Adv. Geom. 1 (2001), 157–163

Advances in Geometry

A short proof of the uniqueness of Kühnel's 9-vertex complex projective plane

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(Communicated by Karl Strambach)

Abstract. We introduce the notion of amicable partitions for combinatorial manifolds with complementarity. We prove that any 4-dimensional combinatorial manifold X_9^4 satisfying complementarity has an amicable partition and any amicable partition determines X_9^4 up to isomorphism. This gives a short proof of the uniqueness of Kühnel's 9-vertex complex projective plane.

2000 Mathematics Subject Classification. 57Q15, 57M15.

1 Introduction

1.1. In [4], Brehm and Kühnel proved that if X is a non-sphere d-dimensional combinatorial manifold on n vertices then $n \geqslant 3d/2 + 3$. In case of equality, the only possibilities are $d = 2^m$, $m \leqslant 4$, and in these cases $|X|$ is a 'manifold like a projective plane'. Arnoux and Marin showed in [1] that in the cases of equality X must have the following *complementarity* property: exactly one of the two cells in any non-trivial bipartition (of the vertex set of X) must be a face of X. In [6], the second-named author proved the following converse: if X is an n-vertex d-dimensional combinatorial manifold with the complementarity property then $n = 3d/2 + 3$ (and hence $d = 2^m$, $m \leqslant 4$).

1.2. Let us say that a non-sphere combinatorial manifold is a *B-K manifold* (B-K stands for Brehm and Kühnel, of course) if it satisfies $n = 3d/2 + 3$. It is well known (and quite easy to prove, see for instance [3]) that there is a unique 2-dimensional B-K manifold, namely the 6-vertex real projective plane $\mathbb{R}P_6^2$. It is also known (and this is much harder to prove) that there is a unique 4-dimensional B-K manifold, namely Kühnel's 9-vertex complex projective plane $\mathbb{C}P_9^2$. In [5], Brehm and Kühnel constructed three distinct 8-dimensional B-K manifolds. These three are combinatorially equivalent and hence their geometric realizations are PL-homeomorphic. (Recall that two simplicial complexes are called combinatorially equivalent if they have isomorphic subdivisions.) It is not known whether these are the only 8-dimensional B-K manifolds, nor is it known whether the common topological manifold triangulated by them is the quaternionic projective plane. No 16-dimensional example is known at present; presumably such an object would triangulate the Cayley projective plane.

1.3. Several proofs of the existence and uniqueness of $\mathbb{C}P_9^2$ are now known. The first was the computer-aided proof of Kühnel and Laßmann [9]. (A beautiful exposition of this paper may be found in [8].) The second proof, due to Arnoux and Marin [1], uses cohomology theory with $\mathbb{Z}_2$ coefficients. The third, a combinatorial proof, is due to the present authors in [2]. In [11], Morin and Yoshida surveyed the known proofs (and added one of their own) of the fact that the topological space triangulated by $\mathbb{C}P_9^2$ is the complex projective plane. Since then, one more proof of the last-named fact has been found by Madahar and Sarkaria [10]. They constructed a 17-vertex 4-ball D_{17}^4 whose boundary is a 12-vertex 3-sphere S_{12}^3 and defined a combinatorial analogue $h : S_{12}^3 \to S_4^2$ of the Hopf map so that the simplicial complex $S_4^2 \cup_h D_{17}^4$ is precisely $\mathbb{C}P_9^2$.

1.4. In [11], Morin and Yoshida presented arguments in support of having so many proofs identifying the geometric realization of $\mathbb{C}P_9^2$. The gist of their argument is that $\mathbb{C}P_9^2$ is such an important and exotic object that it is certainly worth in-depth studies, and different proofs will throw light on different aspects of this object. We believe that this argument applies equally well to proofs of the uniqueness of $\mathbb{C}P_9^2$. Thus encouraged, we present yet another combinatorial proof of the uniqueness. More precisely we prove:

Theorem. *Up to simplicial isomorphism there is a unique* 9*-vertex* 4*-dimensional combinatorial manifold satisfying complementarity.*

1.5. In [7], the second-named author proved that a 4-dimensional weak pseudomanifold (without boundary) satisfying complementarity is automatically a combinatorial manifold on 9 vertices. Therefore, the above theorem may also be stated as saying that: *up to isomorphism there is a unique* 4*-dimensional weak pseudomanifold without boundary satisfying complementarity.* Our proof, presented below, has the virtue of brevity: it is much shorter than all the previous proofs. The proof is based on the notion of amicable partition: in the language of [2], they are just the partitions of the vertex set into blue triangles. We prove that (a) any combinatorial manifold $X = X_9^4$ satisfying complementarity has an amicable partition, (b) up to isomorphism there are two types of amicable partitions, (c) any amicable partition determines X_9^4 up to isomorphism and (d) both types of amicable partitions determine the same combinatorial manifold. The general theory is developed in Section 2, while we specialize these results to $\mathbb{C}P_9^2$ in Section 3. Thus, Section 3 contains the proof of the main theorem.

