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On threefolds admitting a bielliptic curve as abstract complete intersection

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Abstract. We study smooth projective varieties $X \subseteq \mathbb{P}^N$ of dimension 3, such that there are two very ample invertible sheaves $\mathscr{L}$, $\mathscr{M}$ on X, and there exist two sections of $\mathscr{L}$, $\mathscr{M}$ which intersect along a bielliptic curve C. We give a classification of such threefolds X under some hypotheses on the degree of C with respect to the two embeddings given by $\mathscr{L}$, $\mathscr{M}$.

Key words. Threefolds, polarizations, bielliptic curves, special varieties.

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Introduction

The question of classifying projective varieties which possess hyperplane sections with special properties is a classical one in Algebraic Geometry (e.g. see [7], [11], [21], [13]). In particular a problem that has been widely studied also in recent times is that of varieties with hyperelliptic, bielliptic or trigonal curve-sections (e.g. see [25], [6], [5], [12], [22], [4], [2], [9], [10]).

A natural generalization of this kind of problem is to classify projective varieties having particular curves C as intersection of sections of different very ample line bundles, according to the following definition:

Definition 1. Let X be a smooth, irreducible scheme of dimension d, defined over an algebraically closed field k of characteristic zero. Let $\mathscr{L}_1, \dots, \mathscr{L}_r$ be very ample line bundles on X. We say that a subscheme $V \subseteq X$, of dimension $d - r$, is an *abstract complete intersection* of $\mathscr{L}_1, \dots, \mathscr{L}_r$ in X, ab.c.i. for short, if $\mathscr{I}_V \subset \mathscr{O}_X$ is globally generated by r sections $A_1 \in H^0(X, \mathscr{L}_1), \dots, A_r \in H^0(X, \mathscr{L}_r)$.

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A classification of the possibilities for X when $d = 3$ and C is a smooth hyperelliptic curve is given in [8], which is what inspired us for the present paper.

To be precise, in this paper we consider the case of triples $(X, \mathscr{L}, \mathscr{M})$ such that:

($*$) X is a smooth irreducible scheme with $\dim X = 3$, $\mathscr{L}$ and $\mathscr{M}$ are two very ample line bundles such that there is an irreducible, smooth, bielliptic curve $C \subset X$ which is an ab.c.i. in X of two smooth, irreducible sections $A \in |\mathscr{M}|$ and $B \in |\mathscr{L}|$.

We will always assume that $\mathscr{M} \neq \mathscr{L}$, in fact when $\mathscr{M} = \mathscr{L}$ we have that C is a curve-section of X (in the embedding given by $\mathscr{L}$) and this case has already been studied in [10]. Moreover, in view of [8], we assume that C is not hyperelliptic (hence, in particular, we assume that for the genus $g(C)$ of C we have $g(C) > 2$).

In order to introduce our results and to give some examples of the varieties we are concerned with, let us introduce some notation. Let $\mathscr{F}_V$ denote the restriction of a sheaf $\mathscr{F}$ on X to a subscheme $V \subseteq X$. We define $d_A = \mathscr{L}_A^2$, $d_B = \mathscr{M}_B^2$, of course we have also $d_A = \mathscr{M}\mathscr{L}^2$ and $d_B = \mathscr{L}\mathscr{M}^2$. Without loss of generality we can always suppose $d_A \geqslant d_B$.

Then a first example of this kind of varieties is offered by:

Example 1. Let $X \cong \mathbb{P}^3$ and consider a (canonical) bielliptic plane quartic curve $C \subset H$ of genus 3, where H is a plane in $\mathbb{P}^3$. Of course C is the complete intersection of H and a quartic surface, hence if we put $\mathscr{L} = \mathscr{O}(4)$, $\mathscr{M} = \mathscr{O}(1)$ we are in the situation of ($*$), and here $d_A = 16$, $d_B = 4$.

We will be able to describe the triples $(X, \mathscr{L}, \mathscr{M})$ as in ($*$) when either $d_A \geqslant 18$ or $d_B \leqslant 8$; see the statements of Theorems A, B and C.

Notice that if $d_A \geqslant 19$ (Theorem A) then X is a fibration over a curve (either elliptic or bielliptic); this fact allows us to extend our classification to the case $\dim X \geqslant 4$, see the statement of Theorem A$'$.

The case $d_A = 18$ described in Theorem B seems to be a threshold, in fact for $d_A \leqslant 18$ more kinds of varieties satisfying the condition in ($*$) do appear.

Example 2. Let $X \cong \mathbb{P}^3$ and consider a (canonical) bielliptic curve C of genus 4 which is the complete intersection of a cone Λ over a plane (smooth) cubic curve and a quadric not passing through the vertex of Λ. Hence if $\mathscr{L} = \mathscr{O}(3)$, $\mathscr{M} = \mathscr{O}(2)$ we have a situation as in ($*$) with $d_A = 18$, $d_B = 12$.

We remark that for $d_A = 18$, we cannot affirm that all the varieties listed in Theorem B actually possess a curve C as in ($*$). On the other hand, we can see that there are examples of threefolds as in ($*$) with $9 \leqslant d_B \leqslant d_A \leqslant 17$:

Example 3. Let $\pi : X \to \mathbb{P}^3$ be the blowing up of $\mathbb{P}^3$ at a point P, then $\operatorname{Pic} X \cong \mathbb{Z}\langle H, E\rangle$, where H is the strict transform of a generic plane in $\mathbb{P}^3$ (not through P) and E is the exceptional divisor. We have that $\mathscr{M} = \mathscr{O}_X(2H - E)$ and $\mathscr{L} = \mathscr{O}_X(3H - E)$ are very ample on X and we can choose sections A, B of them (see Example 2) such that their intersection is a bielliptic curve C of genus 4. In this case we have $d_B = 11$, $d_A = 17$.

Example 4. Let $X \subseteq \mathbb{P}^6$ be a double covering $\pi : X \to Y$ of the rational normal threefold $Y \cong \mathbb{P}^1 \times \mathbb{P}^2 \to \mathbb{P}^5$, ramified along a divisor of type $\mathcal{O}_Y(2,2)$ (X is a Fano threefold with $\operatorname{Pic} X \cong \mathbb{Z}^2$, see e.g. [19]). Then $\deg X = 6$ and X can be viewed as obtained by taking a cone over Y from a point in $\mathbb{P}^6$ and intersecting it with a quadric not passing through its vertex. Let $\mathcal{M} \cong \pi^*(\mathcal{O}_Y(1,1))$ and $\mathcal{L} \cong \pi^*(\mathcal{O}_Y(1,2))$. We have $\mathcal{O}_Y(a,b) \cdot \mathcal{O}_Y^2(1,1) = a + 2b$, hence $\mathcal{O}_Y(1,2) \cdot \mathcal{O}_Y^2(1,1) = 5$ and the generic intersection $\mathcal{O}_Y(1,2) \cdot \mathcal{O}_Y(1,1)$ is an elliptic normal curve Γ_5 in $\mathbb{P}^4$. Our curve C, an ab.c.i. of $\mathcal{M}$, $\mathcal{L}$, will be a double covering of Γ_5, hence it will be a (canonical) bielliptic curve in $\mathbb{P}^5$.

Here we have $d_B = 10$ and $d_A = 16$ (since $\mathcal{O}_Y(a,b)^2 \cdot \mathcal{O}_Y(1,1) = b^2 + 2ab$).

The main tool we will use in the paper is adjunction theory, via the classification of varieties of small degree in [16], [17] and [18], the results in [20] and those in [9] about surfaces with bielliptic curve sections, also taking into account the new results in [1].

In the following $\cong$ will denote isomorphisms, while $\sim$ will denote linear equivalence of divisors. For the notation not defined in the paper we refer to [15].

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1 Preliminaries

Let us recall some useful results about bielliptic curves. The first lemma will give us a bound for the degree of an embedded bielliptic curve (for a reference see [9], 1.5 and 1.6).

Lemma 1.1. *Let C be a bielliptic curve of genus $g \geqslant 3$ which is birational to some non-degenerate curve of degree d in $\mathbb{P}^n$. Then it must be $d \geqslant n + g - 1$. In particular, no bielliptic plane curve is smooth, unless $g = 3$.*

For bielliptic curves in $\mathbb{P}^3$ we have the following result.

Lemma 1.2. *Let C be a smooth bielliptic curve in $\mathbb{P}^3$ such that either:*

1. *C is contained in a quadric surface, or*
2. *C is a complete intersection.*

Then C is a complete intersection of a quadric and a cubic, i.e., a canonical curve of genus 4 and degree 6.

