Annales Academiæ Scientiarum Fennicæ
Mathematica
Volumen 33, 2008, 387–412

HIGHER INTEGRABILITY FOR WEAK SOLUTIONS OF HIGHER ORDER DEGENERATE PARABOLIC SYSTEMS

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Abstract. We consider a class of higher order nonlinear degenerate parabolic systems, whose easiest model is the parabolic p-Laplacean system

$$\int_{\Omega_T} \big(u \cdot \varphi_t - |D^m u|^{p-2} D^m u \cdot D^m \varphi\big)\, dz = \int_{\Omega_T} \sum_{k=0}^{m-1} B^k(\cdot, D^m u) \cdot D^k \varphi \, dz$$

and show higher integrability for weak solutions, proving that $D^m u \in L^p$ implies that $D^m u \in L^{p+\varepsilon}$ for some $\varepsilon > 0$.

1. Introduction

Let $\Omega \subset \mathbf{R}^n$ be a bounded open set and $\Omega_T \equiv \Omega \times (-T, 0)$ $(T > 0)$ the parabolic cylinder over Ω. We consider weak solutions $u \in L^p(-T, 0; W^{m,p}(\Omega; \mathbf{R}^N)) \cap L^2(\Omega_T; \mathbf{R}^N)$, with $m, N \geq 1$ and $p > \max\{1, \frac{2n}{n+2m}\}$, of higher order degenerate parabolic systems of the form

$$\int_{\Omega_T} \big(u \cdot \varphi_t - A(z, D^m u) \cdot D^m \varphi\big)\, dz = \int_{\Omega_T} B(z, D^m u) \cdot \delta\varphi \, dz \tag{1}$$

for all $\varphi \in C_0^\infty(\Omega_T; \mathbf{R}^N)$. Here and in the following we write $z = (x, t) \in \mathbf{R}^{n+1}$, $\varphi_t = \partial_t \varphi$ denotes the derivative with respect to the time-variable t, whence Du, respectively $D^k u$ denote the derivatives with respect to the space-variable x and $\delta u = (u, Du, \dots, D^{m-1}u)$ is the vector of lower order derivatives. We note that $D^k u = \{D^\alpha u_i\}_{i=1,\dots,N}^{|\alpha|=k}$ is an element of the vectorspace $\odot^k(\mathbf{R}^n, \mathbf{R}^N)$ of k-linear functions with values in $\mathbf{R}^N$, which can be identified with $\mathbf{R}^{N\binom{n+k-1}{k}}$. We shall use the abbreviations $\mathscr{N} = N\binom{n+m-1}{m}$, $\mathscr{M} = N\binom{n+m-1}{m-1} = \sum_{k=0}^{m-1} \mathscr{M}_k$, where $\mathscr{M}_k = N\binom{n+k-1}{k}$, which allow us to write $D^m u \in \mathbf{R}^{\mathscr{N}}$, $D^k u \in \mathbf{R}^{\mathscr{M}_k}$ and $\delta u \in \mathbf{R}^{\mathscr{M}}$.

We consider coefficients $A\colon \Omega_T \times \mathbf{R}^{\mathscr{M}} \times \mathbf{R}^{\mathscr{N}} \to \mathrm{Hom}(\mathbf{R}^{\mathscr{N}}, \mathbf{R})$ and $B \equiv (B^0, \dots, B^{m-1})$ with $B^k\colon \Omega_T \times \mathbf{R}^{\mathscr{M}} \times \mathbf{R}^{\mathscr{N}} \to \mathrm{Hom}(\mathbf{R}^{\mathscr{M}_k}, \mathbf{R})$ for $k = 0, \dots, m-1$, fulfilling p-growth conditions, which are allowed to be degenerate. To be precise, we assume

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2000 Mathematics Subject Classification: Primary 35D10, 35G20, 35K65.
Key words: Higher integrability, degenerate parabolic systems, higher order, parabolic p-Laplacean.

that

$$A(z,q)\cdot q \ge \nu\,|q|^p - b_0, \tag{2}$$
$$|A(z,q)| \le L\,|q|^{p-1} + b_1, \tag{3}$$
$$|B(z,q)| \le L\,|q|^{p-1} + b_2, \tag{4}$$

for all $z \in \Omega_T$, $q \in \mathbf{R}^{\mathscr{N}}$ and some constants $0 < \nu \le 1$ and $1 \le L < \infty$. Let us mention that the restriction $p > \max\{1, \frac{2n}{n+2m}\}$ is necessary in the parabolic framework, because of the embedding $W^{m,\frac{2n}{n+2m}} \hookrightarrow L^2$ (we always have to deal with the L^2-norm of u, coming from the time derivative u_t of u in the Caccioppoli inequality, i.e. Lemma 6). The functions $b_i\colon \Omega_T \to \mathbf{R}$ are assumed to be measurable for $i = 0, 1, 2$ with bounded norm

$$\|b\|_{L^\sigma(\Omega_T)} < \infty \quad \text{for some } \sigma > p, \quad \text{where } \ b \equiv \big(|b_1| + |b_2|\big)^{\frac{1}{p-1}} + |b_0|^{\frac{1}{p}}.$$

The purpose of this paper is to show that $D^m u$ is higher integrable, i.e. that there exists $\varepsilon > 0$ such that $u \in L^{p+\varepsilon}(-T, 0; W^{m,p+\varepsilon}(\Omega; \mathbf{R}^N))$, together with a local estimate for the $L^{p+\varepsilon}$-norm of $D^m u$.

Initially, higher integrability results were achieved for elliptic systems, see [8, 11, 18]. The main point in the proof is to apply in turn a Caccioppoli inequality for the weak solution and the Sobolev–Poincaré inequality to conclude a reverse-Hölder inequality. Then, the gain of the exponent is achieved with the help of Gehring's lemma, [10]. But, unfortunately in the case of parabolic systems neither the Sobolev–Poincaré inequality nor the Poincaré inequality can be applied (even in the case $p = 2$), since weak solutions are only assumed to be L^p-functions with respect to the time-variable t. Nevertheless, it turns out that the weighted means of a weak solution (see (7)) are absolutely continuous with respect to t, which allows us to show a sort of Poincaré inequality valid for weak solutions. This method was introduced by Giaquinta and Struwe [12], proving higher integrability of weak solutions in the case $p = 2$. But this method could not directly be transferred to the case $p \neq 2$, where we have to deal with the additional difficulty that the parabolic system behaves "non-homogeneous", in the sense that solutions of the parabolic p-Laplacean system are not invariant under multiplication by constants. On the other hand, reverse-Hölder inequalities which are essential to apply Gehring's lemma, are indeed invariant under multiplication by constants. The key to come up with this lack of homogeneity is to choose a system of cylinders whose side lengths depend on the size of the solution itself. This idea goes back to DiBenedetto, [5, 6] proving "intrinsic" Harnack estimates and $C^{1,\alpha}$-regularity of solutions of the p-Laplacean equation, respectively system. This method turned out to be fruitful also when considering systems of more general structure and it was used in [13, 14] by Kinnunen and Lewis to show higher integrability for second order parabolic systems in the case $p \neq 2$. However, due to the fact that no uniform system of cylinders is available, the proof is much more involved, compared to the case $p = 2$. For instance, the reverse Hölder inequality (see Lemma 13) is valid only on cylinders fulfilling certain additional assumptions.

In the present paper we extend this result to the case of higher order systems. Regarding higher order parabolic problems, there had to be developed new techniques to overcome the difficulties arising from the lack of regularity of the intermediate derivatives $Du, \ldots, D^{m-1}u$ with respect to the time variable t. In particular, we cannot estimate those integrals in terms of $D^m u$, since the general Poincaré inequality is not applicable. To show nevertheless a suitable Caccioppoli inequality we use an interpolation theorem on the annulus (see Lemma 3), which preserves the right scaling. Moreover, in a certain sense we have to "approximate" the solution up to m-th order. For this aim we exploit the mean value polynomials of u, depending only on the space-variable x. The advantage of choosing polynomials not depending on t is that we need no regularity with respect to t when estimating them. Moreover, we prove a suitable bound for the L^2-norm of u which simplifies the proof of the higher integrability in the case $p < 2$ also for second order systems.

Finally, we want to point out that recently Acerbi and Mingione [1] showed Calderón & Zygmund estimates for a class of degenerate parabolic systems. In the proof, higher integrability of the solutions plays an important role. For similar results in the elliptic case see also the papers [15], [16] with the references therein.

2. Notation and statement of the result

In the case of parabolic systems it is convenient to show the estimates on parabolic cylinders of the form $Q_{z_0}(\varrho, s) \equiv B_{x_0}(\varrho) \times (t_0 - s, t_0 + s) \subset \mathbf{R}^{n+1}$, where $z_0 = (x_0, t_0) \in \mathbf{R}^{n+1}$, $\varrho, s > 0$ and $B_{x_0}(\varrho)$ denotes the open ball in $\mathbf{R}^n$ with center x_0 and radius ϱ. In the case $s = \varrho^{2m}$ we write $Q_{z_0}(\varrho) \equiv Q_{z_0}(\varrho, \varrho^{2m})$. If $z_0 = 0$, we abbreviate $Q(\varrho, s) = Q_0(\varrho, s)$ and $B(\varrho) = B_0(\varrho)$. Moreover, if $v\colon Q_{z_0}(\varrho, s) \to \mathbf{R}^k$, $k \in \mathbf{N}$ is integrable we write $(v)_{z_0;\varrho,s} \equiv (v)_{Q_{z_0}(\varrho,s)} \equiv ⨍_{Q_{z_0}(\varrho,s)} v\,dz$ for its mean-value on $Q_{z_0}(\varrho, s)$, respectively for $w\colon B_{x_0}(\varrho) \to \mathbf{R}^k$ we write $(w)_{x_0;\varrho} \equiv (w)_{B_{x_0}(\varrho)} \equiv ⨍_{B_{x_0}(\varrho)} w\,dz$.

Now, we can state our main result:

Theorem 1. *Let $p > \max\{1, \frac{2n}{n+2m}\}$ and suppose that $u \in L^p(-T, 0; W^{m,p}(\Omega; \mathbf{R}^N)) \cap L^2(\Omega_T; \mathbf{R}^N)$ is a weak solution of the parabolic system (1) under the assumptions (2) – (4). Then there exists $\varepsilon = \varepsilon(n, N, m, p, L/\nu, \sigma) > 0$, such that*

$$u \in L^{p+\varepsilon}_{\mathrm{loc}}(-T, 0; W^{m,p+\varepsilon}_{\mathrm{loc}}(\Omega; \mathbf{R}^N)),$$

and for any parabolic cylinder $Q_{z_0}(2\varrho) \Subset \Omega_T$ there holds

$$⨍_{Q_{z_0}(\varrho)} |D^m u|^{p+\varepsilon}\,dz \le c \left[⨍_{Q_{z_0}(2\varrho)} \left(|D^m u|^p + b^{\,p} \right) dz \right]^{1+\frac{\varepsilon}{d}} + c ⨍_{Q_{z_0}(2\varrho)} \left(1 + b^{\,p+\varepsilon} \right) dz,$$

where $c = c(n, N, m, p, L/\nu)$ *and*

$$d \equiv \begin{cases} 2 & \text{if } p \geq 2, \\ p - \dfrac{n(2-p)}{2m} & \text{if } p < 2. \end{cases}$$

3. Preliminary material

3.1. Technical lemma. In order to "absorb" certain integrals of the right-hand side, we will use the following lemma, which is standard and can be found for instance in [11].

