,n}$ of $\mathrm{PG}(N,K)$ is in canonical bijective correspondence with the set of lines of $\mathrm{PG}(2n+1,K)$ meeting two fixed disjoint subspaces of dimension n, say π_1 and π_2, representing $\mathrm{PG}(n,K)$ and its dual, respectively. Fix any linear projectivity from π_1 to π_2; a Veronese variety on $S_{n,n}$ is the set of (disjoint) lines connecting any point of π_1 with its image on π_2.

Here is the construction of the translation plane.

Theorem 2. *To any flock $\mathscr{F}$ of $S_{n,n}$ there corresponds an (A,B)-regular n-spread of* $\mathrm{PG}(2n+1,K)$, *which defines a translation plane $\Pi(\mathscr{F})$ of dimension at most $n+1$ over the kernel, which contains K. Conversely, any translation plane arising from an (A,B)-regular n-spread of* $\mathrm{PG}(2n+1,K)$ *canonically defines a flock of $S_{n,n}$.*

Moreover, the flocks are isomorphic if and only if the translation planes are.

Proof. The proof follows from Theorem 1, since every point of $\mathrm{PG}(2n+1,K)$ neither on π_1 nor on π_2 is on exactly one line meeting both π_1 and π_2.

A flock $\mathscr{F}$ of $S_{n,n}$ is **linear** if all the $n(n+3)/2$-dimensional subspaces of the Veronese varieties of the flock share an n-dimensional subspace of $\mathrm{PG}(N,K)$.

Theorem 3. *The Segre variety $S_{n,n}$ of* $\mathrm{PG}(N,q)$ *has a linear flock, and the associated translation plane is a nearfield plane.*

Proof. As in the proof of Theorem 1, let ϕ be a projectivity from π_1 to π_2, and choose coordinates in such a way that ϕ is represented by the identity matrix. The Segre variety consists of the matrices $M_{ij}=X_iY_j$ with X_i (resp. Y_j) coordinates in π_1 (resp. π_2), and the ambient space of a Veronesian consists of all symmetric $(n+1)\times(n+1)$-matrices. Compose ϕ with all the elements of the Singer group $\langle S\rangle$ of π_1 to get a partition into Veronese varieties. This flock is linear because, for each symmetric matrix M, if SM is symmetric, then so is $S^k M$ for all natural numbers k: $S^2 M$ is symmetric because $(SSM)^t = (SM)^t S^t = S(MS^t) = S(SM)^t = SSM$, the general case following by induction.

Moreover, $\langle S \rangle$ acts on the Grassmannian $\mathscr{G}_{1,2n+1}$ as the group T generated by $\begin{pmatrix} I & 0 \\ 0 & X \end{pmatrix}$, with $X \in GL(n+1,q)$ of order $(q^{n+1}-1)/(q-1)$, and I the identity $(n+1) \times (n+1)$ matrix; the spread consists of $A = \{(0,y) \mid y \in \mathrm{GF}(q)^{n+1}\}$, $B = \{(x,0) \mid x \in \mathrm{GF}(q)^{n+1}\}$ and the $q^{n+1}-1$ elements $\{(a, a\lambda X^k) \mid a \in \mathrm{GF}(q)^{n+1}\}$ with $\lambda \in \mathrm{GF}(q) \setminus \{0\}$. The group T fixes B pointwise, A setwise, and acts transitively on the elements of the spread different from both A and B. Hence, this spread defines a nearfield plane.

3 Flocks and translation planes

We want to study the connections between flocks and translation planes via (A,B)-regular spreads, for the field $K = \mathrm{GF}(q)$. Note that some results still hold for K an infinite field.

As flocks of $S_{n,n}$ are equivalent to translation planes of dimension at most $n+1$ over the kernel $\mathrm{GF}(q)$, defined by (A,B)-regular n-spreads $\mathscr{S}$ of $\mathrm{PG}(2n+1,q)$, if $q > 2$ there are exactly three families of flocks, characterized by the properties of the coordinatizing quasifield Q (for the relevant definitions and properties, see [4, pp. 131–135], [5], [6]):

a) $\mathscr{S}$ is (A,B)-regular, i.e. $\mathrm{GF}(q)$ is contained in the middle nucleus of Q and $\mathrm{GF}(q)$ is central in Q;

b) $\mathscr{S}$ is A-regular, i.e. Q is a semifield, whose center contains $\mathrm{GF}(q)$;

c) $\mathscr{S}$ is regular, i.e. Q is a field.