Proof. Case 1. If the quadric containing C is smooth and C is a divisor of type (a,b), then C has degree $a + b$ and genus $(a-1)(b-1)$; by Lemma 1.1, this is possible, for non-hyperelliptic curves, only if $(a,b) = (3,3)$. If the quadric containing C is a cone, things do not change much; there are two possible cases according to whether C contains the vertex of the cone or not. Taking into account the degree and genus formulae (e.g. see [15], p. 352) we get a contradiction with Lemma 1.1, except in the case that the curve is the complete intersection of the cone with a cubic not passing through its vertex.

Case 2. If C is a complete intersection of two surfaces of degrees α and β, then $\deg C = \alpha\beta$, and the genus of C, from the exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-\alpha-\beta) \to \mathcal{O}_{\mathbb{P}^3}(-\alpha) \oplus \mathcal{O}_{\mathbb{P}^3}(-\beta) \to \mathcal{I}_C \to 0,$$

is $g(C) = \alpha\beta(\alpha+\beta-4)/2+1$. Again by Lemma 1.1 the only possibility is that $\alpha = 2$, $\beta = 3$.

Let X be as in $(*)$. By the Hodge Index Theorem we have

$$\mathcal{L}_A^2\mathcal{M}_A^2 \leqslant (\mathcal{L}_A \cdot \mathcal{M}_A)^2, \quad \mathcal{L}_B^2\mathcal{M}_B^2 \leqslant (\mathcal{L}_B \cdot \mathcal{M}_B)^2$$

from which we get (on X)

$$(\mathcal{L}^2\mathcal{M})\mathcal{M}^3 \leqslant (\mathcal{L}\cdot\mathcal{M}^2)^2, \quad (\mathcal{M}^2\mathcal{L})\mathcal{L}^3 \leqslant (\mathcal{L}^2\cdot\mathcal{M})^2$$

i.e.,

$$d_A\mathcal{M}^3 \leqslant d_B^2, \quad d_B\mathcal{L}^3 \leqslant d_A^2. \tag{1.1}$$

Remark. From (1.1) and $d_A \geqslant d_B$ we trivially have $\mathcal{M}^3 \leqslant d_B$.

Lemma 1.3. *Let $(X, \mathcal{M}, \mathcal{L})$ and C be as in $(*)$, then $d_A \geqslant 6$.*

Proof. By Lemma 1.1 (since $g(C) \geqslant 3$, $n \geqslant 2$) we have $d_A \geqslant 4$, but the case $d_A = 5$ cannot occur since we should have $n \leqslant 3$ and there are no smooth curves of degree 5 and genus $\geqslant 3$ in $\mathbb{P}^3$ or $\mathbb{P}^2$. If $d_A = 4$, then, by Lemma 1.1 again, C must be a smooth plane quartic, hence A should be a quartic surface in $\mathbb{P}^3$ and the situation is as in Example 1: $X = \mathbb{P}^3$, $\mathcal{L} = \mathcal{O}_{\mathbb{P}^3}(1)$, $\mathcal{M} = \mathcal{O}_{\mathbb{P}^3}(4)$. In this case we would have $d_B = 16$, contradicting our hypothesis that $d_A \geqslant d_B$ (of course we can have this kind of situation interchanging the roles of $\mathcal{L}$ and $\mathcal{M}$).

Lemma 1.4. *Let $(X, \mathcal{M}, \mathcal{L})$ and C be as in $(*)$ and let $h^0(X, \mathcal{M}) = n+1$, i.e., $\mathcal{M}$ embeds X into $\mathbb{P}^n$, then*

$$d_B \geqslant n + g(C) - 2 \geqslant n+1.$$

Proof. The second inequality is trivial since $g(C) \geqslant 3$. For the first inequality, we know that $C \subset \mathbb{P}^{n-1}$, because $C = A \cap B$ is contained in a hyperplane section A of X; in order to obtain the first inequality by applying Lemma 1.1, it is enough to show that C is non-degenerate in $\mathbb{P}^{n-1}$. Suppose the contrary, then also B is contained in a hyperplane section, i.e., we can write $|A| = |B + B'|$ where B' is effective, and we get

$$B^2(B+B') = \mathcal{L}^2\mathcal{M} = d_A \geqslant d_B = \mathcal{L}\mathcal{M}^2 = B(B+B')^2,$$

hence we should have $B^3 + B^2B' \geqslant B^3 + 2B^2B' + BB'^2$, i.e., $0 \geqslant BB'(B+B') = ABB'$, which is impossible since A and B are very ample and B' is effective.

2 The case $d_A \geqslant 18$

The following holds:

Theorem A. *Let* $(X, \mathscr{L}, \mathscr{M})$ *and* C *be as in* $(*)$ *with* $d_A \geqslant 19$. *Then either*:

A.1. *X is a scroll over C with respect to both polarizations, i.e., X is a $\mathbb{P}^2$ bundle over C and on every fiber F we have* $(F, \mathscr{L}_F) \cong (F, \mathscr{M}_F) \cong (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1))$.

A.2. *X is a $\mathbb{P}^2$-bundle over an elliptic curve E and for every fiber $F \cong \mathbb{P}^2$ we have* $\mathscr{M}_F \cong \mathcal{O}_{\mathbb{P}^2}(1)$ *(i.e., $(X, \mathscr{M})$ is a scroll), while* $\mathscr{L}_F \cong \mathcal{O}_{\mathbb{P}^2}(2)$.

A.3. *X is a quadric bundle over an elliptic curve E and for every fiber $F \cong \mathbb{P}^1 \times \mathbb{P}^1$ we have* $\mathscr{M}_F \cong \mathscr{L}_F \cong \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1, 1)$.

Proof. By [9] we know that the possibilities for $(A, \mathscr{L}_A)$ are the following:

1. $(A, \mathscr{L}_A)$ is a scroll on a bielliptic curve C,
2. $(A, \mathscr{L}_A)$ is a conic bundle on an elliptic curve E.

In case 1, from [3], Theorem 5.5.3, we have that the fiber bundle structure on A extends to one on X (in fact the only possible cases in which this does not happen are when A is a quadric, which is not our case). In particular this gives that X is a $\mathbb{P}^2$-bundle over the curve C. More specifically, [3], Theorem 7.9.5 gives that $(X, \mathscr{M})$ is a scroll as required.

By denoting with f a fiber of A, we have

$$\mathcal{O}_{\mathbb{P}^1}(1) = (\mathscr{L}_A)_f = (\mathscr{L}_F)_f = \mathcal{O}_{\mathbb{P}^2}(\alpha)|_f = \mathcal{O}_{\mathbb{P}^1}(\alpha),$$

hence $\alpha = 1$, i.e., $(X, \mathscr{L})$ is a scroll, as required.

In case 2, we proceed very much as in [8], case 3.3; we sketch here an outline of the reasoning. The conic bundle structure $\pi : A \to E$ is given by the Remmert–Stein factorization of $\phi_{K_A+\mathscr{L}_A}$, and by [24], Propositions 3.1 and 3.2, the bundle $K_X+\mathscr{L}+\mathscr{M}$ is spanned with the only possible exception (in our case) that $X \cong \mathbb{P}(\mathscr{E})$, where $\mathscr{E}$ is a rank 3 vector bundle on C and $\mathscr{L}, \mathscr{M}$ are of the form $\xi_{\mathscr{E}} + \mathscr{L}_i$, $i = 1, 2$, where $\xi_{\mathscr{E}}$ is the tautological line bundle and the $\mathscr{L}_i$'s are pull backs of line bundles on C. In this case X is a scroll with respect to both polarizations, and we are in case A.1 of our theorem.

When $K_X + \mathscr{L} + \mathscr{M}$ is spanned, π is induced by a morphism $\Pi : X \to E$ (given by the Remmert–Stein factorization of $\phi_{K_X+\mathscr{L}+\mathscr{M}}$). Let F be a general fiber of Π, then F is a smooth surface and, by [23], Corollary 1.5.2, $(F, A_F) = (F, \mathscr{M}_F)$ is one of the following:

a) $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1))$;
b) $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$;
c) $(\mathbb{F}_e, [\sigma + b\phi])$, where σ is a section of minimal degree $\sigma^2 = -e$ and ϕ is a fibre.

In case a) it follows from [27, Claim p. 194] that every fiber is isomorphic to $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1))$, i.e., that $(X, \mathscr{M})$ is a scroll over $\mathbb{P}^1$ and we are in case A.2.

In case b) we have that $\mathscr{M}_F \cong \mathcal{O}_{\mathbb{P}^2}(t)$ for some $t \geqslant 1$, and recalling that $(A, \mathscr{L}_A)$ is a conic bundle necessarily we have $t = 1$. So up to interchanging the roles of $\mathscr{L}$ and $\mathscr{M}$ we are again in case A.2.