Lemma 2. *Let* $0 < \vartheta < 1$, $A, B \geq 0$, $\alpha > 0$ *and let* $f \geq 0$ *be a bounded function satisfying*

$$f(t) \leq \vartheta f(s) + A(s-t)^{\alpha} + B \quad \text{for all } 0 < r \leq t < s \leq \varrho.$$

Then there exists a constant $c_{\mathrm{tech}} = c_{\mathrm{tech}}(\alpha, \vartheta)$, *such that*

$$f(r) \leq c_{\mathrm{tech}}\big(A(\varrho - r)^{-\alpha} + B\big).$$

3.2. Interpolation lemmata. We now state an interpolation lemma for intermediate derivatives on the annulus, similar to [2], Theorem 4.14. For the proof in this particular situation, i.e. the right scaling on the annulus we refer to [3], Lemma B.1. Later, we will apply this lemma several times on the horizontal time slices.

Lemma 3. *Let* $B(r_1), B(r_2) \subset \mathbf{R}^n$ *be two balls with the same center and radius* r_1 *respectively* r_2, *where* $0 < r_1 < r_2 \leq 1$ *and let* $u \in W^{m,p}(B(r_2))$ *with* $p \geq 1$. *Then for any* $0 \leq k \leq m-1$ *and* $0 < \varepsilon \leq 1$ *there exists* $c = c(n, m, p, 1/\varepsilon)$, *such that*

$$\int_{B(r_2)\setminus B(r_1)} \frac{|D^k u|^p}{(r_2 - r_1)^{p(m-k)}}\, dx \leq \varepsilon \int_{B(r_2)\setminus B(r_1)} |D^m u|^p\, dx + c \int_{B(r_2)\setminus B(r_1)} \frac{|u|^p}{(r_2 - r_1)^{pm}}\, dx.$$

We now state Gagliardo–Nirenberg's inequality (see [17]) in a form, which is convenient for our purpose:

Theorem 4. *Let* $B_{x_0}(\varrho) \subset \mathbf{R}^n$ *with* $\varrho \leq 1$ *and* $u \in W^{m,\vartheta}(B_{x_0}(\varrho))$, $m \in \mathbf{N}$ *and* $1 \leq p, \vartheta, r \leq \infty$ *and* $\theta \in (0,1)$ *and* $0 \leq k \leq m-1$ *with* $k - \frac{n}{p} \leq \theta(m - \frac{n}{\vartheta}) - (1-\theta)\frac{n}{r}$. *Then, there holds*

$$\begin{aligned} &\fint_{B_{x_0}(\varrho)} |D^k u|^p\, dx \\ &\leq c(n,m,p)\; \varrho^{(m\theta - k)p} \Bigg(\sum_{j=0}^{m} \fint_{B_{x_0}(\varrho)} \frac{|D^j u|^{\vartheta}}{\varrho^{\vartheta(m-j)}}\, dx \Bigg)^{\frac{\theta p}{\vartheta}} \Bigg(\fint_{B_{x_0}(\varrho)} |u|^r\, dx \Bigg)^{\frac{(1-\theta)p}{r}}. \end{aligned}$$

3.3. Mean value polynomials. In order to treat regularity problems for elliptic respectively parabolic systems one usually needs to control oscillation quantities of the solutions to measure in a weak sense its regularity. Therefore polynomials, especially the mean value polynomials, will play an important role. In addition we can estimate any polynomial in terms of its mean values.

Lemma 5. *Let* $P\colon \mathbf{R}^n \to \mathbf{R}^N$ *be a polynomial of degree* $\le m-1$ *and* $B_{x_0}(r) \subset \mathbf{R}^n$. *Then for any* $0 \le k \le m-1$ *there holds:*

$$|D^k P(x)| \le c(n,m) \sum_{j=k}^{m} r^{j-k} |(D^j P)_{x_0;r}| \quad \text{for all } x \in B_{x_0}(r).$$

Proof. We will only sketch the proof and refer to [3], Lemma A.1. for a more detailed proof. From [7] we know, that P can be expressed in terms of its mean values as follows:

$$P(x) = \sum_{|\alpha|\le m} \sum_{|\alpha+\beta|\le m} \frac{b_\beta}{\alpha!} (D^{\alpha+\beta} P)_{x_0;r}\, (x-x_0)^\alpha,$$

where

$$b_\beta = \begin{cases} 1, & \text{if } |\beta| = 0, \\ -\displaystyle\sum_{0<\gamma\le\beta} \frac{b_{\beta-\gamma}}{\gamma!} \fint_{B_{x_0}(r)} (y-x_0)^\gamma\, dy, & \text{if } |\beta| \ge 1. \end{cases}$$

We can show that $|b_\beta| \le c(n,m)\, r^{|\beta|}$ for all β with $0 \le |\beta| \le m$. From the above representation of P we then conclude the desired estimate. □

3.4. Steklov-means. Since by their definition, weak solutions do not require any differentiability properties with respect to the time variable t, it is standard to use some mollification in time. Therefore, given a function $f \in L^1(\Omega_T)$ and $0 < h < T$ we define its Steklov-mean by

$$[f]_h(x,t) \equiv \begin{cases} \dfrac{1}{h} \displaystyle\int_t^{t+h} f(x,s)\, ds, & t \in (-T,-h), \\ 0, & t \in (-h, 0). \end{cases}$$

For the Steklov-mean $[u]_h$ of a weak solution u of (1), we get the following equivalent system: For a.e. $t \in (-T, 0)$ there holds

$$\int_\Omega \Big(\partial_t [u]_h(\cdot,t)\cdot\varphi + \big[A(\cdot, D^m u)\big]_h(\cdot,t)\cdot D^m\varphi \Big)\, dx = -\int_\Omega \big[B(\cdot, D^m u)\big]_h(\cdot,t)\cdot\delta\varphi\, dx \tag{5}$$

for all $\varphi \in L^2(\Omega;\mathbf{R}^N) \cap W_0^{m,p}(\Omega;\mathbf{R}^N)$.

4. Caccioppoli inequality

As usual, the first step in proving higher integrability is a suitable Caccioppoli inequality.

Lemma 6. *Suppose that $u \in L^p(-T,0;W^{m,p}(\Omega;\mathbf{R}^N)) \cap L^2(\Omega_T;\mathbf{R}^N)$ is a weak solution of system (1) in Ω_T under the assumptions (2) – (4) and $P\colon \mathbf{R}^n \to \mathbf{R}^N$ is a polynomial of degree $\le m-1$. Then for all parabolic cylinders $Q_{z_0}(R,S) \Subset \Omega_T$ with $0 < R \le 1$, $S > 0$ and for $r \in (R/2,R)$, $s \in (S/2,S)$ there holds*

$$\sup_{t\in(t_0-s,t_0+s)} \fint_{B_{x_0}(r)} \frac{|u(\cdot,t)-P|^2}{s}\,dx + \fint_{Q_{z_0}(r,s)} |D^m u|^p\,dz$$
$$\le c_{Cac}(n,m,p,L/\nu) \fint_{Q_{z_0}(R,S)} \left(\frac{|u-P|^2}{S-s} + \frac{|u-P|^p}{(R-r)^{mp}} + b^{\,p} \right) dz.$$

Proof. Without loss of generality we can assume that $z_0 = (x_0,t_0) = 0$. We choose $r \le r_1 < r_2 \le R$ and $\eta \in C_0^\infty(B(r_2))$, $\zeta \in C^1(\mathbf{R})$ to be two cut-off functions with

$$\begin{cases} \eta \equiv 1 \text{ in } B(r_1), \quad 0 \le \eta \le 1, \quad |D^k \eta| \le c_\eta (r_2-r_1)^{-k} \text{ for all } 0 \le k \le m; \\ \zeta \equiv 0 \text{ on } (-\infty,-S)\,, \quad \zeta \equiv 1 \text{ on } (-s,\infty)\,, \quad 0 \le \zeta \le 1, \quad 0 \le \zeta' \le 2(S-s)^{-1}. \end{cases}$$

Choosing the test-function $\varphi_h \equiv \eta\zeta^2([u]_h - P)$ in the Steklov-formulation (5) of the system we get for a.e. $\tau \in (-S,S)$

$$\int_{B(r_2)} \big(\partial_\tau [u]_h \cdot \varphi_h + [A(\cdot,D^m u)]_h \cdot D^m \varphi_h\big)(\cdot,\tau)\,dx = -\int_{B(r_2)} \big([B(\cdot,D^m u)]_h \cdot \delta\varphi_h\big)(\cdot,\tau)\,dx.$$

Noting that $\partial_t P = 0$ and $\zeta(-S) = 0$, we find for a.e. $t \in (-S,S)$ that

$$\int_{-S}^t \int_{B(r_2)} \partial_\tau [u]_h \cdot \varphi_h\,dx d\tau = \int_{-S}^t \int_{B(r_2)} \big(\tfrac12 \partial_\tau\big(|[u]_h - P|^2 \zeta^2\big)\eta - |[u]_h - P|^2 \eta\zeta\zeta'\big)\,dx d\tau$$
$$= \tfrac12 \int_{B(r_2)} |[u]_h(\cdot,t) - P|^2 \eta\zeta(t)^2\,dx - \int_{-S}^t \int_{B(r_2)} |[u]_h - P|^2 \eta\zeta\zeta'\,dx\,d\tau.$$

Therefore, integrating the above system over $(-S,t)$ and passing to the limit $h \searrow 0$ yields for a.e. $t \in (-S,S)$

$$\tfrac12 \int_{B(r_2)} |u(\cdot,t) - P|^2\,\eta\zeta(t)^2\,dx + \int_{-S}^t \int_{B(r_2)} A(\cdot,D^m u) \cdot D^m u\,\eta\zeta^2\,dz$$
$$= \int_{-S}^t \int_{B(r_2)} \Big(-A(\cdot,D^m u) \cdot \mathrm{LOT}\,\zeta^2 - B(\cdot,D^m u) \cdot \delta\varphi + |u-P|^2\,\eta\zeta\zeta' \Big) dz,$$

where $\varphi \equiv \eta\zeta^2(u-P)$ and $dz = dx\,d\tau$ and we have used the abbreviation

$$D^m \varphi = \Big(\eta\, D^m u + \underbrace{\sum_{k=0}^{m-1} \binom{m}{k} D^{m-k}\eta \odot D^k(u-P)}_{\equiv \mathrm{LOT}} \Big) \zeta^2.$$