Here we show that the flocks of the first family associated with Dickson nearfields are the natural generalization of Thas flocks of $Q^+(3,q) \cong S_{1,1}$, and we give a geometric characterization of the second family in terms of a configurational proposition. Also, we prove that the flock corresponding to the third family is linear, and that the linear flock constructed in Theorem 3 corresponds to a desarguesian plane.

3.1 (A,B)-regular spreads. A *regular nearfield*, or a *Dickson nearfield*, $N(n+1,q)$, is defined as follows (see e.g. [4]). Let $q = p^e$ be a prime power and $n+1$ an integer all of whose prime divisors divide $q-1$. Also, suppose $n+1 \not\equiv 0 \bmod 4$ if $q \equiv 3 \bmod 4$. The pair $(n+1,q)$ is called a Dickson pair, and $n+1$ divides $(q^{n+1}-1)/(q-1)$ ([11, Theorem 6.4]). Let $F = \mathrm{GF}(q^{n+1})$ and c a primitive element of the field. Then $G = \langle c^{n+1} \rangle$ is a subgroup of F^* whose cosets are represented by the elements $c_i = c^{(q^i-1)/(q-1)}$, for $i = 0,1,\dots,n$. Define $\lambda : F^*(\cdot) \mapsto Z_{n+1}(+)$ as $x \mapsto i$ if $xG = c_i G$, and $\sigma : F^* \mapsto F^*$ as $x \mapsto x^q$. Define also a new multiplication $\circ : F \mapsto F$ by $x \circ 0 = 0$ and $x \circ y = x^{\sigma^{\lambda(y)}} y$ for $x, y \in F$ and $y \neq 0$. Both λ and σ are group homomorphisms, and $\lambda(x) = 0$ for all $x \in \mathrm{GF}(q)^*$. Then $N = N(n+1,q) = F(+,\circ)$ is a nearfield with kernel $\mathrm{GF}(q)$. The number of non-isomorphic regular nearfields, for a given Dickson pair $(n+1,q)$, is $\phi(n+1)f^{-1}$, with ϕ the Euler function and f the order of p modulo

$n+1$. Note that the construction of non-isomorphic $N(n+1,q)$'s depends on the choice of λ.

A finite nearfield is either regular or is one of the exceptional nearfields [17], and all exceptional nearfields have dimension 2 over the kernel $\mathrm{GF}(p)$. Hence, a nearfield of dimension $n+1>2$ over the kernel is one of the $N(n+1,q)$'s.

On the other hand, let $a \in \mathrm{GF}(q)$ and $y \in N$. Then $a \circ y = a^{\sigma^{\lambda(y)}} y = ay = ya = y \circ a$, hence the kernel is central. As the quasifield N is a nearfield, the spread associated with the translation plane coordinatized by $N(n+1,q)$ is (A,B)-regular ([9]). Note that an (A,B)-regular spread can arise from a quasifield which is not a nearfield, see e.g. [10, Corollario].

Theorem 4. *Let $(n+1,q)$ be a Dickson pair. The flock of $S_{n,n}$ associated with the regular nearfield $N(n+1,q)$ consists of $n+1$ equivalent families of Veronese surfaces, say $\mathscr{E}_1,\mathscr{E}_2,\dots,\mathscr{E}_{n+1}$. For each $i=1,2,\dots,n+1$, the Veronesians of $\mathscr{E}_i$ belong to spaces which share a fixed n-dimensional space, and each $\mathscr{E}_i$ can be completed to a linear flock.*

Proof. Fix an element $\alpha \in \mathrm{GF}(q^{n+1}) = F$ such that $\lambda(\alpha)=1$. Recall that $n+1$ divides $q^n+\cdots+q+1$, hence $n+1$ divides $q^{n+1}-1$, and α^{n+1} is in the cyclic subgroup of F^* of order $(q^{n+1}-1)/(n+1)$.