2 Amicable partitions

2.1. *Amicable* partitions may be defined for any d-dimensional B-K manifold. These are the partitions of its vertex set into three $(d/2)$-faces A_1, A_2, A_3 such that the link of each A_i is the standard $(d/2-1)$-sphere on A_{i+1} (addition in the suffix is modulo three). We have:

Lemma 1. *Let A be a $(d/2)$-face of a d-dimensional B-K manifold X. Suppose the link of A is a standard sphere. Then A belongs to a unique amicable partition of X.*

Proof. Put $A = A_1$. Let A_2 be the vertex-set of the link of A_1 and let A_3 be the set of vertices outside $A_1 \cup A_2$. Then each A_i contains $d/2+1$ vertices. Note that complementarity implies that any set of $d/2+1$ (or fewer) vertices of X spans a face. In particular, each A_i is a $(d/2)$-face of X. So, to complete the proof, it is sufficient to show that the link of A_2 (respectively A_3) is the standard sphere on A_3 (respectively A_1).

Take any vertex $x \in A_2$. Then $A_3 \cup \{x\}$ is not a face since its complement $A_1 \cup (A_2 \setminus \{x\})$ is a face. Thus no vertex of A_2 belongs to the link of A_3. Therefore, the vertex set of the link of A_3 is contained in A_1. Since this link has at least $d/2+1$ vertices, it follows that the link of A_3 is the standard sphere on A_1. Replacing A_1 by A_3 (and hence A_2 by A_1, A_3 by A_2) in this argument, we see that the link of A_2 is the standard sphere on A_3.

In particular, this lemma shows that each edge of $\mathbb{R}P^2_6$ is a cell of a unique amicable partition. Hence there are five amicable partitions in $\mathbb{R}P^2_6$, and this fact trivialises the existence and uniqueness of $\mathbb{R}P^2_6$. From [2] it can be read off that $\mathbb{C}P^2_9$ has seven amicable partitions. (But this fact will not be used in what follows.) We observe that each of the three known 8-dimensional B-K manifolds has amicable partitions. (In fact, these three B-K manifolds have five, nine and eleven amicable partitions, respectively.) But we see no way to prove (or disprove!) the following:

Conjecture. Every B-K manifold has an amicable partition.

2.2. If U is an n-vertex m-sphere $(n > m+2)$ then clearly each vertex x of U is of degree $\geqslant m+1$ (i.e., x is in at least $m+1$ edges). If x is a vertex of degree $m+1$, we can construct an $(n-1)$-vertex m-sphere V as follows. Delete the vertex x (and all faces through x); introduce the set of neighbours of x as a new facet (i.e., maximal face). We shall say that V is obtained from U by *collapsing* the vertex x. Conversely, U can be recovered from V by starring a vertex x in the new facet.

Let X be a d-dimensional B-K manifold with an amicable partition $\{A_1, A_2, A_3\}$. Say, $A_1 = \{x_0, \ldots, x_{d/2}\}$. Then the link in X of the $(d/2-1)$-face $A_1 \setminus \{x_i\}$ is a $(d/2)$-sphere on the vertex set $\{x_i\} \cup A_2 \cup A_3$ wherein x_i is a vertex of degree $d/2+1$ and its neighbours are the vertices in A_2. Let X_i be the $(d/2)$-sphere obtained by collapsing x_i. The set $\{X_i : 0 \leqslant i \leqslant d/2\}$ of $(d/2)$-spheres thus obtained will be called a *layer* of the given amicable partition with respect to the cell A_1. Thus, any amicable partition has three layers of $(d/2)$-spheres corresponding to its three cells.

2.3. For any combinatorial sphere U, we shall use $\Gamma(U)$ to denote the graph with the vertices of U as vertices, such that two distinct vertices x and y are adjacent in

$\Gamma(U)$ if and only if $\{x, y\}$ is not a face of U. In other words, the edges of $\Gamma(U)$ are precisely the missing edges of U. Thus $\Gamma(U)$ is just the graph theoretic complement of the 1-skeleton of U.