From the ellipticity (2) of A, the growth conditions (3) of A and (4) of B, Young's inequality and the fact that $\zeta' \le (S-s)^{-1}$ and $0 \le \eta, \zeta \le 1$, we infer for $\varepsilon > 0$ that

$$\tfrac12 \int_{B(r_2)} |u(\cdot,t) - P|^2\, \eta\zeta^2(t)\, dx + \nu \int_{-S}^{t}\int_{B(r_2)} \big(|D^m u|^p - |b_0|\big)\, \eta\zeta^2\, dz$$
$$\le \varepsilon \int_{-S}^{t}\int_{B(r_2)} |D^m u|^p \zeta^2\, dz + c \int_{-S}^{t}\int_{B(r_2)} \bigg(|\mathrm{LOT}|^p\zeta^2 + |\delta\varphi|^p + \frac{|u-P|^2}{S-s} + b^{\,p}\bigg) dz,$$

where $c = c(p, L, 1/\varepsilon)$. To estimate the term involving the terms of lower order, we exploit the fact that $D^k\eta = 0$ on $B(r_1)$ for $k \ge 1$ and apply the Interpolation Lemma 3 "slicewise" on the annulus $B(r_2)\backslash B(r_1)$ to obtain for $0 < \mu \le 1$

$$\int_{-S}^{t}\int_{B(r_2)} |\mathrm{LOT}|^p\, \zeta^2\, dz \le c \sum_{k=0}^{m-1} \int_{-S}^{t}\int_{B(r_2)\backslash B(r_1)} \frac{|D^k(u-P)|^p}{(r_2-r_1)^{p(m-k)}}\, \zeta^2\, dz$$
$$\le \int_{-S}^{t}\int_{B(r_2)\backslash B(r_1)} \bigg(\mu\, |D^m u|^p + c(n,m,\tfrac{1}{\mu})\, \frac{|u-P|^p}{(r_2-r_1)^{mp}}\bigg)\zeta^2\, dz.$$

Similarly we obtain

$$\int_{-S}^{t}\int_{B(r_2)} |\delta\varphi|^p\, dz \le c \sum_{k=0}^{m-1}\sum_{j=0}^{k} \int_{-S}^{t}\int_{B(r_2)} |D^j(u-P)|^p\, |D^{k-j}\eta|^p\, \zeta^2\, dz$$
$$\le c \sum_{k=0}^{m-1}\sum_{j=0}^{k} \int_{-S}^{t}\int_{\operatorname{spt} D^{k-j}\eta} \frac{|D^j(u-P)|^p}{(r_2-r_1)^{p(k-j)}}\, \zeta^2\, dz$$
$$\le \int_{-S}^{t}\int_{B(r_2)\backslash B(r_1)} \bigg(\mu\, |D^m u|^p + c(n,m,\tfrac{1}{\mu})\, \frac{|u-P|^p}{(r_2-r_1)^{mp}}\bigg)\zeta^2\, dz,$$

where we have taken into account that $r_2 - r_1 \le 1$. Inserting the two previous estimates above, choosing $\mu \ll 1$ with respect to p, L and ε and noting that $\eta \equiv 1$ on $B(r_1)$ we infer for a.e. $t \in (-S, S)$ that

$$\tfrac12 \int_{B(r_1)} |u(\cdot,t) - P|^2\, \zeta^2(t)\, dx + \nu \int_{-S}^{t}\int_{B(r_1)} |D^m u|^p\, \zeta^2\, dz$$
$$\le 2\varepsilon \int_{-S}^{t}\int_{B(r_2)} |D^m u|^p\, \zeta^2\, dz + c \int_{Q(R,S)} \bigg(\frac{|u-P|^p}{(r_2-r_1)^{mp}} + \frac{|u-P|^2}{S-s} + b^{\,p}\bigg) dz,$$

where $c = c(n, m, p, L, 1/\varepsilon)$. We take in the first term on the left-hand side the supremum over $t \in (-s, s)$ (note that $\zeta \equiv 1$ on $(-s, s)$) and take $t = S$ in the second term. Then we multiply with $\frac{2}{\nu}$ and take $\varepsilon = \frac{\nu}{8}$ to obtain

$$\sup_{t\in(-s,s)} \int_{B(r_1)} |u(\cdot,t) - P|^2\, dx + \int_{Q(r_1,S)} |D^m u|^p \zeta^2\, dz$$
$$\le \tfrac12 \int_{Q(r_2,S)} |D^m u|^p \zeta^2\, dz + c \int_{Q(R,S)} \bigg(\frac{|u-P|^p}{(r_2-r_1)^{mp}} + \frac{|u-P|^2}{S-s} + b^{\,p}\bigg)\, dz,$$

where $c = c(n, m, p, L/\nu)$. Applying Lemma 2 we get rid of the term involving $|D^m u|$ on the right-hand side and recalling that $\zeta \equiv 1$ on $(-s, s)$ we conclude the desired Caccioppoli inequality. □

5. Poincaré type estimates

Since a weak solution u is a priori only an L^p-function with respect to the time-variable t, the Poincaré inequality cannot be applied. Nevertheless, we can prove a sort of Poincaré inequality, valid for weak solutions (see Lemma 8). It is shown by considering the space and time direction separately. In x-direction we can apply the general Poincaré inequality. In t-direction we will gain the needed regularity from the parabolic system. Namely, in the next Lemma we will show a suitable bound for the difference in time of the weighted means $(D^k u)_{\widetilde{\eta}}(t)$ of $D^k u(x,t)$—defined below—proving that they are absolutely continuous.

We say that $\widetilde{\eta} \in C_0^\infty(B_{x_0}(\varrho))$ is a nonnegative weight-function on $B_{x_0}(\varrho) \subset \mathbf{R}^n$, if

$$\widetilde{\eta} \geq 0, \quad \int_{B_{x_0}(\varrho)} \widetilde{\eta}\, dx = 1 \quad \text{and} \quad \|D^\ell \widetilde{\eta}\|_\infty \leq c_{\widetilde{\eta}}\, \varrho^{-(n+\ell)} \text{ for } \ 0 \leq \ell \leq 2m. \tag{6}$$

Note that the smallest possible value of $c_{\widetilde{\eta}}$ depends on n and m. Let $Q_{z_0}(\varrho, s) \subset \mathbf{R}^{n+1}$ be a parabolic cylinder and $v \in L^1(Q_{z_0}(\varrho, s); \mathbf{R}^k)$, $k \in \mathbf{N}$. Then, we define the weighted mean of $v(\cdot, t)$ on $B_{x_0}(\varrho)$ for a.e. $t \in (t_0 - s, t_0 + s)$ by

$$(v)_{\widetilde{\eta}}(t) \equiv \int_{B_{x_0}(\varrho)} v(\cdot, t)\, \widetilde{\eta}\, dx. \tag{7}$$

Lemma 7. *Suppose that $u \in L^p(-T, 0; W^{m,p}(\Omega; \mathbf{R}^N))$ is a weak solution of system (1) with (3) and (4) and $Q_{z_0}(\varrho, s) \Subset \Omega_T$ is a parabolic cylinder with $0 < \varrho \leq 1$, $s > 0$. Let $\widetilde{\eta} \in C_0^\infty(B_{x_0}(\varrho))$ be a nonnegative weight-function satisfying (6). Then for the weighted means of $D^k u$, $0 \leq k \leq m-1$ defined in (7) there holds for a.e. $t_1, t_2 \in (t_0 - s, t_0 + s)$ that*

$$|(D^k u)_{\widetilde{\eta}}(t_2) - (D^k u)_{\widetilde{\eta}}(t_1)| \leq c(N, L, c_{\widetilde{\eta}})\, \frac{s}{\varrho^{m+k}} \fint_{Q_{z_0}(\varrho, s)} \big(|D^m u| + b\big)^{p-1} dz.$$

Proof. Without loss of generality we assume that $z_0 = 0$. For $i \in \{1, \ldots N\}$ we choose $\varphi\colon \mathbf{R}^{n+1} \to \mathbf{R}^N$ with $\varphi_i = \widetilde{\eta}$, $\varphi_j = 0$ for $j \neq i$ as test-function in the Steklov-formulation (5) of the parabolic system and for a.e. $t_1, t_2 \in (t_0 - s, t_0 + s)$ we get

$$\begin{aligned}
\big([u_i]_h\big)_{\widetilde{\eta}}(t_2) - \big([u_i]_h\big)_{\widetilde{\eta}}(t_1) &= \int_{t_1}^{t_2} \partial_t \big([u_i]_h\big)_{\widetilde{\eta}}\, dt \\
&= -\int_{t_1}^{t_2} \int_{B(\varrho)} \Big(\big[A_i(\cdot, D^m u)\big]_h \cdot D^m \widetilde{\eta} + \big[B_i(\cdot, D^m u)\big]_h \cdot \delta \widetilde{\eta}\Big)\, dx\, dt.
\end{aligned}$$

Using the growth conditions (3) and (4) for A and B, and the fact that $\|D^j\widetilde{\eta}\|_\infty \le c\varrho^{-(n+j)} \le c\varrho^{-(n+m)}$ for $0 \le j \le m-1$ we find after passing to the limit $h \searrow 0$

$$\begin{aligned}
&|(u_i)_{\widetilde{\eta}}(t_2) - (u_i)_{\widetilde{\eta}}(t_1)| \\
&\le \int_{t_1}^{t_2}\int_{B(\varrho)} \Big(\big(L|D^m u|^{p-1} + |b_1|\big)|D^m\widetilde{\eta}| + \big(L|D^m u|^{p-1} + |b_2|\big)|\delta\widetilde{\eta}|\Big)\, dz \\
&\le c(L, c_{\widetilde{\eta}})\ \varrho^{-(n+m)} \int_{Q(\varrho,s)} \big(|D^m u| + b\big)^{p-1}\, dz.
\end{aligned}$$

Summing over $i = 1, \dots N$ we infer the assertion for the case $k = 0$. For the general case we have for a multiindex α of order k with integration by parts that

$$(D^\alpha u)_{\tilde{\eta}}(t) = \int_{B(\varrho)} D^\alpha u(\cdot,t)\tilde{\eta}\, dx = (-1)^k \int_{B(\varrho)} u(\cdot,t) D^\alpha\tilde{\eta}\, dx = (-1)^k (u)_{D^\alpha\tilde{\eta}}(t).$$

Therefore the assertion follows from the case $k = 0$ by exchanging $\widetilde{\eta}$ with $D^\alpha\widetilde{\eta}$ and summing over $|\alpha| = k$. □

Lemma 8. *Suppose that $u \in L^p(-T, 0; W^{m,p}(\Omega; \mathbf{R}^N))$ is a weak solution of (1) with (3) and (4) and $Q_{z_0}(\varrho, s) \Subset \Omega_T$ with $0 < \varrho \le 1$ and $s > 0$. Then for all $0 \le k \le m-1$ and $1 \le \vartheta \le p$ there holds*

$$\begin{aligned}
&\fint_{Q_{z_0}(\varrho,s)} |D^k(u - P_Q)|^\vartheta\, dz \\
&\le c\ \varrho^{(m-k)\vartheta}\left[\fint_{Q_{z_0}(\varrho,s)} |D^m u|^\vartheta\, dz + \left(\frac{s}{\varrho^{2m}}\fint_{Q_{z_0}(\varrho,s)} \big(|D^m u| + b\big)^{p-1}\, dz\right)^\vartheta\right],
\end{aligned}$$

where $c = c(n, N, m, L, \vartheta)$ and $P_Q\colon \mathbf{R}^n \to \mathbf{R}^N$ denotes the mean value polynomial of u (depending only on x) of degree $\le m-1$, defined by $(\delta P_Q)_{x_0;\varrho} = (\delta u)_{z_0;\varrho,s}$.