For any $y \in F$, let $C(y)$ be defined by $xy = x \circ C(y)$. Put

$$\mathscr{C}_0 = \{C(y) \mid \lambda(y)=0,\ y\in F\},$$
$$\mathscr{C}_1 = \{C(y\circ\alpha) \mid \lambda(y)=0,\ y\in F\},$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\mathscr{C}_n = \{C(y\circ\alpha^n) \mid \lambda(y)=0,\ y\in F\}.$$

The cardinality of each $\mathscr{C}_j$ is $(q^{n+1}-1)/(n+1)$ for $j=0,1,\dots,n$.

Define $T: F \mapsto F$, $z \mapsto z\circ\alpha = z^\sigma\alpha$ and note that $\mathscr{C}_1 = \mathscr{C}_0 T$, $\mathscr{C}_2 = \mathscr{C}_1 T, \dots, \mathscr{C}_0 = \mathscr{C}_n T$. Let $\mathscr{S}$ be the spread associated with $N(n+1,q)$, and let $\mathscr{S}_j \subset \mathscr{S}$, for $j=0,1,\dots,n$, be the partial spread corresponding to $\mathscr{C}_j$. Clearly, $\{\mathscr{S}_0,\mathscr{S}_1,\dots,\mathscr{S}_n\}$ is a partition of $\mathscr{S}$ and $\mathscr{S}_0$ is the union of $(q^n+\cdots+q+1)/(n+1)$ reguli, each containing the elements of the spread corresponding to (0) and (∞) in the given coordinatization. Also, $\mathscr{S}_0$ is contained in the regular spread $\mathscr{F}_0$ associated with the field F.

The map $(x,y)\mapsto(x,y^\sigma\alpha)$ fixes $\mathscr{S}$ (setwise) and acts as a cycle on $\{\mathscr{S}_0,\mathscr{S}_1,\dots,\mathscr{S}_n\}$. Hence $\mathscr{S}_1$, $\mathscr{S}_2,\dots,\mathscr{S}_n$, as well as $\mathscr{S}_0$, are the union of $(q^n+\cdots+q+1)/(n+1)$ reguli, each containing the elements of the spread corresponding to (0) and (∞), and are contained in regular spreads $\mathscr{F}_1,\mathscr{F}_2,\dots,\mathscr{F}_n$, which are the images of $\mathscr{F}_0$ under $T,T^2,\dots,T^n$, respectively. Hence, the theorem is proved.

Note that flocks of $S_{n,n}$ associated with Dickson nearfield planes are the natural generalization of Thas flocks of $Q^+(3,q)\cong S_{1,1}$.

3.2 *A*-regular spreads. Let $n > 1$. If $\mathcal{S}$ is an (A, B)-regular spread of $\mathrm{PG}(2n+1, q)$, then each line incident with A and B is a transversal of some regulus of $\mathcal{S}$ containing A and B. Two lines l and m incident with both A and B are called (A, B)-parallel if they are transversals of the same regulus (containing A and B) of $\mathcal{S}$, and we write $l \parallel m$.

Define a configurational proposition $(\mathcal{L})$ in the following way:
$(\mathcal{L})$ Let l_1, l_2, l_3 (resp. m_1, m_2, m_3) be three lines on the point P (resp. Q) in B and incident with A, such that $l_i \parallel m_i$ for $i = 1, 2, 3$. If l_1, l_2, l_3 are in a plane, then m_1, m_2, m_3 are in a plane.

Theorem 5 ([9]). *An (A, B)-regular spread $\mathcal{S}$ is A-regular if and only if the configurational proposition $(\mathcal{L})$ holds in $\mathcal{S}$.*

We now characterize flocks associated with A-regular spreads in terms of a configurational proposition on the Veronesians of the flock.