In case c), since $\mathcal{M}_F$ is very ample, we must have $(\sigma + b\phi) \cdot \sigma \geqslant 1$, i.e., $b \geqslant 1 + e$, and, for the same reason, if $\mathcal{L}_F \cong \alpha\sigma + \beta\phi$, we must have $\alpha > 0$ and $-e\alpha + \beta \geqslant 1$. Since A is a conic bundle we have $\mathcal{M}_F \cdot \mathcal{L}_F = -e\alpha + \beta + b\alpha = 2$ which implies $e = 0$ and $\alpha = b = \beta = 1$, i.e., $\mathcal{L}_F = \mathcal{M}_F \cong \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1, 1)$ and $(X, \mathcal{L})$, $(X, \mathcal{M})$ are both quadric fibrations, i.e., we are in case A.3.

If we restrict to threefolds of minimal degree, Theorem A yields the following result.

Proposition 2.1. *Let $(X, \mathcal{L}, \mathcal{M})$ and C be as in $(*)$ and suppose that $(X, \mathcal{M})$ is a threefold of minimal degree. Then there are only three possible cases*:

i) *X is a quadric in $\mathbb{P}^4$ and $\mathcal{L} \cong \mathcal{O}_X(3)$, $\mathcal{M} = \mathcal{O}_X(1)$ (here $d_A = 18$, $d_B = 6$, $g(C) = 4$).*

ii) *$X \cong \mathbb{P}^3$ as in Example* 1: *$\mathcal{L} = \mathcal{O}(4)$, $\mathcal{M} = \mathcal{O}(1)$ (here $d_A = 16$, $d_B = 4$, $g(C) = 3$).*

iii) *$X \cong \mathbb{P}^1 \times \mathbb{P}^2$ and $\mathcal{L} = \mathcal{O}_X(3, 1)$, $\mathcal{M} = \mathcal{O}_X(1, 1)$ (here $d_A = 15$, $d_B = 7$, $g(C) = 3$).*

Proof. Let $(X, \mathcal{M})$ be a threefold of minimal degree (i.e., a threefold of degree $n - 2$ in $\mathbb{P}^n$), hence $A \in |\mathcal{M}| = |\mathcal{O}_X(1)|$ is a surface of minimal degree. If $A \cong \mathbb{P}^2$, then we are in case ii), so let $\operatorname{Pic} A = \langle \sigma, f \rangle$: then we have $\mathcal{L}_A \cong \mathcal{O}_A(a\sigma + bf)$ and suppose that $C \sim a\sigma + bf$ is bielliptic. Since $\mathcal{L}$ is very ample we have $b > ae$ and $a > 1$, where $e = -\sigma^2$; we also have $a \geqslant 3$ since otherwise C would be rational or hyperelliptic.

From Theorem A, we have that $\mathcal{L}_A^2 = -a^2 e + 2ab = a(2b - ae) \leqslant 18$. Hence, by easy computations, we get

$$2ae + 2 \leqslant 2b \leqslant ae + \frac{18}{a}.$$

If $e = 0$, then $A \cong \mathbb{P}^1 \times \mathbb{P}^1$, i.e., a quadric surface, so, by Lemma 1.2, we get that $\mathcal{L}_A \cong \mathcal{O}_A(3, 3)$ and $\mathcal{L}_A^2 = 18$, hence we are in case i) (see also Theorem B).

If $e > 0$, from the above inequalities it follows that we can only have $e = 1$, $a = 3$, $b = 4$. In this case we should have $\mathcal{L}_A \cong \mathcal{O}_A(3\sigma + 4f)$, so our problem is to determine if there is a very ample invertible sheaf $\mathcal{L}$ on X such that $\mathcal{L}_A \cong \mathcal{O}_A(3\sigma + 4f)$. Since $(A, \mathcal{M}_A)$ is a scroll, we have $\mathcal{M}_A \cong \mathcal{O}_A(\sigma + kf)$ and $X \subseteq \mathbb{P}^n$ with $n = 2k + 1$. Moreover, $\operatorname{Pic} X = \langle H, F \rangle$, where $H \in |\mathcal{M}|$ and F is a fiber, so we will have $\mathcal{L} \cong \mathcal{O}_X(\alpha H + \beta F)$. From $H \cdot A \sim \sigma + kf$ and $F \cdot A \sim f$, we get $\alpha = 3$ and $\beta = 4 - 3k$.

If $\mathcal{L} \cong \mathcal{O}_X(3H + (4 - 3k)F)$, then

$$\mathcal{L}^3 = \alpha^3 H^3 + 3\alpha^2 \beta H^2 F = 81 - 27k$$

and

$$d_B = \mathcal{L}\mathcal{M}^2 = 3k + 1.$$

From the first equality we get $k \leqslant 2$, hence either $k = 1$ and $(A, \mathcal{M}_A) = (\mathbb{F}_1, \mathcal{O}_A(\sigma + f))$, but this contradicts the very ampleness of $\mathcal{M}$, or $k = 2$ and X is embedded by $\mathcal{M}$ in

$\mathbb{P}^5$, so X is the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^2$, with $\mathcal{M} \cong \mathcal{O}_X(1,1)$ and we are in case iii), since $\mathcal{L} \cong \mathcal{O}_X(3H - 2F) \cong \mathcal{O}_X(3,1)$.

We can generalize the result in Theorem A to the case when $\dim X = d \geqslant 3$; namely, suppose X is as in Definition 1, and $C \subset X$ is an ab.c.i. of $\mathcal{L}_1, \dots, \mathcal{L}_{d-1}$. Let $d_i = \deg \mathcal{L}_i|_C$ (without loss of generality we can suppose $d_1 \geqslant d_2 \geqslant \cdots \geqslant d_{d-1}$), let $A_1, \dots, A_{d-1}$ be sections of $\mathcal{L}_1, \dots, \mathcal{L}_{d-1}$ which realize C as an ab.c.i. and suppose that all the varieties $S_1 = A_2 \cap \cdots \cap A_{d-1}$, $S_{i_2,\dots,i_k} = \bigcap_{j \neq i_2,\dots,i_k} A_j$, where $\{i_2, \dots, i_k\} \subset \{2, \dots, d-1\}$ and $k = 2, \dots, d-2$, are smooth and irreducible. Then the following holds.

Theorem A′. *Let $(X, \mathcal{L}_1, \dots, \mathcal{L}_{d-1})$ and C be as above and suppose C to be a smooth irreducible bielliptic curve. Then if $d_1 \geqslant 19$ either:*

A′.1. *X is a scroll over C with respect to all the polarizations, i.e., X is a $\mathbb{P}^{d-1}$ bundle over C and on every fiber F we have $(F, \mathcal{L}_i|_F) \cong (\mathbb{P}^{d-1}, \mathcal{O}_{\mathbb{P}^{d-1}}(1))$, or*

A′.2. *X is a $\mathbb{P}^{d-1}$-bundle over an elliptic curve E, and for every fiber $F \cong \mathbb{P}^{d-1}$ we have: $\mathcal{L}_i|_F \cong \mathcal{O}_{\mathbb{P}^{d-1}}(1)$, for all $i = 2, \dots, d-1$ (i.e., $(X, \mathcal{L}_i)$ is a scroll), and $\mathcal{L}_1|_F \cong \mathcal{O}_{\mathbb{P}^{d-1}}(2)$, or*

A′.3. *X is a quadric bundle over an elliptic curve E, and for every fiber F we have $(F, \mathcal{L}_i|_F) \cong (F, \mathcal{M}_i|_F) \cong (Q_{d-1}, \mathcal{O}_{Q_{d-1}}(1))$, where Q_r is an r-dimensional hyperquadric in $\mathbb{P}^{r+1}$.*

Proof. The proof works by complete induction on d. For $d = 3$ this is just Theorem A. When $d \geqslant 4$, suppose that the result is known for every $d' \leqslant d-1$ and consider the varieties $S_{i_2,\dots,i_k}$. We can apply the result in [9] to the surface $(S_1, \mathcal{L}_1|_{S_1})$, as we did at the beginning of the proof of Theorem A, in order to get that either $(S_1, \mathcal{L}_1|_{S_2})$ is a scroll on a bielliptic curve C or $(S_1, \mathcal{L}_1|_{S_2})$ is a conic bundle on an elliptic curve E.