Proof. Without loss of generality we assume that $z_0 = 0$. Let $\widetilde{\eta} \in C_0^\infty(B(\varrho))$ be a nonnegative weight-function satisfying (6). In order to apply Poincaré's inequality "slicewise" with respect to x, we use the weighted means of $D^j(u - P_Q)$, defined in (7) and consider for $k \le j \le m-1$ and a.e. $t \in (-s, s)$ the following decomposition

$$\begin{aligned}
&\fint_{B(\varrho)} |D^j(u(\cdot,t) - P_Q)|^\vartheta\, dx \\
&\le 3^\vartheta\left[\fint_{B(\varrho)} \big|D^j(u(\cdot,t) - P_Q) - \big(D^j(u(\cdot,t) - P_Q)\big)_{\widetilde{\eta}}\big|^\vartheta dx \right. \\
&\qquad \left. + \left|\fint_{-s}^{s} \big((D^j u)_{\widetilde{\eta}}(t) - (D^j u)_{\widetilde{\eta}}(\tau)\big)\, d\tau\right|^\vartheta + \left|\fint_{-s}^{s} (D^j u)_{\widetilde{\eta}}(\tau)\, d\tau - (D^j P_Q)_{\widetilde{\eta}}\right|^\vartheta\right] \\
&= 3^\vartheta\big(I(t) + II(t) + III\big),
\end{aligned} \tag{8}$$

with the obvious meaning of $I(t)$, $II(t)$ and III.

Estimate for $I(t)$: Applying Poincaré's inequality "slicewise" to $D^j(u-P_Q)(\cdot,t)$ we find for a.e. $t\in(-s,s)$ that

$$I(t)\le c(n,\vartheta)\ \varrho^\vartheta \fint_{B(\varrho)} |D^{j+1}(u(\cdot,t)-P_Q)|^\vartheta\,dx.$$

Estimate for III: Here, we exploit the fact that $\fint_Q(D^ju-D^jP_Q)\,dz=0$ and apply Poincaré's inequality "slicewise" to $D^j(u-P_Q)$ completely similar to the estimate for $I(t)$ and infer that

$$\begin{aligned}III&\le \fint_{-s}^{s}\fint_{B(\varrho)} \big|D^j(u-P_Q)-\big(D^j(u-P_Q)\big)_{\widetilde\eta}\big|^\vartheta\,dx\,d\tau\\ &\le c(n,\vartheta)\ \varrho^\vartheta \fint_{-s}^{s}\fint_{B(\varrho)} \big|D^{j+1}(u-P_Q)\big|^\vartheta\,dx\,d\tau.\end{aligned}$$

Estimate for $II(t)$: The estimate for differences in time of weighted means from Lemma 7 yields for a.e. $t\in(-s,s)$ that

$$II(t)\le \fint_{-s}^{s} |(D^ju)_{\widetilde\eta}(t)-(D^ju)_{\widetilde\eta}(\tau)|^\vartheta\,d\tau\le c\left(\frac{s}{\varrho^{m+j}}\fint_{Q(\varrho,s)}(|D^mu|+b\,)^{p-1}\,dz\right)^\vartheta,$$

where $c=c(n,N,m,L)$.

Combining the previous estimates for $I(t)$, $II(t)$ and III with (8) and integrating with respect to t over $(-s,s)$ we infer for $k\le j\le m-1$ that

$$\begin{aligned}&\fint_{-s}^{s}\fint_{B(\varrho)} |D^j(u-P_Q)|^\vartheta\,dx\,dt\\ &\le c\ \varrho^\vartheta \fint_{-s}^{s}\fint_{B(\varrho)} |D^{j+1}(u-P_Q)|^\vartheta dx\,dt + c\left(\frac{s}{\varrho^{m+j}}\fint_{Q(\varrho,s)}(|D^mu|+b\,)^{p-1}dz\right)^\vartheta,\end{aligned}$$

where $c=c(n,N,m,L,\vartheta)$. Iterating this estimate for $j=k,\dots,m-1$ we find that

$$\begin{aligned}&\fint_{Q(\varrho,s)} |D^k(u-P_Q)|^\vartheta\,dz\\ &\le c\ \varrho^\vartheta \fint_{Q(\varrho,s)} |D^{k+1}(u-P_Q)|^\vartheta\,dz + c\left(\frac{s}{\varrho^{m+k}}\fint_{Q(\varrho,s)}\big(|D^mu|+b\,\big)^{p-1}\,dz\right)^\vartheta\\ &\le c\ \varrho^{2\vartheta} \fint_{Q(\varrho,s)} |D^{k+2}(u-P_Q)|^\vartheta\,dz + c\left(\frac{s}{\varrho^{m+k}}\fint_{Q(\varrho,s)}\big(|D^mu|+b\,\big)^{p-1}dz\right)^\vartheta\\ &\ \vdots\\ &\le c\ \varrho^{\vartheta(m-k)} \fint_{Q(\varrho,s)} |D^mu|^\vartheta\,dz + c\left(\frac{s}{\varrho^{m+k}}\fint_{Q(\varrho,s)}\big(|D^mu|+b\,\big)^{p-1}dz\right)^\vartheta,\end{aligned}$$

where $c=c(n,N,m,L,\vartheta)$. This proves the asserted Poincaré type inequality. □

In the previous Poincaré type inequality we have the "wrong exponent" of $|D^mu|$ on the right-hand side, namely $(\fint|D^mu|^{p-1}\,dz)^\vartheta$. Roughly speaking, in the following lemma, we "compensate" this wrong exponent, introducing a special scaling of the parabolic cylinders, which depends on the solution itself.

Corollary 9. *Let $u \in L^p(-T, 0; W^{m,p}(\Omega; \mathbf{R}^N))$ be a weak solution of (1) with (3) and (4) and $Q_{z_0}(\varrho, s) \Subset \Omega_T$ with $0 < \varrho \le 1$, $\lambda > 0$ and $s = \lambda^{2-p}\varrho^{2m}$. Suppose that there is a constant $\kappa \ge 1$, such that*

$$\kappa^{-1}\,\lambda^p \le \fint_{Q_{z_0}(\varrho,s)} (|D^m u|^p + b^{\,p})\, dz \le \kappa\,\lambda^p. \tag{9}$$

Then for all $0 \le k \le m-1$ and $1 \le \vartheta \le p$ there holds

$$\fint_{Q_{z_0}(\varrho,s)} |D^k(u - P_Q)|^\vartheta\, dz \le c\ \varrho^{\vartheta(m-k)} \left(\fint_{Q_{z_0}(\varrho,s)} (|D^m u| + b)^q\, dz \right)^{\frac{\vartheta}{q}},$$

where $q \equiv \max\{\vartheta, p-1\}$, $c = c(n, N, m, L, \vartheta, \kappa)$ and $P_Q\colon \mathbf{R}^n \to \mathbf{R}^N$ denotes the mean value polynomial of u of degree $\le m-1$, defined by $(\delta P_Q)_{x_0;\varrho} = (\delta u)_{z_0;\varrho,s}$.

Proof. We can assume $z_0 = 0$. Applying the Poincaré type inequality from Lemma 8 and noting that $s/\varrho^{2m} = \lambda^{2-p}$, we obtain

$$\begin{aligned}
&\fint_{Q(\varrho,s)} |D^k(u - P_Q)|^\vartheta\, dz \\
&\le c\ \varrho^{\vartheta(m-k)} \left[\fint_{Q(\varrho,s)} |D^m u|^\vartheta\, dz + \left(\lambda^{2-p} \fint_{Q(\varrho,s)} (|D^m u| + b\,)^{p-1}\, dz \right)^{\vartheta} \right],
\end{aligned}$$

where $c = c(n, N, m, L, \vartheta)$. To estimate the second term on the right-hand side we use Hölder's inequality and the hypothesis (9) to find that

$$\begin{aligned}
&\lambda^{2-p} \fint_{Q(\varrho,s)} \big(|D^m u| + b\,\big)^{p-1}\, dz = \lambda^{2-p}\, \Big(\dots\Big)^{1-\frac{1}{p-1}} \Big(\dots\Big)^{\frac{1}{p-1}} \\
&\le \lambda^{2-p} \left(\fint_{Q(\varrho,s)} \big(|D^m u| + b\,\big)^{p}\, dz \right)^{\frac{p-2}{p}} \left(\fint_{Q(\varrho,s)} \big(|D^m u| + b\,\big)^{q}\, dz \right)^{\frac{1}{q}} \\
&\le c(\kappa)\ \lambda^{2-p}\, \lambda^{p\frac{p-2}{p}} \left(\fint_{Q(\varrho,s)} \big(|D^m u| + b\,\big)^{q}\, dz \right)^{\frac{p}{q}} \\
&= c(\kappa)\, \left(\fint_{Q(\varrho,s)} \big(|D^m u| + b\,\big)^{q}\, dz \right)^{\frac{1}{q}}.
\end{aligned}$$

Inserting this above to bound the second term on the right-hand side and using once again Hölder's inequality for the first term, we conclude the asserted estimate. □

Corollary 10. *Let $u \in L^p(-T, 0; W^{m,p}(\Omega; \mathbf{R}^N))$ be a weak solution of (1) with (3) and (4) and $Q_{z_0}(\varrho, s) \Subset \Omega_T$ with $0 < \varrho \le 1$, $\lambda > 0$ and $s = \lambda^{2-p}\varrho^{2m}$. Suppose* *that there is a constant $\kappa \ge 1$, such that*

$$\fint_{Q_{z_0}(\varrho,s)} \left(|D^m u|^p + b^p\right) dz \le \kappa\, \lambda^p. \tag{10}$$

Then for all $0 \le k \le m-1$ and $1 \le \vartheta \le p$ there holds

$$\fint_{Q_{z_0}(\varrho,s)} |D^k(u - P_Q)|^\vartheta\, dz \le c(n, N, m, L, \vartheta, \kappa)\ \varrho^{\vartheta(m-k)} \lambda^\vartheta.$$

where $P_Q\colon \mathbf{R}^n \to \mathbf{R}^N$ denotes the mean value polynomial of u of degree $\le m-1$, defined by $(\delta P_Q)_{x_0;\varrho} = (\delta u)_{z_0;\varrho,s}$.