Let $S_{n,n}$ be the Segre variety representing on $\mathcal{G}_{1,2n+1}$ the lines of $\mathrm{PG}(2n+1, q)$ incident with both A and B. Denote by $\mathcal{M}_1$ and $\mathcal{M}_2$ the two systems of $S_{n,n}$. The n-dimensional subspaces of one of the systems, say $\mathcal{M}_1$, represent the lines incident with A and a fixed point of B, while the n-dimensional subspaces of the other system, say $\mathcal{M}_2$, represent the lines incident with B and a fixed point of A. A Veronese variety $\mathcal{V}$, intersection of $S_{n,n}$ with a subspace of dimension $n(n+3)/2$, is the representation on $\mathcal{G}_{1,2n+1}$ of the transversals of a regulus $\mathcal{R}$ containing A and B. Therefore, as through any point of A, respectively B, there is exactly one transversal line of $\mathcal{R}$ incident with it, each subspace of $\mathcal{M}_1$, resp. $\mathcal{M}_2$, intersects $\mathcal{V}$ in exactly one point.

Let $\mathcal{F}$ be a flock of $S_{n,n}$ and let $\mathcal{M}$ be any of the systems of $S_{n,n}$. Define the configurational proposition:

$(\mathcal{L}')$ Let $\mathcal{V}_1, \mathcal{V}_2, \mathcal{V}_3$ be three Veronesians of $\mathcal{F}$. For any $X, Y \in \mathcal{M}$, let $p_i = X \cap \mathcal{V}_i$ and $q_i = Y \cap \mathcal{V}_i$ with $i = 1, 2, 3$. If p_1, p_2, p_3 are on a line, then q_1, q_2, q_3 are on a line.

From Theorem 5, it follows

Theorem 6. *The (A, B)-regular spread associated with a flock $\mathcal{F}$ of $S_{n,n}$ is A-regular if and only if the configurational proposition $(\mathcal{L}')$ holds in $\mathcal{F}$.*

Note that for $n = 1$ both $(\mathcal{L})$ and $(\mathcal{L}')$ are trivial. On the other hand, no proper semifield of dimension two over the center $\mathrm{GF}(q)$ exists.

3.3 Regular spreads. First, observe that the linear flock constructed in Theorem 3 is associated with a desarguesian plane.

Theorem 7. *The linear flock of $S_{n,n}$ constructed in Theorem* 3 *corresponds to a regular n-spread of* $\mathrm{PG}(2n+1, q)$.

Proof. If $n = 1$ the result is known. If $n \geqslant 2$, the flock constructed in Theorem 3 defines a nearfield plane, and (proper) nearfields of dimension greater than two over the kernel are Dickson nearfields (see e.g. [4, pp. 229–232]), which are associated, by Theorem 4, with flocks which are not linear.

On the other hand, we can prove that the flock arising from a regular spread is the linear flock of Theorem 3.

Theorem 8. *The desarguesian n-spread of* $\mathrm{PG}(2n+1,q)$ *corresponds to the linear flock of* $S_{n,n}$ *constructed in Theorem* 3.

Proof. The multiplicative group of the field coordinatizing the translation plane is cyclic, hence it contains an element, say X, of order $q^{n+1}-1$. With the spread as in Theorem 3, the group generated by the collineation $\begin{pmatrix} I & 0 \\ 0 & X \end{pmatrix}$ fixes B pointwise, A setwise, and is transitive on the reguli; its image on $\mathscr{G}_{1,2n+1}$ represents a Singer cycle, say S. Then, one can regard S as the identity on one of the generators and as a Singer cycle on the other one. The flock is therefore constructed exactly as in Theorem 3.

Remark. For $n=1$, all linear flocks are isomorphic, as each one is defined by all the planes of a 3-dimensional space containing a fixed exterior line to $Q^+(3,q)$. For $n>1$, the spaces which actually contribute to the flock are some of all those sharing the fixed n-dimensional space. Hence, a priori, linear flocks might exist associated with non-desarguesian translation planes. Consequently, it is still an open problem to determine whether linear flocks associated with non-desarguesian planes exist.