By Theorem A, we get that, for any $i_2 = 2, \dots, d-1$, we can have three cases:

1. the threefolds $(S_{i_2}, \mathcal{L}_{i_2}|_{S_{i_2}})$ and $(S_{i_2}, \mathcal{L}_1|_{S_{i_2}})$ are scrolls;
2. the threefolds $(S_{i_2}, \mathcal{L}_{i_2}|_{S_{i_2}})$ are scrolls and $(S_{i_2}, \mathcal{L}_1|_{S_{i_2}})$ is a Veronese bundle (i.e., the fibers are embedded as Veronese surfaces);
3. $(S_{i_2}, \mathcal{L}_{i_2}|_{S_{i_2}})$ and $(S_{i_2}, \mathcal{L}_1|_{S_{i_2}})$ are all quadric bundles on an elliptic curve.

In cases 1 and 2, by using [3], Theorem 5.5.2, we can extend the $\mathbb{P}^i$-bundle structure from $S_{i_2,\dots,i_k}$ to $S_{i_2,\dots,i_k,i_{k+1}}$, and from $S_{i_2,\dots,i_{d-2}}$ to X to get that X is either as in A′.1 or as in A′.2 (in order to check what is the value a for which $\mathcal{L}_{i_2}|_F \cong \mathcal{O}_{\mathbb{P}^{d-1}}(a)$ one can proceed as in the proof of Theorem A).

In case 3 we can use [26], Proposition III (as in the analogous case in [10], Theorem A) to extend the quadric fibration from $S_{i_2,\dots,i_k}$ to $S_{i_2,\dots,i_k,i_{k+1}}$, and from $S_{i_2,\dots,i_{d-2}}$ to X in order to get that X is as in A′.3.

As we noticed in the introduction, $d_A = 18$ seems to be a threshold (as it is in the case of varieties with a bielliptic curve-section, [9], [10]), in fact in this case we have many possibilities for our threefolds, as the following shows.

Theorem B. *Let $(X, \mathscr{L}, \mathscr{M})$ and C be as in $(*)$ with $d_A = 18$ and C bielliptic. Then, if X is not as in Theorem A, it is one of the following*:

B.1. *$X \cong \mathbb{P}^3$ as in Example 2: $\mathscr{L} = \mathscr{O}(3)$, $\mathscr{M} = \mathscr{O}(2)$,*

B.2. *$X \cong Q$, where Q is a quadric hypersurface in $\mathbb{P}^4$ and $\mathscr{L} = \mathscr{O}_Q(3)$, $\mathscr{M} = \mathscr{O}_Q(1)$,*

B.3. *X is a Fano threefold of principal series with $\rho = \operatorname{Pic} X = 1$ and $\pi : X \to \mathbb{P}^3$ is a double covering with a sextic surface as discriminant divisor; $\mathscr{M} = -K_X = \pi^*\mathscr{O}(1)$ and $\mathscr{L} = \pi^*\mathscr{O}(3)$,*

B.4. *X is a Fano threefold of principal series with $\rho = 2$ and $\mathscr{M} = -K_X$,*

B.5. *$(X, \mathscr{M})$ is a conic bundle on a smooth surface,*

B.6. *$(X, \mathscr{M})$ is a quadric bundle over $\mathbb{P}^1$,*

B.7. *$(X, \mathscr{M})$ is a scroll, either over $\mathbb{P}^2$, or $\mathbb{P}^1 \times \mathbb{P}^1$, or $\mathbb{F}_1$,*

B.8. *$(X, \mathscr{M})$ is the blow up at two points of its reduction $(Q, \mathscr{O}_Q(2))$, Q as in* B.2 *and the two points not lying on a line of Q,*

B.9. *$(X, \mathscr{M})$ is the blow up at one point of its reduction $(\mathbb{P}(\mathscr{E}), 2\eta - p^*\mathscr{O}_{\mathbb{P}^1}(1))$, where $\mathscr{E} = \mathscr{O}_{\mathbb{P}^1}(1) \oplus \mathscr{O}_{\mathbb{P}^1}(1) \oplus \mathscr{O}_{\mathbb{P}^1}(1)$, η is the tautological bundle of $\mathscr{E}$ and $p : \mathbb{P}(\mathscr{E}) \to \mathbb{P}^1$ is the bundle projection.*

Proof. Under our hypotheses it follows by [9], Theorem 3.5, that $(A, \mathscr{L}_A)$ is either $(\mathbb{P}^1 \times \mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(3,3))$ or a double plane. Since also $\mathbb{P}^1 \times \mathbb{P}^1$ has a double plane structure, the two cases can be treated together.

Let $g = g(C)$, from Lemma 1.1 we have that $d_A = 18 \geqslant g + 2$, hence $g \leqslant 16$. Moreover, since $\mathscr{L}_A \cong \pi^*(\mathscr{O}_{\mathbb{P}^2}(3))$, where $\pi : A \to \mathbb{P}^2$ is the double covering, again from [9] we get that $\pi|_C$ is a 2:1 morphism onto an elliptic curve, hence the cardinality of $\pi(C) \cap \Gamma$, where Γ is the ramification curve of π, is exactly $2g - 2$. Then, by Bezout, $3 \deg \Gamma = 2g - 2$, and so $g - 1 \equiv 0 \bmod 3$. Thus the only possible values for g are: 4, 7, 10, 13, and 16. Since $K_A \cong \pi^*(\mathscr{O}_{\mathbb{P}^2}(a))$, with $a \geqslant -2$, and $\deg \Gamma = 2(a + 3)$, the values of a corresponding to the five possible values of g are, respectively, -2, -1, 0, 1, 2. Now, $(X, \mathscr{M})$ is a threefold with a very ample divisor which is a double covering of $\mathbb{P}^2$. From the classification of such threefolds in [20], we get: cases B.1 and B.2 for $g = 4$, $a = -2$; cases B.3, B.4 for $g = 10$, $a = 0$; case B.5 for $g = 13, 16$, $a = 1, 2$ and cases B.6 to B.9 when $g = 7$, $a = -1$.

In order to prove the theorem we only have to show that the only two other cases which appear for $a = -1$ in [20, Theorem 3.2], namely cases 3.2.3 and 3.2.5, cannot occur in our case.

In case 3.2.3, X is as in Example 3, with $\mathscr{M} \cong \mathscr{O}_X(3H - E)$. Then we should have $\mathscr{L} \cong \mathscr{O}_X(2H - E)$ by Lemma 1.2, but this is not possible because it would yield $d_B = 17$, $d_A = 11$.

In case 3.2.5, $(X, \mathscr{M}) \cong (\mathbb{P}^1 \times \mathbb{P}^2, \mathscr{O}_{\mathbb{P}^1 \times \mathbb{P}^2}(2,2))$, and it cannot occur for degree reasons. In fact if $\mathscr{M} = \mathscr{O}_{\mathbb{P}^1 \times \mathbb{P}^2}(2,2)$, say (2,2) for short, and $\mathscr{L} = (a, b)$, then

$$18 = d_A = \mathscr{M}\mathscr{L}^2 = (2,2)(a,b)(a,b) = 2b(b + 2a),$$

i.e., $9 = b(b + 2a)$, whose only solutions are $(0, 3)$, which does not correspond to a very ample divisor on $X \cong \mathbb{P}^1 \times \mathbb{P}^2$, and $(4, 1)$, which should imply that C is hyperelliptic.

3 The case $d_B \leqslant 8$

We have the following result.

Theorem C. *Let* $(X, \mathscr{L}, \mathscr{M})$ *and* C *be as in* $(*)$ *with* $d_B \leqslant 8$. *If* X *is not as in Theorem A, then it is one of the following*:

C.1. $X \cong \mathbb{P}^3$ *and* $\mathscr{M} = \mathscr{O}(1)$, $\mathscr{L} = \mathscr{O}(4)$.

C.2. $X \cong Q$, *where* Q *is a quadric hypersurface in* $\mathbb{P}^4$ *and* $\mathscr{L} = \mathscr{O}_Q(3)$, $\mathscr{M} = \mathscr{O}_Q(1)$ (*this is also case* B.2, *since here* $d_A = 18$).

C.3. $X \subset \mathbb{P}^4$ *is a cubic hypersurface and* $\mathscr{L} = \mathscr{O}_X(2)$, $\mathscr{M} = \mathscr{O}_X(1)$.

C.4. $X \subset \mathbb{P}^5$, $X \cong \mathbb{P}^1 \times \mathbb{P}^2$ *and* $\mathscr{L} = \mathscr{O}_X(3,1)$, $\mathscr{M} = \mathscr{O}_X(1,1)$ (*see Proposition* 2.1).

C.5. $X \subset \mathbb{P}^5$ *is a complete intersection of two hyperquadrics and* $\mathscr{L} = \mathscr{O}_X(2)$, $\mathscr{M} = \mathscr{O}_X(1)$.