Proof. Once again we assume that $z_0 = 0$. Similarly to the proof of the previous Corollary, we infer the assertion from Lemma 8 (note that $s/\varrho^{2m} = \lambda^{2-p}$), Hölder's inequality and the hypothesis (10):

$$\begin{aligned}
&\fint_{Q(\varrho,s)} |D^k(u - P_Q)|^\vartheta dz \\
&\le c\ \varrho^{\vartheta(m-k)} \left[\fint_{Q(\varrho,s)} |D^m u|^\vartheta dz + \left(\lambda^{2-p} \fint_{Q(\varrho,s)} \big(|D^m u| + b\big)^{p-1} dz\right)^\vartheta\right] \\
&\le c\ \varrho^{\vartheta(m-k)} \big(\lambda^\vartheta + (\lambda^{2-p}\,\lambda^{p-1})^\vartheta\big) = c\ \varrho^{\vartheta(m-k)} \lambda^\vartheta.
\end{aligned}$$

□

Corollary 11. *Let $u \in L^p(-T, 0; W^{m,p}(\Omega_T; \mathbf{R}^N))$ be a weak solution of (1) with (3) and (4) and $Q_{z_0}(R, \lambda^{2-p} R^{2m}) \Subset \Omega_T$ with $0 < R \le 1$, $\lambda > 0$. Suppose that there is a constant $\kappa \ge 1$, such that*

$$\fint_{Q_{z_0}(R,\lambda^{2-p}R^{2m})} \left(|D^m u|^p + b^p\right) dz \le \kappa\, \lambda^p.$$

Moreover, let $R/2 \le r < R$ and $P_r, P_R\colon \mathbf{R}^n \to \mathbf{R}^N$ be the mean value polynomials of u of degree $\le m-1$, defined by $(\delta P_r)_{x_0;r} = (\delta u)_{z_0;r,\lambda^{2-p}r^{2m}}$, respectively $(\delta P_R)_{x_0;R} = (\delta u)_{z_0;R,\lambda^{2-p}R^{2m}}$. Then

$$|P_r(x) - P_R(x)| \le c(n, N, m, L, \kappa)\ R^m \lambda \quad \text{for all } x \in B_{x_0}(R).$$

Proof. Applying in turn Lemma 5, Corollary 10 and recalling that $R/2 \le r \le R$, we infer the asserted estimate:

$$\begin{aligned}
|(P_r - P_R)(x)| &\le c \sum_{j=0}^{m-1} r^j \left|\fint_{B_{x_0}(r)} D^j(P_r - P_R)\, dy\right| \\
&= c \sum_{j=0}^{m-1} r^j \left|\fint_{Q_{z_0}(r,\lambda^{2-p}r^{2m})} D^j(u - P_R)\, dz\right| \\
&\le c\ \sum_{j=k}^{m-1} R^j \fint_{Q_{z_0}(R,\lambda^{2-p}R^{2m})} |D^j(u - P_R)|\, dz \\
&\le c\ R^m \lambda,
\end{aligned}$$

where $c = c(n, N, m, L, \kappa)$. □

In the case $p < 2$ we cannot take $\vartheta = 2$ in Corollary 10. Nevertheless, we will have to estimate the L^2-norm of u, since it appears on the right-hand side of the Caccioppoli inequality from Lemma 6. Therefore we need the following Lemma.

Lemma 12. *Suppose that* $\max\{1, \frac{2n}{n+2m}\} < p < 2$ *and* $u \in L^p(-T, 0; W^{m,p}(\Omega_T; \mathbf{R}^N)) \cap L^2(\Omega; \mathbf{R}^N)$ *is a weak solution of (1) with (2) – (4). Let* $Q_{z_0}(2\varrho, \lambda^{2-p}(2\varrho)^{2m}) \Subset \Omega_T$ *with* $0 < \varrho \leq 1$, $\lambda > 0$. *Supposed that there is a constant* $\kappa \geq 1$, *such that*

$$\fint_{Q_{z_0}(2\varrho,\lambda^{2-p}(2\varrho)^{2m})} (|D^m u|^p + b^p)\, dz \leq \kappa\, \lambda^p, \tag{11}$$

then there holds

$$\fint_{Q_{z_0}(\varrho,\lambda^{2-p}\varrho^{2m})} |u - P_\varrho|^2\, dz \leq c(n, N, m, p, L/\nu, \kappa)\ \varrho^{2m}\lambda^2,$$

where $P_\varrho\colon \mathbf{R}^n \to \mathbf{R}^N$ *denotes the mean value polynomial of* u *of degree* $\leq m-1$, *defined by* $(\delta P_\varrho)_{x_0;\varrho} = (\delta u)_{z_0;\varrho,\lambda^{2-p}\varrho^{2m}}$.

Proof. Without loss of generality we assume that $z_0 = 0$. Let $q = \max\{1, \frac{2n}{n+2m}\}$. We choose $\varrho \leq r < R \leq 2\varrho$ and denote by $P_r, P_R\colon \mathbf{R}^n \to \mathbf{R}^N$ the mean value polynomials of u of degree $\leq m-1$, defined by $(\delta P_r)_{0;r} = (\delta u)_{0;r,\lambda^{2-p}r^{2m}}$, respectively $(\delta P_R)_{0;R} = (\delta u)_{0;R,\lambda^{2-p}R^{2m}}$. Applying Gagliardo–Nirenberg's inequality, i.e. Theorem 4 with $(p, \vartheta, \theta, r, k)$ replaced by $(2, q, \frac{q}{2}, 2, 0)$ "slicewise" to $(u - P_r)(\cdot, t)$ we obtain:

$$\begin{aligned}
&\fint_{Q(r,\lambda^{2-p}r^{2m})} |u - P_r|^2\, dz \\
&\leq c\, r^{mq} \fint_{-\lambda^{2-p}r^{2m}}^{\lambda^{2-p}r^{2m}} \fint_{B(r)} \sum_{k=0}^{m} \frac{|D^k(u - P_r)|^q}{r^{q(m-k)}}\, dx \left(\fint_{B(r)} |u - P_r|^2 dx \right)^{1-\frac{q}{2}} dt \\
&\leq c(n, m)\ r^{mq} \fint_{Q(r,\lambda^{2-p}r^{2m})} \sum_{k=0}^{m} \frac{|D^k(u - P_r)|^q}{r^{q(m-k)}}\, dz \cdot J^{1-\frac{q}{2}},
\end{aligned}$$

where

$$J \equiv \sup_{t \in (-\lambda^{2-p}r^{2m}, \lambda^{2-p}r^{2m})} \fint_{B(r)} |u(\cdot, t) - P_r|^2\, dx.$$

We note that the assumption (11) is also valid for r instead of 2ϱ, with a possibly larger constant, since $|Q(2\varrho, \lambda^{2-p}(2\varrho)^{2m})|/|Q(r, \lambda^{2-p}r^{2m})| \leq 2^{n+2m}$. Therefore, we can apply Corollary 10 on the cylinder $Q(r, \lambda^{2-p}r^{2m})$ to estimate the first term in the above inequality and obtain

$$\fint_{Q(r,\lambda^{2-p}r^{2m})} |u - P_r|^2\, dz \leq c(n, m, N, L, \kappa)\ r^{mq}\lambda^q \cdot J^{1-\frac{q}{2}}. \tag{12}$$

Estimate of J: Applying Corollary 11 (enlarging the involved cylinder just as we did above, we see that the assumption of the Corollary is fulfilled due to hypothesis

(11)) we can bound the difference of the mean value polynomials by

$$|P_r(x) - P_R(x)| \le c(n, N, m, L, \kappa)\ R^m \lambda \quad \text{for all } x \in B(R).$$

From the Caccioppoli inequality in Lemma 6 we get for a.e. $t \in (-\lambda^{2-p} r^{2m}, \lambda^{2-p} r^{2m})$

$$\fint_{B(r)} \frac{|u(\cdot,t) - P_R|^2}{\lambda^{2-p} r^{2m}}\, dx \le c_{Cac} \fint_{Q(R,\lambda^{2-p}R^{2m})} \left(\frac{|u - P_R|^2}{\lambda^{2-p}(R^{2m} - r^{2m})} + \frac{|u - P_R|^p}{(R-r)^{mp}} + b^{\,p} \right) dz,$$

where $c_{Cac} = c_{Cac}(n, m, p, L/\nu)$. From Young's inequality (note that $p < 2$) and hypothesis (11) we get

$$\fint_{Q(R,\lambda^{2-p}R^{2m})} \left(\frac{|u - P_R|^p}{(R-r)^{mp}} + b^{\,p} \right) dz \le \fint_{Q(R,\lambda^{2-p}R^{2m})} \frac{|u - P_R|^2}{\lambda^{2-p}(R-r)^{2m}}\, dz + c\ \lambda^p,$$

with $c = c(n, m, \kappa)$. Inserting this above and noting that $p < 2$ and $r < R$ yields

$$\fint_{B(r)} |u(\cdot,t) - P_R|^2\, dx \le c\ \left(\frac{R}{R-r}\right)^{2m} \left(\fint_{Q(R,\lambda^{2-p}R^{2m})} |u - P_R|^2\, dz + R^{2m}\lambda^2 \right).$$

Combining this with the estimate for the difference of the mean value polynomials above, we obtain the following estimate for J:

$$J \le c(n, m, p, L/\nu, \kappa)\ \left(\frac{R}{R-r}\right)^{2m} \left(\fint_{Q(R,\lambda^{2-p}R^{2m})} |u - P_R|^2\, dz + R^{2m}\lambda^2 \right).$$

Using this estimate for J in (12) and applying Young's inequality, we obtain (note that $R \le 2\varrho$ and $\frac{\varrho}{R-r} \ge 1$)

$$\fint_{Q(r,\lambda^{2-p}r^{2m})} |u - P_r|^2\, dz \le \tfrac{1}{2} \fint_{Q(R,\lambda^{2-p}R^{2m})} |u - P_R|^2\, dz + c\ \left(\frac{\varrho}{R-r}\right)^{2m(\frac{2}{q}-1)} \varrho^{2m}\lambda^2,$$

where $c = c(n, N, m, p, L/\nu, \kappa)$. Applying Lemma 2 we obtain the desired estimate. □

6. Reverse-Hölder type inequality

In order to prove higher integrability one typically first shows a reverse-Hölder inequality. In the case of parabolic systems which are degenerate ($p > 2$) or singular ($p < 2$), a general reverse-Hölder inequality is not expected to hold. The reason for this is, that the Poincaré type inequality from Lemma 8 has an integral involving $|D^m u|^{p-1}$ on the right-hand side. Consequently we would end up with the "wrong exponent" of $|D^m u|$, namely with $(\fint |D^m u|^{p-1}\, dz)^p$ on the right-hand side. Therefore, we exploit the scaling of the parabolic cylinders, introduced in the previous chapter, which depends (via hypothesis (13) and (14)) on the solution itself, to "compensate" the degeneracy.