The spheres in a layer of an amicable partition are far from arbitrary; they satisfy some strong compatibility requirements:

Lemma 2. *Let $\{X_i : 0 \leqslant i \leqslant d/2\}$ be a layer of an amicable partition $\{A_1, A_2, A_3\}$ of a d-dimensional B-K manifold X, say with respect to the cell A_1. Then*

(a) *A_2 and A_3 are common facets of all the X_i, $0 \leqslant i \leqslant d/2$; and $\{A_2, A_3\}$ gives a partition of the common vertex set of these spheres. It follows that for each i, $\Gamma(X_i)$ is a bipartite graph (with A_2, A_3 as its parts).*

(b) *$\{\Gamma(X_i) : 0 \leqslant i \leqslant d/2\}$ is an edge-partition of the complete bipartite graph $K_{d/2+1, d/2+1}$ with parts A_2, A_3.*

(c) *For $0 \leqslant i \neq j \leqslant d/2$, any facet C of X_i intersects any facet D of X_j, provided $\{C, D\} \neq \{A_2, A_3\}$.*

Proof. A_2 is a facet of each X_i by construction. Since $\mathrm{Lk}_X(A_3)$ is the standard sphere on A_1, $A_3 \cup (A_1 \backslash \{x_i\})$ is a facet of X, and hence A_3 is a facet of X_i. Since A_2 (or A_3) is a facet of X_i, no two vertices in A_2 (or in A_3) are adjacent in $\Gamma(X_i)$. So, $\Gamma(X_i)$ is bipartite. This proves (a).

Let $\{x, y\}$ be an edge of $K_{d/2+1, d/2+1}$. Say $x \in A_2$, $y \in A_3$. Then $(A_2 \backslash \{x\}) \cup (A_3 \backslash \{y\})$ is a $(d-1)$-face of X. One of the two facets of X containing this face is $A_2 \cup (A_3 \backslash \{y\})$. The other facet cannot be $(A_2 \backslash \{x\}) \cup A_3$ (since the vertex set of $\mathrm{Lk}_X(A_3)$ is A_1). So, there is a unique vertex x_i in A_1 such that $(A_2 \backslash \{x\}) \cup (A_3 \backslash \{y\}) \cup \{x_i\}$ is a facet of X. By complementarity, x_i is the unique vertex in A_1 for which $(A_1 \backslash \{x_i\}) \cup \{x, y\}$ is not a face of X. Thus $\{x, y\}$ is not a face of X_i for a uniquely determined index i. This proves (b).

If $C \cap D = \varnothing$, C a facet of X_i, D is a facet of X_j, then $C \cup D = A_2 \cup A_3$. If, further $\{C, D\} \neq \{A_2, A_3\}$ then it follows that $C \neq A_2$ and $D \neq A_2$. Hence $C \cup (A_1 \backslash \{x_i\})$ and $D \cup (A_1 \backslash \{x_j\})$ are two facets of X which together cover the vertex set of X (as $i \neq j$). Therefore, the complement of either of these two facets of X is a face of X—contradicting complementarity. This proves (c).

2.4. If $\{X_i : 0 \leqslant i \leqslant d/2\}$ is one of the layers of an amicable partition, then the set $\{\Gamma(X_i) : 0 \leqslant i \leqslant d/2\}$ will be called the *frame* of the layer. Thus the frame is an edge partition of a complete bipartite graph by spanning subgraphs.

Lemma 3. *Each layer of an amicable partition of a B-K manifold determines the other two frames.*

Proof. Let the cells of the amicable partition be $A_i = \{x_{ij} : 0 \leqslant j \leqslant d/2\}$ with corresponding layer $\{X_{ij} : 0 \leqslant j \leqslant d/2\}$ and frame $\{\Gamma_{ij} = \Gamma(X_{ij}) : 0 \leqslant j \leqslant d/2\}, 1 \leqslant i \leqslant 3$. Suppose the layer $\{X_{1j} : 0 \leqslant j \leqslant d/2\}$ is known. Then $\{x_{1j}, x_{3l}\}$ is an edge of Γ_{2k} if and only if $(A_2\backslash\{x_{2k}\})\cup\{x_{1j},x_{3l}\}$ is not a face of the B-K manifold X; by complementarity this happens if and only if $(A_1\backslash\{x_{1j}\})\cup(A_3\backslash\{x_{3l}\})\cup\{x_{2k}\}$ is a facet of X, i.e., if and only if $(A_3\backslash\{x_{3l}\})\cup\{x_{2k}\}$ is a facet of X_{1j}. Similarly $\{x_{1j},x_{2k}\}$ is an edge of Γ_{3l} if and only if $(A_2\backslash\{x_{2k}\})\cup\{x_{3l}\}$ is a facet of X_{1j}.