C.6. $X \subset \mathbb{P}^5$ *is a rational quadric bundle*, $\mathscr{M} = \mathscr{O}_X(1)$ *and* $\mathscr{L} = \mathscr{O}_X(2A - F)$, *where* $A \in |\mathscr{M}|$ *and* F *is a fiber. Here* $d_B = 8$ *and* $d_A = 12$.

Cases C.1 *to* C.5 *actually occur.*

Proof. We will work by considering $d_A \leqslant 18$ since the other cases are covered by Theorem A. By the remark in section 1, we have that if $8 \geqslant d_B$ then $\mathscr{M}^3 \leqslant 8$. All the varieties of degree $\leqslant 8$ are classified in [16] and [17], taking into account also the missed case considered in [1], hence we have to check which are the ones that can possess a bielliptic curve as an ab.c.i. with $d_B \leqslant 8$ and $d_A \leqslant 18$. Notice that from (1.1) we also have that

$$d_A \mathscr{M}^3 \leqslant d_B^2 \leqslant 64, \tag{3.1}$$

which gives a better bound on d_A as soon as $\mathscr{M}^3 \geqslant 4$.

We will proceed by examining the possibilities for X with respect to the degree $\mathscr{M}^3$ and the codimension s with respect to the embedding given by $\mathscr{M}$.

We notice that if $\mathscr{M}^3 \geqslant 4$ the cases when $(X, \mathscr{M})$ is a hypersurface in $\mathbb{P}^4$ or a rational normal threefold are ruled out by Lemma 1.2 and by Proposition 2.1, respectively.

$\mathscr{M}^3 = 1$. Here the only possibility is trivially case C.1 (see Proposition 2.1 and Example 1).

$\mathscr{M}^3 = 2$. By Lemma 1.2, the only possibility is trivially C.2.

$\mathscr{M}^3 = 3$. The only possibilities for a threefold of degree 3 are either a cubic hypersurface in $\mathbb{P}^4$ (and then by Lemma 1.2 we are in case C.3), or a rational normal scroll $X \subset \mathbb{P}^5$ and then we are in case C.4 by Proposition 2.1.

$\mathscr{M}^3 = 4$. The only possibility for X is to be the complete intersection of two quadric hypersurfaces in $\mathbb{P}^5$; in this case, by the Lefschetz theorem (e.g. see [15]), we have $\operatorname{Pic} X \cong \mathbb{Z}\langle \mathscr{O}_X(1) \rangle$, hence $\mathscr{L} \cong \mathscr{O}_X(a)$ with $a > 1$ and C is embedded by $\mathscr{M}$ as a complete intersection curve of type $(2,2,a)$ in $\mathbb{P}^4$ with degree $d_B = 4a \leqslant 8$. Then the only possibility is $a = 2$ which corresponds to case C.5.

$\mathscr{M}^3 = 5$. According to the classification in [16], our threefold X can only have codimension $s = 3, 2$.

$s = 3$. $X \subseteq \mathbb{P}^6$ is a Del Pezzo threefold which is a section of the Grassmannian $G(1,4)$, $\mathscr{M} \cong \mathscr{O}_X(1)$ and $\operatorname{Pic} X \cong \mathbb{Z}\mathscr{M}$ (see [19]). Moreover $(A, \mathscr{M}_A)$ is a Del Pezzo surface and $\mathcal{M}_A \cong -K_A$, hence $\mathcal{L}_A \cong \mathcal{M}_A^\alpha$ and then

$$d_B = \mathcal{M}_A\mathcal{L}_A = \alpha\mathcal{M}^2 = 5\alpha \leqslant 8,$$

which implies $\alpha = 1$. So $\mathcal{M} \cong \mathcal{L}$ and C should be elliptic, a contradiction.

$s = 2$. $X \subseteq \mathbb{P}^5$ is a rational quadric bundle, and $A \in |\mathcal{M}|$ is a Del Pezzo surface of degree 1, i.e., it is isomorphic to the blow up of $\mathbb{P}^2$ at eight points (and $\mathcal{M}_A$ is given by the linear system of the quartic curves passing at least doubly through one point and simply through the others). Since $\operatorname{Pic} X \cong \mathbb{Z}\langle A, F\rangle$, where F is a fiber (e.g., see [19], Theorem 1.4.3 and [16] 0.6)), let $\mathcal{L} \cong \mathcal{O}_X(\alpha A + \beta F)$ and consider $\operatorname{Pic} A \cong \mathbb{Z}\langle E_0, E_1, E_2, \dots, E_8\rangle$, with $\mathcal{M}_A \cong \mathcal{O}_A(4E_0 - E_1 - E_2 - \cdots - 2E_8)$, $F|_A \sim E_0 - E_8$ and $\mathcal{L}_A = \mathcal{O}_A(\alpha A_A + \beta F_A) = \mathcal{O}_A((4\alpha+\beta)E_0 - \alpha E_1 - \cdots - \alpha E_7 - (2\alpha+\beta)E_8)$. From $\mathcal{M}^3 = 5$, $\mathcal{M}^2\mathcal{O}_X(F) = \mathcal{M}_A\mathcal{O}_A(F_A) = 2$, by Lemma 1.4 and $d_B \leqslant 8$, we have

$$8 \geqslant d_B = \mathcal{L}\mathcal{M}^2 = \alpha\mathcal{M}^3 + \beta\mathcal{O}_X(F)\mathcal{M}^2 = 5\alpha + 2\beta \geqslant 6.$$

By computing the genus g of C as a divisor in $|\mathcal{L}_A|$ we have that these inequalities only hold for $d_B = 8$, $\alpha = 2$, $\beta = -1$, $g = 5$, $d_A = 12$.

This situation corresponds to case C.6.

$\mathcal{M}^3 = 6$. According to the classification in [16], we can only have $s = 4, 3, 2$.

$s = 4$. One possibility is that $X \cong \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ (a Segre variety) and $\mathcal{M} \cong \mathcal{O}_X(1,1,1)$. Let $\mathcal{L} \cong \mathcal{O}_X(a_1, a_2, a_3)$, with $1 \leqslant a_3 \leqslant a_2 \leqslant a_1$. We can rule out this case by computing $d_B = \mathcal{L}\mathcal{M}^2$. We consider plurihomogeneous coordinates $\langle x_0, x_1; y_0, y_1; z_0, z_1\rangle$ on X, two divisors in $|\mathcal{M}|$ are given e.g. by $x_0y_0z_0$ and $x_1y_1z_1$ and their intersection is given by the following six lines (given parametrically):

$$\Gamma_1 = (a,b;0,1;1,0), \quad \Lambda_1 = (a,b;1,0;0,1),$$
$$\Gamma_2 = (0,1;a,b;1,0), \quad \Lambda_2 = (1,0;a,b;0,1),$$
$$\Gamma_3 = (0,1;1,0;a,b), \quad \Lambda_3 = (1,0;0,1;a,b).$$

So we have that Γ_i and Λ_i intersect a divisor of $|\mathcal{L}|$ in a_i points. Summing up we get $d_B = 2(a_1 + a_2 + a_3)$ and $d_B \leqslant 8$ implies that $(a_1, a_2, a_3) = (2,1,1)$ (since $\mathcal{L} \neq \mathcal{M}$), but in this case the curve C would be hyperelliptic (for any point P in the first factor, there are 2 points on C in the corresponding $\mathbb{P}^1 \times \mathbb{P}^1$, so when P varies in $\mathbb{P}^1$ it describes a g_2^1 on C). Hence, as claimed, this case is not possible.

Another possibility is that $X \cong \mathbb{P}(T_{\mathbb{P}^2})$. Then X can also be viewed as a hyperplane section of the Segre variety of $\mathbb{P}^2 \times \mathbb{P}^2 \subset \mathbb{P}^8$. In this case (e.g., see [14]), $\operatorname{Pic} X \cong \operatorname{Pic}(\mathbb{P}^2 \times \mathbb{P}^2) \cong \mathbb{Z}^2$ (where the isomorphism is given by the restriction map). With obvious notation, we have that $\mathcal{M} \cong \mathcal{O}_X(1,1)$. Let $\mathcal{L} \cong \mathcal{O}_X(a,b)$, we should have $d_B = \mathcal{L} \cdot \mathcal{M}^2 = 3a + 3b \leqslant 8$ which is impossible for positive values of $(a,b) \neq (1,1)$, hence also this case cannot occur.