Lemma 13. *Let $p > \max\{1, \frac{2n}{n+2m}\}$ and $u \in L^p(-T, 0; W^{m,p}(\Omega; \mathbf{R}^N)) \cap L^2(\Omega_T; \mathbf{R}^N)$ be a weak solution of (1) with (2) – (4) and let $Q_{z_0}(10\varrho, \lambda^{2-p}(10\varrho)^{2m}) \Subset \Omega_T$*

with $0 < \varrho \le 1$, $\lambda > 0$, $s = \lambda^{2-p}\varrho^{2m}$. *Suppose that there is a constant* $\kappa \ge 1$, *such that*

$$\lambda^p \le \kappa \fint_{Q_{z_0}(\varrho,s)} (|D^m u|^p + b^p)\, dz \tag{13}$$

and

$$\fint_{Q_{z_0}(10\varrho,10^{2m}s)} (|D^m u|^p + b^p)\, dz \le \kappa\, \lambda^p. \tag{14}$$

Then there exists a constant $c = c(n, N, m, p, L/\nu, \kappa)$, *such that*

$$\fint_{Q_{z_0}(10\varrho,10^{2m}s)} |D^m u|^p\, dz \le c \left(\fint_{Q_{z_0}(2\varrho,2^{2m}s)} |D^m u|^q\, dz \right)^{\frac{p}{q}} + c \fint_{Q_{z_0}(2\varrho,2^{2m}s)} b^p\, dz,$$

where $q = \max\{1, \frac{2n}{n+2m}\}$ *if* $p < 2$ *and* $q = \max\{p-1, \frac{np}{n+2m}\}$ *if* $p \ge 2$.

Proof. Without loss of generality we assume that $z_0 = 0$. For convenience of the reader we set $B = B(\varrho)$, $Q \equiv Q(\varrho, s)$ and $\alpha B = B(\alpha\varrho)$, $\alpha Q \equiv Q(\alpha\varrho, \alpha^{2m}s)$ for $\alpha > 0$. By $P_{\alpha Q}\colon \mathbf{R}^n \to \mathbf{R}^N$ we denote the mean value polynomials of u of degree $\le m-1$, defined by $(\delta P_{\alpha Q})_{\alpha B} = (\delta u)_{\alpha Q}$.

From the Caccioppoli inequality, i.e. Lemma 6 we obtain

$$\begin{aligned} \fint_Q |D^m u|^p\, dz &\le c_{\mathrm{Cac}} \fint_{2Q} \left(\frac{|u - P_{2Q}|^2}{s} + \frac{|u - P_{2Q}|^p}{\varrho^{mp}} + b^p \right) dz \\ &= c_{\mathrm{Cac}}(n, m, p, L/\nu) \left(I_2 + I_p + \fint_{2Q} b^p\, dz \right), \end{aligned} \tag{15}$$

with the obvious meaning of I_2 and I_p. We now distinguish the cases $p \ge 2$ and $p < 2$.

In the case $p \ge 2$ we first estimate the term I_2 by I_p. Therefore we note that $s = \lambda^{2-p}\varrho^{2m}$ and get from Young's inequality (with exponents $\frac{p}{2}$, $\frac{p}{p-2}$) for $\varepsilon > 0$ that

$$I_2 = \lambda^{p-2} \fint_{2Q} \frac{|u - P_{2Q}|^2}{\varrho^{2m}}\, dz \le \varepsilon\lambda^p + c(p, 1/\varepsilon)\; I_p.$$

We set $\vartheta \equiv \max\{1, \frac{np}{n+2m}\}$. In order to estimate I_p we apply Gagliardo–Nirenberg's inequality, i.e. Theorem 4 with (r, θ, k) replaced by $(2, \frac{\vartheta}{p}, 0)$ "slicewise" to $(u - P_{2Q})(\cdot, t)$:

$$\begin{aligned} I_p &\le c\; \varrho^{m(\vartheta-p)} \fint_{-2^{2m}s}^{2^{2m}s} \fint_{2B} \sum_{k=0}^{m} \frac{|D^k(u - P_{2Q})|^\vartheta}{\varrho^{\vartheta(m-k)}}\, dx \left(\fint_{2B} \left|u - P_{2Q}\right|^2 dx \right)^{\frac{p-\vartheta}{2}} dt \\ &\le c(n, m, p)\; \varrho^{m(\vartheta-p)} \fint_{2Q} \sum_{k=0}^{m} \frac{|D^k(u - P_{2Q})|^\vartheta}{\varrho^{\vartheta(m-k)}}\, dz \cdot J^{\frac{p-\vartheta}{2}}, \end{aligned}$$

where

$$J \equiv \sup_{t\in(-2^{2m}s,2^{2m}s)} \fint_{2B} |u(\cdot,t) - P_{2Q}|^2\,dx. \tag{16}$$

We note that due to (13) and (14) also hypothesis (9) of Corollary 9 is fulfilled on $2Q$ with $5^{n+2m}\kappa$ instead of κ, since $|10Q|/|2Q| = 5^{n+2m}$ and $|2Q|/|Q| = 2^{n+2m}$. The application yields that

$$I_p \le c(n,N,m,p,L,\kappa)\ \varrho^{m(\vartheta-p)}\left(\fint_{2Q}(|D^m u| + b)^q\,dz\right)^{\frac{\vartheta}{q}} \cdot J^{\frac{p-\vartheta}{2}}. \tag{17}$$

Estimate of J: Applying Corollary 11 and assumption (14) we get for the difference of the polynomials

$$|P_{2Q}(x) - P_{4Q}(x)| \le c(n,m,N,L,\kappa)\ \varrho^m\lambda \quad \text{for all } x \in 4B. \tag{18}$$

From the Caccioppoli inequality in Lemma 6 we infer for a.e. $t \in (-2^{2m}s, 2^{2m}s)$

$$\begin{aligned}\fint_{2B} |u(\cdot,t) - P_{4Q}|^2\,dx &\le c_{Cac}\fint_{4Q}\left(|u - P_{4Q}|^2 + \frac{s}{\varrho^{mp}}|u - P_{4Q}|^p + s\,b^p\right)dz\\ &= c(n,m,p,L/\nu)\left(J_2 + J_p + s\fint_{4Q} b^p\,dz\right),\end{aligned}$$

with the obvious meaning of J_2 and J_p. For J_2 we use Corollary 10, which is applicable due to the hypothesis (14) and since $10^{2m}s = \lambda^{2-p}(10\varrho)^{2m}$. Therefore, we have

$$J_2 \le c(n,N,m,L,\kappa)\ \varrho^{2m}\lambda^2.$$

Similarly, we get for J_p

$$J_p \le c\ \frac{s}{\varrho^{mp}}\ \varrho^{mp}\lambda^p = c\ s\lambda^p = c\ \varrho^{2m}\lambda^{2-p}\lambda^p = c\ \varrho^{2m}\lambda^2.$$

Combining the estimates for J_2, J_p and (18) we arrive at:

$$J \le c\left(\varrho^{2m}\lambda^2 + s\fint_{4Q} b^p\,dz\right) \le c(n,N,m,p,L/\nu,\kappa)\ \varrho^{2m}\lambda^2,$$

where we again have used that $s = \lambda^{2-p}\varrho^{2m}$ and the assumption (14). Inserting this in (17) and applying Young's inequality (with exponents $\frac{p}{\vartheta}$, $\frac{p}{p-\vartheta}$), we obtain for $\mu > 0$

$$I_p \le c\left(\fint_{2Q}(|D^m u| + b)^q\,dz\right)^{\frac{\vartheta}{q}}\lambda^{p-\vartheta} \le \mu\lambda^p + c\left(\fint_{2Q}(|D^m u| + b)^q\,dz\right)^{\frac{p}{q}},$$

where $c = c(n, N, m, p, L/\nu, \kappa, 1/\mu)$. Inserting this in (15) and using once again Hölder's inequality (note that $q = \max\{p-1, \vartheta\}$) we arrive at

$$\begin{aligned}\fint_Q |D^m u|^p\,dz &\le c_{Cac}\Big(\varepsilon\lambda^p + \big(c(p,1/\varepsilon)+1\big)I_p + \fint_{2Q} b^{\,p}\,dz\Big)\\ &\le \tfrac{1}{2\kappa}\,\lambda^p + c(n,N,m,p,L/\nu,\kappa)\bigg(\Big(\fint_{2Q}|D^m u|^q\,dz\Big)^{\frac{p}{q}} + \fint_{2Q} b^{\,p}\,dz\bigg),\end{aligned}$$

where we have chosen ε and μ small enough in the last line. Combining this with the assumption (13) we obtain

$$\lambda^p \le \kappa\fint_Q \big(|D^m u|^p + b^p\big)\,dz \le \tfrac12\,\lambda^p + c\Big(\fint_{2Q}|D^m u|^q\,dz\Big)^{\frac{p}{q}} + c\fint_{2Q} b^{\,p}\,dz.$$

Now we can absorb $\frac12\lambda^p$ on the left-hand side. Exploiting once again hypothesis (14) we conclude the desired reverse-Hölder inequality.

In the case $p<2$ we estimate I_p by I_2 in (15). This is achieved using $s = \lambda^{2-p}\varrho^{2m}$ and Young's inequality (with exponents $\frac{2}{p}$, $\frac{2}{2-p}$) and we obtain for $\varepsilon > 0$

$$I_p = \lambda^{\frac{p(2-p)}{2}}\fint_{2Q}\frac{|u-P_{2Q}|^p}{s^{p/2}}\,dz \le \varepsilon\lambda + c(p,1/\varepsilon)\ I_2.$$

To estimate I_2 proceed similar to the case $p \ge 2$, i.e. the derivation of (17). In turn we apply Gagliardo-Nirenberg's inequality, i.e. Theorem 4 with $(p, \vartheta, r,\ \theta, k)$ replaced by $(2, q, 2, \frac{q}{2}, 0)$ "slicewise" to $(u-P_{2Q})(\cdot,t)$ and and Corollary 9 to conclude that

$$\begin{aligned}I_2 &\le c\ s^{-1}\varrho^{mq}\fint_{-2^{2m}s}^{2^{2m}s}\fint_{2B}\sum_{k=0}^{m}\frac{|D^k(u-P_{2Q})|^q}{\varrho^{q(m-k)}}\,dx\,\Big(\fint_{2B}|u-P_{2Q}|^2dx\Big)^{1-\frac{q}{2}}\,dt\\ &\le c\ s^{-1}\varrho^{mq}\fint_{2Q}(|D^m u|+b)^q\,dz\cdot J^{1-\frac{q}{2}},\end{aligned} \tag{19}$$

with J defined in (16).