4 Thas' Lemma

The classification of flocks of $Q^+(3,q)$, q odd, relies on the following property:

Theorem 9 ([14], Theorem 2). *Let* $\mathscr{F}$ *be a flock of* $Q^+(3,q)$, *q an odd prime power. For any plane of the flock, there exists an involutorial collineation of* $Q^+(3,q)$ *fixing the plane pointwise and stabilizing the flock.*

We want to remark explicitly that even a weaker statement of the above result is, in general, not true for the Segre variety $S_{n,n}$ with $n>1$. Indeed, suppose there is an involutorial collineation ψ of $S_{n,n}$ fixing pointwise a particular element of the flock and stabilizing the flock. As each collineation of the Segre variety is induced by a collineation of the Grassmannian $\mathscr{G}_{1,2n+1}$, ψ defines an involutorial collineation $\bar{\psi}$ of $\mathrm{PG}(2n+1,q)$ stabilizing the set of the transversals to an (A,B)-regular spread (associated with the flock), and fixing each transversal of a particular regulus, say $\mathscr{R}_0$. Therefore, there are two possibilities: either $\bar{\psi}$ fixes A and B, or $\bar{\psi}$ interchanges A and B. If $\bar{\psi}$ fixes both A and B, then it is the identity on A and B (because each transversal of $\mathscr{R}_0$ is fixed); hence, all the transversals of A and B are fixed, i.e., $\bar{\psi}$ is the identity on $S_{n,n}$. Thus, $\bar{\psi}$ interchanges A and B. Hence, any A-regular spread is regular, i.e. any semifield, whose center contains $\mathrm{GF}(q)$, of dimension $n+1>2$ over the center, is a field, a contradiction, as e.g. Albert twisted fields are examples of such semifields.

Finally, observe that Thas' Lemma holds for a particular regulus if $(ab)b^{-1} = a$ for all a, b in the quasifield ([10]).

5 Flocks and ovoids

Recall that, for q odd, associated with a flock of $Q^+(3,q)$ there is the ovoid of $Q^+(5,q)$ consisting of the points of the $q+1$ conics with two common points which corresponds to the lines of the (A,B)-regular spread. Hence, one can ask for a possible generalization of this configuration related with flocks, precisely: given q^n+1 points of $Q^+(2n+1,q)$, q odd, lying on $q^{n-1}+q^{n-2}+\cdots+q+1$ conics with two common points, do they form an ovoid?

These ovoids do not exist, as they are related with maximal exterior sets of hyperbolic quadrics, as we show here.

We discuss the case $n=3$, the general case following by a similar argument. By way of contradiction, suppose there exists an ovoid $\mathcal{O}$ of $Q^+(7,q)$ consisting of q^2+q+1 conics C_i through the points a and b, and let π' and π'' be the planes containing two of these conics. Denote by $\perp$ the polarity defined by $Q^+(7,q)$. The 3-dimensional space $\langle \pi' \cup \pi'' \rangle$ intersects $Q^+(7,q)$ in some $Q^-(3,q)$, as $\mathcal{O}$ is an ovoid, hence $\langle \pi' \cup \pi'' \rangle^\perp$ also intersects $Q^+(7,q)$ in some $Q^-(3,q)$. Also, for any plane π containing a conic C_i, $\pi^\perp$ intersects $Q^+(7,q)$ in a quadric $Q(4,q)$ contained in the 5-dimensional space polar to the line joining a and b, and $\langle a,b \rangle^\perp \cap Q^+(7,q)$ is some $Q^+(5,q)$ because $\langle a,b \rangle$ is a secant line. Hence, we have a set of q^2+q+1 quadrics $Q(4,q)$, contained in some $\mathrm{PG}(5,q)$, which pairwise intersect in some elliptic 3-dimensional quadric, and the polar points of these quadrics (with respect to the polarity defined by $Q^+(5,q)$), form a set of q^2+q+1 points of $\mathrm{PG}(5,q)$ such that the line joining any two of them is external to $Q^+(5,q)$. Such a set of points is, by definition, a maximal exterior set of $Q^+(5,q)$, which does not exist by [15], as q is odd.

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