$s = 3$. X is a Fano threefold with $\operatorname{Pic} X \cong \mathbb{Z}^2$, which is a double covering $\pi : X \to Y$ of the rational normal threefold $Y \cong \mathbb{P}^1 \times \mathbb{P}^2 \to \mathbb{P}^5$, ramified along a divisor of type $\mathcal{O}_Y(2,2)$ (see also Example 4). We have $\mathcal{M} \cong \pi^*(\mathcal{O}_Y(1,1))$. Let $\mathcal{L} \cong \pi^*(\mathcal{O}_Y(a,b))$, the inequality $d_B = \mathcal{L} \cdot \mathcal{M}^2 \leqslant 8$ implies $\mathcal{O}_Y(a,b) \cdot \mathcal{O}_Y^2(1,1) \leqslant 4$, which is possible only when $(a,b) = (2,1)$. But in this case the curve $\mathcal{O}_Y(a,b) \cdot \mathcal{O}_Y(1,1)$ (on Y) is a rational normal quartic, hence C (which is a double covering of it, via π) would be hyperelliptic. Thus also this case cannot occur.

$s = 2$. We have two possibilities for X. The first is $X \cong \mathbb{P}(\mathscr{E})$, where $\mathscr{E}$ is a rank 2 locally free sheaf on $\mathbb{P}^2$ given by the exact sequence $0 \to \mathcal{O}_{\mathbb{P}^2} \to \mathscr{E} \to \mathscr{I}_Y(4) \to 0$, Y is a set of 10 general points in $\mathbb{P}^2$, and $\mathscr{M}$ is the tautological sheaf on $\mathbb{P}(\mathscr{E})$. In this case A is isomorphic to the blow-up of $\mathbb{P}^2$ along Y and $\mathscr{M}_A$ is associated to the linear system of quartic curves passing through Y. We have that $\operatorname{Pic} X \cong \mathbb{Z}\langle A, \Pi\rangle$, where Π denotes the divisor over a generic line in $\mathbb{P}^2$ in the bundle structure of X, so $A^2 \cdot \Pi = 4$, $A \cdot \Pi^2 = 1$ and $\Pi^3 = 0$ on X. Let $\mathscr{L} \cong \mathcal{O}_A(\alpha A + \beta\Pi)$, then we must have

$$8 \geqslant d_B = \mathscr{L}\mathscr{M}^2 = 6\alpha + 4\beta$$

(which, since $\alpha > 0$, implies that $\beta \leqslant 0$) and

$$0 \leqslant \mathscr{L}^3 = 3\alpha(2\alpha^2 + 4\alpha\beta + \beta^2),$$

which yields $\alpha \geqslant \dfrac{-2\beta + \sqrt{2\beta^2}}{2}$. These inequalities have integer solutions only for $\beta = 0, -1$. If $\beta = 0$, the only possibility is that $\mathscr{L} = \mathscr{M}$, that we do not consider. If $\beta = -1$, then $\alpha \leqslant 2$ by the first inequality, hence $\alpha = 2$ because for $\alpha = 1$ we would have $\mathscr{L}$ not very ample (this can be easily seen on $\mathscr{L}_A$). Therefore we get $d_A = \mathscr{L}^2\mathscr{M} = 6\alpha^2 - 4\alpha + 1 = 17$ and this is not possible because $d_A \leqslant 10$ by (3.1). Hence this case cannot occur.

The other possibility is that X is a complete intersection of type $(2, 3)$. In this case, since $\operatorname{Pic} X \cong \mathbb{Z}$, we have $\mathscr{M} \cong \mathcal{O}_X(1)$ and $\mathscr{L} \cong \mathcal{O}_X(b)$. Then we should have that C is a complete intersection of type $(2, 3, b)$ in $\mathbb{P}^4$; a simple computation (e.g., using the resolution of the ideal sheaf $\mathscr{I}_C$) shows that such curves have genus $3b^2 + 1$ (and degree $6b$), hence they cannot be bielliptic by Lemma 1.1.

$\mathscr{M}^3 = 7$. In this case, according to [16], we can only have $s = 5, 4, 3, 2$.

$s = 5$. X is the blowing up $\pi : X \to \mathbb{P}^3$ of $\mathbb{P}^3$ at one point P (see also Example 3). We have that $\mathscr{M} = \mathcal{O}_X(2H - E)$. Let $\mathscr{L} = \mathcal{O}_X(\alpha H - \beta E)$, we must have $d_B = \mathscr{M}^2\mathscr{L} = 4\alpha - \beta \leqslant 8$. Since $\alpha \geqslant \beta$ and, for the very ampleness of $\mathscr{L}$, we must also have $\beta \geqslant 1$, we get that either $(\alpha, \beta) = (2, 1)$, and this yields $\mathscr{L} = \mathscr{M}$, or $(\alpha, \beta) = (2, 2)$ in which case $\mathscr{L}$ is not very ample ($\mathscr{L}$ would contract every line passing through P). So this case cannot occur too.

$s = 4$. X is the blowing up $\pi : X \to \mathbb{P}^3$ of $\mathbb{P}^3$ along an elliptic normal curve Γ. We have that $\mathscr{M} = \mathcal{O}_X(3H - E)$ where H is the strict transform of a generic plane of $\mathbb{P}^3$ and E is the exceptional divisor. If A is a general element in $|\mathscr{M}|$, i.e., A is isomorphic to a smooth cubic surface containing Γ, let $\operatorname{Pic} A \cong \mathbb{Z}\langle E_0, E_1, \dots, E_6\rangle$. We can choose the generators of $\operatorname{Pic} A$ in order to have that $\Gamma \sim 3E_0 - E_1 - E_2 - \cdots - E_5$, hence $\mathscr{M}_A \cong \mathcal{O}_A(9E_0 - 3E_1 - \cdots - 3E_6 - \Gamma) \cong \mathcal{O}_A(6E_0 - 2E_1 - \cdots - 2E_5 - 3E_6)$.

If $\mathscr{L} \cong \mathcal{O}_X(\alpha H - \beta E)$ (since $\operatorname{Pic} X \cong \mathbb{Z}\langle H, E\rangle$) we have $\mathscr{L}_A \cong \mathcal{O}_A(3(\alpha - \beta)E_0 - (\alpha - \beta)E_1 - \cdots - (\alpha - \beta)E_5 - \alpha E_6)$, hence $d_B = \mathscr{M}^2\mathscr{L} = 5\alpha - 8\beta \leqslant 8$. On the other hand we must also have $\alpha \geqslant 2\beta$ (since the ideal of Γ is the complete intersection of two quadric forms). Hence we get $8 \geqslant d_B \geqslant 2\beta$, i.e., $\beta \leqslant 4$ (recall also that $\beta > 0$ to have very ampleness), moreover $\alpha \geqslant 2\beta + 1$ can satisfy $5\alpha - 8\beta \leqslant 8$ only for

$(\alpha, \beta) = (3, 1)$, i.e., for $\mathscr{L} = \mathscr{M}$, and we are not interested in this case. Thus we only have to consider $(\alpha, \beta) = (2\beta, \beta)$, $\beta = 1, 2, 3, 4$, but for these values $\mathscr{L}$ is not very ample (it is given by the generators of $(I_\Gamma)^\beta$). So this case does not occur.

$s = 3$. The first possibility is that X is a scroll over an elliptic curve Γ. Let us consider $A \in |\mathscr{M}|$, which is a ruled surface on Γ with $\operatorname{Pic} A \cong \mathbb{Z}\langle \Gamma_0, F\rangle$ and $\mathscr{M}_A = \mathscr{O}_A(\Gamma_0 + bF)$. We must have $\mathscr{M}_A^2 = 2b - e = 7$ and (for the very ampleness) $b \geqslant e + 3$, hence either $e = 1$, $b = 4$ or $e = -1$, $b = 3$. Let $\mathscr{L}_A \cong \mathscr{O}_A(\alpha A + \beta F) \cong \mathscr{O}_A(\alpha\Gamma_0 + (\alpha b + \beta)F)$, we must have $(\alpha A + \beta F)\Gamma_0 \geqslant 3$ hence $-\alpha e + \alpha b + \beta \geqslant 3$, moreover $d_B = \mathscr{L}_A \cdot \mathscr{M}_A \leqslant 8$. If $e = -1$, $b = 3$ these two inequalities yield $3 - 4\alpha \leqslant \beta \leqslant 8 - 7\alpha$, while if $e = 1$, $b = 4$ they yield $3 - 3\alpha \leqslant \beta \leqslant 8 - 7\alpha$. Both cases imply $\alpha = 1$, which is absurd since C would be elliptic.

Another possibility is that X is a quadric bundle over $\mathbb{P}^1$ and $A \cong \mathbb{A}_e$, where $\mathbb{A}_e$ is the blow-up of a Hirzebruch surface $\mathbb{F}_e$ at 9 points, with $e = 0, 1, 2$ or 3. Then $\operatorname{Pic} X \cong \mathbb{Z}\langle A, F\rangle$, where F is a generic fiber, and, with obvious notation, $\operatorname{Pic} A \cong \mathbb{Z}\langle C_0, F_A, E_1, \dots, E_9\rangle$. Let $\mathscr{M}_A = \mathscr{O}_A(2C_0 + bF_A - E_1 - \cdots - E_6)$, where $b = 4 + e$, and $\mathscr{L} = \mathscr{O}_A(\alpha A + \beta F)$, so that $L_A = \mathscr{O}_A(2\alpha C_0 + (b + \beta)F_A - \alpha E_1 - \cdots - \alpha E_9)$.