Estimate of J: For the difference of the polynomials $|P_{2Q}-P_{4Q}|$ we can once again use (18). From the Caccioppoli inequality in Lemma 6 and the assumption (14) we obtain for a.e. $t \in (-2^{2m}s, 2^{2m}s)$

$$\begin{aligned}\fint_{2B}|u(\cdot,t)-P_{4Q}|^2\,dx &\le c_{Cac}\fint_{4Q}\Big(|u-P_{4Q}|^2 + \frac{s}{\varrho^{mp}}|u-P_{4Q}|^p + s\,b^{\,p}\Big)\,dz\\ &\le c(n,m,p,L/\nu)\ \ (J_2 + J_p + \kappa s\,\lambda^p)\,.\end{aligned}$$

For J_2 we obtain from Lemma 12 (note that the assumption of the Lemma is fulfilled by (14), enlarging the domain of integration from $4Q$ to $10Q$)

$$J_2 = \fint_{4Q}|u-P_{4Q}|^2\,dz \le c\ \varrho^{2m}\lambda^2 = c\ s\lambda^p.$$

J_p is estimated exactly as in the case $p \geq 2$ with the help of Corollary 10 and we obtain using once again the assumption (14)

$$J_p \leq c\ \frac{s}{\varrho^{mp}} \varrho^{mp} \lambda^p = c(n, N, m, p, L/\nu, \kappa)\ s\lambda^p.$$

Thus, we infer that

$$J^{1-\frac{q}{2}} \leq c\ (s\lambda^p)^{1-\frac{q}{2}} = c\ s\lambda^p (s\lambda^p)^{-\frac{q}{2}} = c\ s\lambda^p (\varrho^{2m}\lambda^2)^{-\frac{q}{2}} = c\ \frac{s}{\varrho^{mq}} \lambda^{p-q}.$$

Inserting this in (19) and using Young's inequality, we obtain for $\mu > 0$ that

$$\begin{aligned} I_2 &\leq c\ \fint_{2Q} (|D^m u| + b)^q \, dz\ \lambda^{p-q} \\ &\leq \mu\lambda^p + c(n, N, m, p, L/\nu, \kappa, 1/\mu) \left(\fint_{2Q} |D^m u|^q \, dz \right)^{\frac{p}{q}}. \end{aligned}$$

Thus, we conclude from (15) that

$$\begin{aligned} \fint_Q |D^m u|^p \, dz &\leq c_{Cac} \Big(\varepsilon\lambda^p + \big(c(p, 1/\varepsilon) + 1\big) I_2 + \fint_{2Q} b^{\,p} \, dz \Big) \\ &\leq \tfrac{1}{2\kappa}\, \lambda^p + c(n, m, N, L, p, \kappa) \bigg(\Big(\fint_{2Q} |D^m u|^q \, dz \Big)^{\frac{p}{q}} + \fint_{2Q} b^{\,p} \, dz \bigg), \end{aligned}$$

where we have chosen ε and μ small enough in the last line. Now, we can proceed completely similar to the case $p \geq 2$ to conclude the desired reverse-Hölder inequality. □

7. A version of Gehring's Theorem

We will conclude the higher integrability of $|D^m u|$ from a version of Gehring's Theorem (see Lemma 15). It is a consequence of the one dimensional Gehring Theorem (see [11], Chapter V, Lemma 1.2):

Lemma 14. *Let $\varphi, \omega : [0, \infty] \to [0, \infty)$, φ non-decreasing, $\lim_{\lambda\to\infty} \varphi(\lambda) = 0$ and*

$$-\int_\lambda^\infty \mu^{p-q} \, d\varphi(\mu) \leq A\big(\lambda^{p-q}\varphi(\lambda) + \omega(\lambda)\big)$$

for all $\lambda \geq \lambda_1$, where $0 < q < p < \infty$, $A > 0$ and $\lambda_1 > 0$. Then there exists $\varepsilon = \varepsilon(A, p - q) > 0$, such that

$$-\int_{\lambda_1}^\infty \lambda^{p+\varepsilon-q} \, d\varphi(\lambda) \leq -2\lambda_1^\varepsilon \int_{\lambda_1}^\infty \lambda^{p-q} \, d\varphi(\lambda) - 2A \int_{\lambda_1}^\infty \lambda^\varepsilon \, d\omega(\lambda).$$

Lemma 15. *Let $\lambda_1 \geq 1$, $\kappa \geq 1$, $1 \leq q < p < \sigma$ and $f \in L^p_{\mathrm{loc}}(Q_2)$, $k \in L^\sigma_{\mathrm{loc}}(Q_2)$ with $Q_2 \equiv Q(2, 2^{2m})$. Suppose that for each $\lambda \geq \lambda_1$ and a.e. $\tilde{z} \in Q_2$ with $f(\tilde{z}) > \lambda$*

there exists a parabolic cylinder $Q \equiv Q_{\tilde z}(\varrho, s)$ *around* $\tilde z$ *such that*

$$\kappa^{-1}\lambda^p \le \fint_{5Q} f^p\,dz \le \kappa\Big(\fint_Q f^q\,dz\Big)^{\frac{p}{q}} + \kappa \fint_Q k^p\,dz \le \kappa^2\lambda^p, \tag{20}$$

where $5Q \equiv Q_{\tilde z}(5\varrho, 5^{2m}s)$ *denotes the 5-times enlarged cylinder around* $\tilde z$*. Then there exists* $\varepsilon_0 = \varepsilon_0(\kappa, p-q) \in (0, \sigma - p)$ *such that* $f \in L^{p+\varepsilon_0}_{\mathrm{loc}}(Q_2)$ *and*

$$\int_{Q_2} f^{p+\varepsilon}\,dz \le c\,\lambda_1^{\varepsilon}\int_{Q_2} f^p\,dz + c\int_{Q_2} k^{p+\varepsilon}\,dz \quad \forall\, \varepsilon \in (0, \varepsilon_0],$$

where $c = c(\kappa, p-q)$*.*

Proof. Let $\lambda \ge \lambda_1$ and suppose that $\tilde z$ is a point in Q_2 with $f(\tilde z) > \lambda$. Then, from the assumptions we know that there exists a cylinder Q around $\tilde z$ such that (20) holds. We now will infer an estimate for the L^p-norm of f on the cylinder $5Q$. For $\eta > 0$ we decompose the domain of integration into the lower and upper level set $Q\backslash\Phi_{\eta\lambda}$ and $Q \cap \Phi_{\eta\lambda}$ of f, where

$$\Phi_{\eta\lambda} \equiv \{\tilde z \in Q_2 : f(\tilde z) > \eta\lambda\}.$$

Then we use the chain of inequalities from (20) to find that

$$\begin{aligned}
\Big(\fint_Q f^q\,dz\Big)^{\frac{p}{q}} &\le 2^{\frac{p}{q}-1}(\eta\lambda)^p + 2^{\frac{p}{q}-1}\Big(\tfrac{1}{|Q|}\int_{Q\cap\Phi_{\eta\lambda}} f^q\,dz\Big)^{\frac{p}{q}}\\
&\le 2^{\frac{p-q}{q}}\eta^p\lambda^p + 2^{\frac{p-q}{q}}\Big(\fint_Q f^q\,dz\Big)^{\frac{p}{q}-1}\Big(\tfrac{1}{|Q|}\int_{Q\cap\Phi_{\eta\lambda}} f^q\,dz\Big)\\
&\le c\,\eta^p\Big(\fint_Q f^q\,dz\Big)^{\frac{p}{q}} + c\,\eta^p\fint_Q k^p\,dz + c\,\lambda^{p-q}\tfrac{1}{|Q|}\int_{Q\cap\Phi_{\eta\lambda}} f^q\,dz,
\end{aligned}$$

where $c = c(p-q, \kappa)$. Similarly we see that

$$\begin{aligned}
\fint_Q k^p\,dz &\le (\eta\lambda)^p + \tfrac{1}{|Q|}\int_{Q\cap\Phi_{\eta\lambda}} k^p\,dz\\
&\le c\,\eta^p\Big(\fint_Q f^q\,dz\Big)^{\frac{p}{q}} + c\,\eta^p\fint_Q k^p\,dz + c\,\tfrac{1}{|Q|}\int_{Q\cap\Phi_{\eta\lambda}} k^p\,dz,
\end{aligned}$$

where $c = c(\kappa)$. Adding up the last two inequalities we obtain

$$\begin{aligned}
&\Big(\fint_Q f^q\,dz\Big)^{\frac{p}{q}} + \fint_Q k^p\,dz\\
&\le c\,\eta^p\bigg(\Big(\fint_Q f^q\,dz\Big)^{\frac{p}{q}} + \fint_Q k^p\,dz\bigg) + c\,\tfrac{1}{|Q|}\int_{Q\cap\Phi_{\eta\lambda}}\big(\lambda^{p-q}f^q + k^p\big)\,dz,
\end{aligned}$$

where $c = c(p-q,\kappa)$. Choosing $\eta = \eta(p-q,\kappa) > 0$ small enough, we can absorb the first two integrals of the right-hand side on the left-hand side and conclude that

$$\left(\fint_Q f^q \, dz\right)^{\frac{p}{q}} + \fint_Q k^p \, dz \le c(p-q,\kappa)\ \frac{1}{|Q|} \int_{Q\cap\Phi_{\eta\lambda}} \left(\lambda^{p-q} f^q + k^p\right) \, dz.$$

Together with the second inequality in (20) this yields:

$$\int_{5Q} f^p \, dz \le c(p-q,\kappa) \ \int_{Q\cap\Phi_{\eta\lambda}} \left(\lambda^{p-q} f^q + k^p\right) \, dz. \tag{21}$$