3 Uniqueness of $\mathbb{C}P_9^2$

Throughout this section, Y is a 4-dimensional B-K manifold. Hence Y satisfies complementarity. From complementarity and Dehn–Sommerville equations, it readily follows that the number f_i of i-faces of Y are given by: $f_0=9$, $f_1=\binom{9}{2}=36$, $f_2=\binom{9}{3}=84$, $f_3=90$ and $f_4=36$. Further, we have:

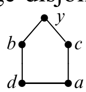
Lemma 4. *Y has an amicable partition.*

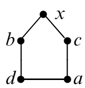
Proof. By Lemma 1, it is sufficient to show that there is at least one triangle (i.e., 2-face) in Y whose link is an S_3^1. Suppose not. Then the link of each triangle has $\geqslant 4$ vertices. Fix any facet σ of Y. By complementarity, the complement of σ induces an S_4^2. Therefore, the link of each of the four triangles in the complement of σ is contained in σ and hence (by our assumption) has four or five vertices. Let a of them have 5-vertex links, hence the remaining $4-a$ have 4-vertex links. Therefore, the total number of tetrahedra meeting σ in a singleton is $5a+4(4-a)=16+a$. But, by complementarity, this number is $\binom{5}{1}\binom{4}{3}$ minus the number of facets meeting σ in a 3-face $=20-5=15$. Hence $a=-1$, a contradiction.

Next we determine the possibilities for the layers of an amicable partition of Y. Each layer consists of three 6-vertex 2-spheres (S_6^2's). It is well known (and immediate from the classifications of S_{d+4}^d's in [3]) that up to isomorphism there are two S_6^2's. Their Γ-graphs are $3K_2$ (the disjoint union of three edges) and the three path $P_3 = \bullet\!\!-\!\!\!-\!\!\bullet\!\!-\!\!\!-\!\!\bullet\!\!-\!\!\!-\!\!\bullet$ (plus two isolated vertices) respectively. We need the following stronger statement:

Lemma 5. *Given a graph $\Gamma=3K_2$ or P_3, there is a unique 6-vertex 2-sphere U with $\Gamma(U)=\Gamma$ (not merely unique up to automorphism of Γ).*

Proof. Note that any 6-vertex 2-sphere U has eight facets and they are 3-cocliques of $\Gamma(U)$. (Recall that a coclique in a graph is a set of pairwise non-adjacent vertices.) Since $\Gamma=3K_2$ has exactly eight 3-cocliques, the lemma is immediate in this case.

In the second case, let $\Gamma = \overset{a}{\bullet}\!\!-\!\!\!-\!\!\overset{b}{\bullet}\!\!-\!\!\!-\!\!\overset{c}{\bullet}\!\!-\!\!\!-\!\!\overset{d}{\bullet}$ with isolated vertices x and y. Then the link of x in U is a pentagon. This pentagon induces a 3-path on $\{a,b,c,d\}$ which is edge disjoint from Γ. Hence this 3-path is $\overset{b}{\bullet}\!\!-\!\!\!-\!\!\overset{d}{\bullet}\!\!-\!\!\!-\!\!\overset{a}{\bullet}\!\!-\!\!\!-\!\!\overset{c}{\bullet}$. Thus, the link of x in U is 

. Similarly, the link of y is 

. This determines all the facets of U.

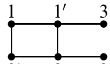
Lemma 6. *Y is uniquely determined (not merely up to isomorphism) by any of the frames of any given amicable partition.*

Proof. Since the graphs in any frame are isomorphic to $3K_2$ or P_3, Lemma 5 shows that the frame determines the corresponding layer. Then Lemma 3 determines all the frames of the given amicable partition. Another appeal to Lemma 5 determines all three layers. The known links of the three cells of the amicable partition give us 9 facets of Y. The known layers give $((8-2)\times 3\times 3)/2 = 27$ more. We now have all the $9+27=36$ facets of Y.