By Lemma 1.4 we have that $d_B \geqslant 7$, so

$$7 \leqslant d_B = \mathscr{M}_A\mathscr{L}_A = -4\alpha e + 2(b + \beta) - 2\alpha b - 9\alpha \leqslant 8.$$

From this condition we get

$$\frac{2(\beta + e) + 1}{2e + 1} \leqslant \alpha \leqslant \frac{2(\beta + e)}{2e + 1},$$

which yields $\beta \geqslant e + 1$. From $d_A \geqslant d_B$ we must have:

$$7 \leqslant d_A = \mathscr{L}_A^2 = -4\alpha^2 e + 4\alpha(b + \beta) - 9\alpha^2.$$

Simple but tedious computations show that the former condition contradicts the latter one, for all $e \in \{0, 1, 2, 3\}$.

The last possibility is that $X \cong \mathbb{P}(\mathscr{E})$, where $\mathscr{E}$ is a rank 2 locally free sheaf on $\mathbb{P}^2$ given by the exact sequence $0 \to \mathscr{O}_{\mathbb{P}^2} \to \mathscr{E} \to \mathscr{I}_Y(4) \to 0$ with Y a set of 9 general points in $\mathbb{P}^2$, and $\mathscr{M}$ is the tautological sheaf on $\mathbb{P}(\mathscr{E})$ (see also the similar case for $\mathscr{M}^3 = 6$, $s = 2$). We have that $\operatorname{Pic} X \cong \mathbb{Z}\langle A, \Pi\rangle$, where Π is the divisor over a generic line in $\mathbb{P}^2$ in the bundle structure of X, so $A^2 \cdot \Pi = 4$, $A \cdot \Pi^2 = 1$ and $\Pi^3 = 0$ on X. If $\mathscr{L} \cong \mathscr{O}_X(\alpha A + \beta\Pi)$, then we have

$$8 \geqslant d_B = \mathscr{L}\mathscr{M}^2 = 7\alpha + 4\beta$$

which, since $\alpha > 0$, implies that $\beta \leqslant 0$. On the other hand for $\beta = 0$ we must have $\alpha = 1$ which yields $\mathscr{M} = \mathscr{L}$. So actually we have $\beta < 0$ and $\alpha \geqslant 2$. Since C is in $\mathbb{P}^5$, by Lemma 1.1 we get $d_B \geqslant 5 + g(C) - 1 \geqslant 7$, which, since $d_B \leqslant d_A \leqslant d_B^2/\mathscr{M}^3 = 7$, implies that either $d_B = d_A = 7$, or $d_B = 8$ and $8 \leqslant d_A \leqslant 9$. From

$$d_A = \mathscr{L}^2\mathscr{M} = 7\alpha^2 + 8\alpha\beta + \beta^2$$

it is just a computation to show that no values for α, β can give the required values of d_A, d_B.

$s = 2$. We have three possibilities for X. A first one is $X \cong \mathbb{P}(\mathscr{E})$, where $\mathscr{E}$ is a rank 2 locally free sheaf on a smooth cubic surface $S \subset \mathbb{P}^3$ given by the exact sequence $0 \to \mathscr{O}_S \to \mathscr{E} \to \mathscr{I}_{Y,S}(2) \to 0$ with Y a set of 5 general points on S, and $\mathscr{M}$ is the tautological sheaf on $\mathbb{P}(\mathscr{E})$ (see also the case above). So $\pi : X \to S$ is a scroll structure with respect to $\mathscr{M}$, and $\operatorname{Pic} X \cong \mathbb{Z}\langle A, F_0, \dots, F_6\rangle$, where $A \in |\mathscr{M}|$, $\operatorname{Pic} S \cong \mathbb{Z}\langle E_0, E_1, \dots, E_6\rangle$ (e.g. see [15]) and $F_i = \pi^{-1}(E_i)$. We have that A is the blow up of S at Y, so if $E_7, \dots, E_{11}$ are its exceptional divisors and (with a slight abuse of notation) $\operatorname{Pic} A \cong \mathbb{Z}\langle E_0, E_1, \dots, E_{11}\rangle$, we have $\mathscr{M}_A \cong \mathscr{O}_A(6E_0 - 2E_1 - \cdots - 2E_6 - E_7 - \cdots - E_{11})$.

Let $\mathscr{L} \cong \mathscr{O}_X(\alpha A + \beta F_0 - \gamma_1 F_1 - \cdots - \gamma_6 E_6)$, then $\mathscr{L}_A \cong \mathscr{O}_A((6\alpha + \beta)E_0 - (2\alpha + \gamma_1)E_1 - \cdots - (2\alpha + \gamma_6)E_6 - \alpha E_7 - \cdots - \alpha E_{11})$; we have $d_B = \mathscr{M}_A \mathscr{L}_A = 7\alpha + \beta - 2\sum_{i=1}^6 \gamma_i$ and $d_A = \mathscr{L}_A^2 = 7\alpha^2 + 12\alpha\beta + \beta^2 - 4\alpha\sum_{i=1}^6 \gamma_i - \alpha\sum_{i=1}^6 \gamma_i^2$. By Lemma 1.4 we have $8 \geqslant d_B \geqslant g + 3$, while from (3.1) we get $d_A \leqslant 9$. Moreover for the genus of C, we have

$$g = \binom{6\alpha + \beta - 1}{2} - \sum_{i=1}^{6} \binom{2\alpha + \gamma_i}{2} - 5\binom{\alpha}{2},$$

which gives $2g = d_A - \alpha - 3\beta + \sum_{i=1}^6 \gamma_i + 2$. From the bound on d_A we get

$$2\alpha + 6\beta - 2\sum_{i=1}^{6} \gamma_i \geqslant 22 - 4g$$

while from the bounds on d_B we have $3 \leqslant g \leqslant 5$. Hence we get

$$7 \leqslant 5\alpha + \left(2\alpha + 6\beta - 2\sum_{i=1}^{6} \gamma_i\right) \leqslant 8,$$

which is clearly impossible for $g = 3$ or 4, since $\alpha \geqslant 1$ and the part in parentheses is $\geqslant 22 - 4g$. When $g = 5$, which implies $d_B = 8$, the bound above can be satisfied only for $\alpha = 1$, but in this case $d_B = 7 + 6\beta - 2\sum_{i=1}^6 \gamma_i$, which cannot be eight. So also this case cannot occur.

A second possibility for $s = 2$ is that X is the blowing up $\pi : X \to Y$ of a smooth 3-fold $Y \subset \mathbb{P}^6$, which is the complete intersection of three quadrics, at a point $P \in Y$ (i.e., X is obtained by projecting Y into $\mathbb{P}^5$ from P). Here $\operatorname{Pic} X \cong \mathbb{Z}\langle H, E\rangle$, where H is the strict transform of a generic hyperplane section of Y and E is the exceptional divisor. We have $\mathscr{M} \cong \mathscr{O}_X(1) \cong \mathscr{O}_X(H - E)$, and from $(H - E)^3 = 7$, together with $H^3 = H \cdot (H - E)^2 = 8$ we get $H^2 \cdot E = H \cdot E^2 = 0$ and $E^3 = 1$. Now let $\mathscr{L} \cong \mathscr{O}_X(aH - bE)$, since $0 < d_B \leqslant 8$ and $d_A > 0$, we have:

$$0 < d_B = (H - E)^2(aH - bE) = 8a - b \leqslant 8,$$

$$0 < d_A = (H - E)(aH - bE)^2 = 8a^2 - b^2.$$

So $b \geqslant 8a - 8$ and $b^2 < 8a^2$, hence $64a^2 - 32a + 64 < 8a^2$, which is never true, and this case is impossible.