Lemma 7. *Up to isomorphism there are two possible types of frames for Y.*

Proof. If two of the graphs in a frame are $3K_2$, they must consist of alternating edges of a hexagon. Then the third graph in the frame is determined as the relative complement of this hexagon with respect to $K_{3,3}$. This third graph is the $3K_2$ whose edges are the long diagonals of the hexagon. This yields the *first type* of frames—consisting of three edge-disjoint copies of $3K_2$.

Next let the frame consist of one $3K_2$ and hence two P_3's. Then the relative complement of the $3K_2$ is a hexagon and each P_3 must consist of three consecutive edges of the hexagon. Say, the edges of the $3K_2$ are $1 - 1'$, $2 - 2'$, $3 - 3'$. Then, without loss, the P_3's in the frame are $1 - 2' - 3 - 1'$ and $1' - 2 - 3' - 1$. Then, from the proof of Lemma 5, we see that $\{1,1',2\}$ is a face of the S^2_6 with the first P_3 as Γ-graph while $\{2',3,3'\}$ is a face of the S^2_6 with the second P_3 as Γ-graph. Since these two triangles are disjoint and distinct from the parts $\{1,2,3\}$, $\{1',2',3'\}$ of the $K_{3,3}$, this contradicts Lemma 2 (c).

So, in the remaining case, the frame must be an edge partition of $K_{3,3}$ into three copies of P_3. Let the parts of the $K_{3,3}$ be $\{1,2,3\}$ and $\{1',2',3'\}$. Without loss, let the first graph in the frame be $1 - 2' - 3 - 3'$. The relative complement (with respect to $K_{3,3}$) of this graph is . It is obvious that the last graph has a unique edge partition into two P_3's. So the remaining two graphs in the frame must be $2 - 3' - 1 - 1'$ and $3 - 1' - 2 - 2'$. This gives the *second* isomorphism type of frames, consisting of three copies of P_3.

Lemma 8. *Y has an amicable partition one of whose frames is of the first type (i.e., consists of three copies of $3K_2$).*

Proof. Take an amicable partition $\{1,2,3\}$, $\{1',2',3'\}$, $\{1'',2'',3''\}$ of Y. (This exists by Lemma 4.) If the frame corresponding to the cell $\{1'',2'',3''\}$ is not of the first type, then (by Lemma 7) it is of the second type. Hence, without loss, this frame consists of $\Gamma(X_1) = 2 - 1' - 3 - 3'$, $\Gamma(X_2) = 3 - 2' - 1 - 1'$, $\Gamma(X_3) = 1 - 3' - 2 - 2'$, where X_i is the 2-sphere obtained from the link of $\{1'',2'',3''\}\setminus\{i''\}$ by collapsing i''. Thus, following the proof of Lemma 6, Y is uniquely determined. Hence one finds that $\{\{1,1',1''\}, \{2,2',2''\}, \{3,3',3''\}\}$ is also an amicable partition of Y and the frame corresponding to the part $\{1, 1', 1''\}$ consists of $\{\overset{2}{\bullet}\!\!-\!\!\!-\!\!\!-\!\!\overset{3'}{\bullet}, \overset{2'}{\bullet}\!\!-\!\!\!-\!\!\!-\!\!\overset{3}{\bullet}, \overset{2''}{\bullet}\!\!-\!\!\!-\!\!\!-\!\!\overset{3''}{\bullet}\}$, $\{\overset{2}{\bullet}\!\!-\!\!\!-\!\!\!-\!\!\overset{3}{\bullet}, \overset{2'}{\bullet}\!\!-\!\!\!-\!\!\!-\!\!\overset{3''}{\bullet}, \overset{2''}{\bullet}\!\!-\!\!\!-\!\!\!-\!\!\overset{3'}{\bullet}\}$ and $\{\overset{2}{\bullet}\!\!-\!\!\!-\!\!\!-\!\!\overset{3''}{\bullet}, \overset{2'}{\bullet}\!\!-\!\!\!-\!\!\!-\!\!\overset{3'}{\bullet}, \overset{2''}{\bullet}\!\!-\!\!\!-\!\!\!-\!\!\overset{3}{\bullet}\}$. This frame is of the first type.

Proof of the theorem. By Lemma 6 and Lemma 8, Y is uniquely determined up to isomorphism.

Remark. It can be seen that all three frames of any amicable partition of $\mathbb{C}P_9^2$ are of the same type. $\mathbb{C}P_9^2$ contains a unique amicable partition of type one and six of type two.

Acknowledgement. We thank W. Kühnel and the referee for their suggestions leading to significant improvements in the presentation of this paper.

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Received 1 September, 2000; revised 6 February, 2001

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