Eventually, the last possibility is that X is a cubic fibration on $\mathbb{P}^1$, where $\mathcal{M} \cong \mathcal{O}_X(1)$, and this structure is given by the adjunction map $\phi_{|K_X+\mathcal{M}|} \to \mathbb{P}^1$, with cubic surfaces S (in a $\mathbb{P}^3$) as generic fibers. We have that $(A, \mathcal{M}_A)$ is fibered by elliptic curves on $\mathbb{P}^1$ and no fiber splits, see [16]. Hence also the fibers of X do not split and $\operatorname{Pic} X \cong \mathbb{Z}\langle A, S\rangle$. We have $A^3 = 7$, $A^2S = 3$, $AS^2 = S^3 = 0$. Let $\mathcal{L} \cong \mathcal{O}_X(aA + bS)$, by Lemma 1.4, we have $d_B \geqslant 6$, so

$$6 \leqslant d_B = \mathcal{L}\mathcal{M}^2 = 7a + 3b \leqslant 8$$

and, by the inequalities (1.1) and (3.1) we get

$$0 < d_A = \mathcal{L}^2\mathcal{M} = a(7a + 6b) \leqslant 9,$$

$$0 < \mathcal{L}^3 = 7a^3 + 9a^2b = a^2(7a + 9b) \leqslant 13.$$

From the first inequalities we get $b < 0$, $a \geqslant 2$ (for $b = 0$, $a = 1$ we would have $\mathcal{L} = \mathcal{M}$), from the third we get $a = 2$ or $a = 3$ and $7a + 9b > 0$. With $a = 2$ the first inequalities gives $-8 \leqslant 3b \leqslant -6$, i.e., $b = -2$, but this contradicts $7a + 9b > 0$. If $a = 3$, then $-15 \leqslant 3b \leqslant -13$, so $b = -5$, but this is impossible since $d_A > 0$.

$\mathcal{M}^3 = 8$. The bounds $\mathcal{M}^3 \leqslant d_B \leqslant 8$ imply $d_B = 8$ while the bounds $d_B \leqslant d_A \leqslant d_B^2/\mathcal{M}^3$ imply $d_A = 8$. Moreover, $d_B\mathcal{L}^3 \leqslant d_A^2$, gives $\mathcal{L}^3 \leqslant 8$. We can exclude that $\mathcal{L}^3 < 8$ by looking at all the cases we have seen before (we have considered all the polarized threefolds of degree $\leqslant 7$), so we only have to study the case $d_B = d_A = \mathcal{M}^3 = \mathcal{L}^3 = 8$.

Now, let $A \in |\mathcal{M}|$ as always, we have that $(A, \mathcal{L}_A)$ is a surface of degree 8 with a bielliptic curve section. Such surfaces are classified in [9], Theorem 4.1, with the exception of the elliptic conic bundles discovered in [1], and we will use these results to complete our proof.

We can easily check that under the degree assumptions above X cannot be a quadric bundle on $\mathbb{P}^1$. In fact in this case, see e.g., [16], 0.6, the fibers are all irreducible and $\operatorname{Pic} X \cong \mathbb{Z}\langle A, F\rangle$, where F is a fiber, $A^3 = 8$, $A^2F = 2$ and $AF^2 = F^3 = 0$. Hence, if $\mathcal{L} \cong \mathcal{O}_X(aA + bF)$, we should have $\mathcal{L}^3 = 8a^3 + 6a^2b = 2a^2(4a + 3b) = 8$ which is easily seen to be impossible.

Now we proceed as in the previous cases. According to [17] we can only have $s = 6, 5, 4, 3, 2$.

$s = 6$. In this case X is the double embedding of $\mathbb{P}^3$ into $\mathbb{P}^9$, i.e., $X \cong \mathbb{P}^3$ and $\mathcal{M} \cong \mathcal{O}_{\mathbb{P}^3}(2)$ (see also Example 2). By Lemma 1.2 we should have $\mathcal{L} \cong \mathcal{O}_{\mathbb{P}^3}(3)$, but then we would have $d_B = 12$, $d_A = 18$.

$s = 5$. X is a hyperplane section of the Segre embedding of $\mathbb{P}^1 \times Q^3$ into $\mathbb{P}^9$, where Q^3 is a quadric hypersurface in $\mathbb{P}^4$. In this case X would be a quadric bundle, but we have just seen that this is impossible.

$s = 4$. We have four possibilities for X. First, X is a scroll on an elliptic curve E. This would imply that also A is an elliptic scroll on E, then its irregularity would be $q(A) = 1$, but this is impossible by [9], Theorem 4.1.

Two other possibilities are that X is either the complete intersection of a hyperquadric with a Segre variety V which is the embedding of $\mathbb{P}^1 \times \mathbb{P}^3$, or a double covering of a hyperplane section of V. But in both cases X would be a rational quadric bundle and we have already excluded this possibility.

The last case is that $X \cong \mathbb{P}(\mathscr{E})$, where $\mathscr{E}$ is a rank 2 locally free sheaf on $\mathbb{P}^2$ given by the exact sequence $0 \to \mathcal{O}_{\mathbb{P}^2} \to \mathscr{L} \to \mathscr{I}_Y(4) \to 0$, Y is a set of 8 general points in $\mathbb{P}^2$, and $\mathscr{M}$ is the tautological sheaf on $\mathbb{P}(\mathscr{E})$. In this case A is isomorphic to the blow-up of $\mathbb{P}^2$ along Y and $\mathscr{M}_A$ is associated to the linear system of quartic curves passing through Y. We should have that $(A, \mathscr{L}_A)$ is a surface of degree 8 which appears in the classification of [9], Theorem 4.1 (since its hyperplane section is a bielliptic curve), but this is possible only for $\mathscr{M}_A = \mathscr{L}_A$, which implies $\mathscr{M} = \mathscr{L}$, so also this case cannot occur.

$s = 3$. We have three possibilities for X. First, X could be a rational quadric bundle, but this is the case we have excluded.

The second case is that $\pi : X \to Q$ is a scroll on a quadric surface Q. So $\operatorname{Pic} X \cong \mathbb{Z}\langle A, F_1, F_2\rangle$, where $A \in |\mathscr{M}|$ and $F_i = \pi^{-1}(G_i)$, $\operatorname{Pic} Q \cong \mathbb{Z}\langle G_1, G_2\rangle$. We have $A^2F_i = AF_1F_2 = 1$ and $AF_i^2 = F_i^2F_j = 0$, $i, j = 1, 2$. Let $\mathscr{L} \cong \mathcal{O}_X(\alpha A + \beta F_1 + \gamma F_2)$, then we must have

$$d_B = \mathscr{L}\mathscr{M}^2 = 8\alpha + \beta + \gamma = 8,$$

$$d_A = \mathscr{L}^2\mathscr{M} = 2\alpha(4\alpha + \beta + \gamma) + 2\beta\gamma = 8,$$

$$\mathscr{L}^3 = \alpha(8\alpha^2 + 3\alpha + 3\beta + 6\beta\gamma) = 8.$$

By the third equality α must divide 8 and it is easy to check that any such value of α does not satisfy the first and second equations in β, γ.

The last possibility is that X is the complete intersection of three quadric hypersurfaces in $\mathbb{P}^6$. In this case $\operatorname{Pic} X \cong \mathbb{Z}$, then $\mathscr{L} \cong \mathcal{O}_X(a)$ and $\mathscr{L}^3 = 8a^2 = 8$ which yields $\mathscr{L} = \mathscr{M}$.

$s = 2$. X could be the complete intersection of a quadric and a quartic hypersurface. Then, since $\mathscr{L} \cong \mathcal{O}_X(a)$ and $\mathscr{L}^3 = 8a^2 = 8$, we can exclude this case as we did above.

Another possibility is that X is a Del Pezzo fibration on $\mathbb{P}^1$ given by its adjunction map $\phi_{|K_X + \mathscr{M}|} : X \to \mathbb{P}^1$. The generic fiber of ϕ is a Del Pezzo surface S isomorphic to a complete intersection of two quadrics in $\mathbb{P}^4$. We have that $\operatorname{Pic} X \cong \mathbb{Z}\langle A, S\rangle$, $A^3 = 8$, $A^2S = 4$, $AS^2 = S^3 = 0$, and let $\mathscr{L} \cong \mathcal{O}_X(aA + bS)$. Hence: $d_A = \mathscr{L}^2\mathscr{M} = 8a^2 + 8ab + 8a(a + b) = 8$, which is possible only for $b = 0$, $a = 1$, but this would imply $\mathscr{L} = \mathscr{M}$ once more.

The last case to be considered is the one missed in [17], [18] (and hence also in [9]) which we mentioned at the beginning of the proof, namely when X is such that $(A, \mathscr{M}_A)$ is a degree 8 conic bundle on an elliptic curve (see [1]). In this case, by working in a similar way as we did in the proof of Theorem A, case 2, we get that X must be as in Theorem A, but these cases have been excluded by our hypotheses.

To complete the proof of our theorem, we have only to notice that the existence of threefolds X as described in the first four cases is obvious and the case $C.5$ occurs when C is a canonical bielliptic curve of genus 5. Unfortunately we have not been able to determine whether a threefold X as in case $C.6$ exists or not.

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