For $\lambda \ge \lambda_1$ we find a family $\mathscr{F}$ of parabolic cylinders that fulfill (21) and cover Φ_λ. From Vitali's covering theorem for parabolic cylinders (see [4], [9], [3], Theorem C.1) we infer that there exists a countable subfamily $(Q_i)_{i=1}^\infty = (Q_{z_i}(\varrho_i, s_i))_{i=1}^\infty \subset \mathscr{F}$ of disjoint parabolic cylinders, such that the 5-times enlarged cylinders $5Q_i = Q_{z_i}(5\varrho_i, 5^{2m}s_i)$ cover the set Φ_λ, i.e. there holds (note that $5Q_i \subset Q_2$ by assumption)

$$\Phi_\lambda \subset \bigcup_{i=1}^\infty 5Q_i \subset Q_2.$$

Covering Φ_λ with $(5Q_i)_{i=1}^\infty$ and recalling that the cylinders $(Q_i)_{i=1}^\infty$ are pairwise disjoint, we infer from (21) that

$$\begin{aligned}\int_{\Phi_\lambda} f^p \, dz &\le \sum_{i=1}^\infty \int_{5Q_i} f^p \, dz \le c \sum_{i=1}^\infty \int_{Q_i\cap\Phi_{\eta\lambda}} \left(\lambda^{p-q} f^q + k^p\right) \, dz \\ &\le c(p-q,\kappa) \int_{\Phi_{\eta\lambda}} \left(\lambda^{p-q} f^q + k^p\right) \, dz.\end{aligned}$$

Moreover, from the definition of the level sets we have $f \le \lambda$ on $\Phi_{\eta\lambda}\backslash\Phi_\lambda$ and therefore

$$\int_{\Phi_{\eta\lambda}\backslash\Phi_\lambda} f^p \, dz \le \lambda^{p-q} \int_{\Phi_{\eta\lambda}\backslash\Phi_\lambda} f^q \, dz \le \lambda^{p-q} \int_{\Phi_{\eta\lambda}} f^q \, dz.$$

Together we infer the following reverse-Hölder inequality on the level sets $\Phi_{\eta\lambda}$ of f, valid for any $\lambda \ge \lambda_1$

$$\int_{\Phi_{\eta\lambda}} f^p \, dz \le c(p-q,\kappa) \int_{\Phi_{\eta\lambda}} \left(\lambda^{p-q} f^q + k^p\right) \, dz.$$

Replacing $\eta\lambda$ with λ and recalling that $\eta = \eta(p-q,\kappa) \le 1$ we obtain for all $\lambda \ge \eta\lambda_1 \equiv \lambda_2$ that

$$\int_{\Phi_\lambda} f^p \, dz \le c\ \eta^{q-p} \int_{\Phi_\lambda} \left(\lambda^{p-q} f^q + k^p\right) \, dz \le c \ \int_{\Phi_\lambda} \left(\lambda^{p-q} f^q + k^p\right) \, dz. \tag{22}$$

where $c = c(p-q,\kappa)$. Setting

$$\varphi(\lambda) \equiv \int_{\Phi_\lambda} f^q \, dz \quad \text{and} \quad \omega(\lambda) \equiv \int_{\Phi_\lambda} k^p \, dz$$

we can reduce the problem of higher integrability to the one dimensional case. Inequality (22) can then be rewritten as

$$-\int_{\lambda}^{\infty} \mu^{p-q}\, d\varphi(\mu) \le c(p-q,\kappa)\, \big(\lambda^{p-q}\varphi(\lambda) + \omega(\lambda)\big)$$

for all $\lambda \ge \lambda_2$. Because φ fulfills the conditions of Gehring's Theorem 14, i.e. φ is non-decreasing and $\lim_{\lambda\to\infty}\varphi(\lambda) = 0$, Gehring's theorem ensures the existence of $\varepsilon_0 = \varepsilon_0(p-q,\kappa) \in (0,\sigma)$ such that for all $\varepsilon \in (0,\varepsilon_0)$ there holds

$$-\int_{\lambda_2}^{\infty} \lambda^{p+\varepsilon-q}\, d\varphi(\lambda) \le -2\lambda_2^{\varepsilon}\int_{\lambda_2}^{\infty} \lambda^{p-q}\, d\varphi(\lambda) - 2c(p-q,\kappa)\, \int_{\lambda_2}^{\infty} \lambda^{\varepsilon}\, d\omega(\lambda).$$

By the definition of φ and ω this inequality can be rewritten as

$$\int_{\Phi_{\lambda_2}} f^{p+\varepsilon}\, dz \le 2\lambda_2^{\varepsilon}\int_{\Phi_{\lambda_2}} f^{p}\, dz + c(p-q,\kappa)\, \int_{\Phi_{\lambda_2}} k^{p+\varepsilon}\, dz.$$

Decomposing $Q_2 = Q_2\backslash\Phi_{\lambda_2} \cup \Phi_{\lambda_2}$ we obtain

$$\int_{Q_2} f^{p+\varepsilon}\, dz \le \lambda_2^{\varepsilon}\int_{Q_2\backslash\Phi_{\lambda_2}} f^{p}\, dz + \int_{\Phi_{\lambda_2}} f^{p+\varepsilon}\, dz \le 3\,\lambda_2^{\varepsilon}\int_{Q_2} f^{p}\, dz + c\int_{Q_2} k^{p+\varepsilon}\, dz,$$

with $c = c(\kappa, p-q)$. Recalling that $\lambda_2 = \eta\lambda_1 \le \lambda_1$ and the definition of λ_1 this proves the assertion of the lemma. □

8. Proof of the higher integrability

As we have seen, in the degenerate respectively singular case there is no general reverse-Hölder inequality available. We only have Lemma 13, which needs the additional conditions (13) and (14) to hold. Therefore, the main task in the following proof will be to find such appropriate disjoint parabolic cylinders, covering the upper level sets of $|D^m u|$. For this we will use the "stopping time argument". It exploits the continuous dependence of the integral on the domain of integration to "find" (see (24)) the right cylinder radius between a point and a too big cylinder.

Proof of Theorem 1. Without loss of generality we may assume that $\varrho = 1$ and $z_0 = 0$. Otherwise we consider $v(x,t) = \varrho^{-m}u(x_0 + \varrho x, t_0 + \varrho^{2m}t)$ on $Q(1,1)$ and get the general result by rescaling to $Q_{z_0}(\varrho,\varrho^{2m})$.

We set $Q_1 \equiv Q(1,1)$ and $Q_2 \equiv Q(2,2^{2m})$ and define the parabolic distance of $z \in Q_2$ to the boundary of Q_2 by

$$\mathrm{d}_{\mathscr{P}}(z) = \inf_{\bar z\in \mathbf{R}^{n+1}\backslash Q_2} \min\big\{|x-\bar x|,\ \sqrt[2m]{|t-\bar t|}\big\}. \tag{23}$$

Furthermore on Q_2 we define the function

$$g \equiv |D^m u| + b\,.$$

In the case $p \ge 2$ we set

$$f \equiv \hat c^{-1}\ \mathrm{d}_{\mathscr{P}}^{\alpha}\, g, \quad \text{with } \alpha = \frac{n+2m}{2},$$

where $\hat{c} \geq 1$ is some constant, which will be chosen appropriately later. We also choose

$$\lambda \geq \lambda_1 \equiv \max\{\lambda_0, 2^\alpha\}, \quad \text{with} \;\; \lambda_0 \equiv \left(\fint_{Q_2} g^p \, dz \right)^{\frac{1}{2}}.$$

Suppose that $\tilde{z}$ is a point in Q_2 with $f(\tilde{z}) > \lambda$. We then we set $r_{\tilde{z}} \equiv \mathrm{d}_{\mathscr{P}}(\tilde{z})$ for the parabolic distance of $\tilde{z}$ to the boundary of Q_2 and define as scaling factor for the parabolic cylinders (note that $r_{\tilde{z}}^\alpha \leq 2^\alpha \leq \lambda_1 \leq \lambda$ and $p \geq 2$ imply that $(r_{\tilde{z}}^{-\alpha}\lambda)^{2-p} = (r_{\tilde{z}}^{\alpha}\lambda^{-1})^{p-2} \leq 1$)

$$\gamma \equiv \gamma(\tilde{z}) \equiv \left(r_{\tilde{z}}^{-\alpha}\lambda\right)^{2-p} \leq 1.$$

With the help of the stopping time argument we now want to find an appropriate cylinder around $\tilde{z}$ on which we may apply Lemma 13, i.e. on which the conditions (13) and (14) are fulfilled.

Therefore we first note that for R with $\frac{r_{\tilde{z}}}{20} \leq R \leq r_{\tilde{z}}$ and $\hat{c}$ big enough there holds

$$\begin{aligned}\fint_{Q_{\tilde{z}}(R,\gamma R^{2m})} g^p \, dz &\leq \frac{|Q_2|}{|Q(R,\gamma R^{2m})|} \fint_{Q_2} g^p \, dz = 2^{n+2m} \, R^{-(n+2m)} \gamma^{-1} \lambda_0^2 \\ &\leq 40^{n+2m} \, r_{\tilde{z}}^{-(n+2m)} \gamma^{-1} \lambda^2 = \hat{c}^p \left(r_{\tilde{z}}^{-\alpha}\lambda\right)^p,\end{aligned}$$

where we have used the definitions of λ, γ, α and $\lambda \geq \lambda_0$. This fixes $\hat{c} = \hat{c}(n,m,p)$. Furthermore the Lebesgue differentiation theorem ensures that for a.e. $\tilde{z} \in Q_2$ with $f(\tilde{z}) > \lambda$ there holds

$$\lim_{r \to 0} \fint_{Q_{\tilde{z}}(r,\gamma r^{2m})} g^p \, dz = g(\tilde{z})^p = \hat{c}^p \, \mathrm{d}_{\mathscr{P}}(\tilde{z})^{-\alpha p} f(\tilde{z})^p > \hat{c}^p \left(r_{\tilde{z}}^{-\alpha}\lambda\right)^p,$$

where we have used that $\mathrm{d}_{\mathscr{P}}(\tilde{z}) = r_{\tilde{z}}$ and the definition of f. So the last two estimates yield on one hand a cylinder, namely $Q_{\tilde{z}}(R, \gamma R^{2m})$, which is too large, and on the other hand a cylinder which is too small. By the continuous dependence of the integral on the domain of integration there must be at least one cylinder in between, for which equality holds, i.e. there exists a radius $\varrho = \varrho(\tilde{z})$ with $0 < \varrho \leq \frac{r_{\tilde{z}}}{20}$, such that

$$\fint_{Q_{\tilde{z}}(\varrho,\gamma\varrho^{2m})} g^p \, dz = \hat{c}^p \left(r_{\tilde{z}}^{-\alpha}\lambda\right)^p \quad \text{and} \quad \fint_{Q_{\tilde{z}}(R,\gamma R^{2m})} g^p \, dz \leq \hat{c}^p \left(r_{\tilde{z}}^{-\alpha}\lambda\right)^p \tag{24}$$

for all R with $\varrho \leq R \leq r_{\tilde{z}}$. We now set $s \equiv s(\tilde{z}) \equiv \gamma\varrho^{2m}$ and $Q \equiv Q_{\tilde{z}}(\varrho, s)$ and $\alpha Q \equiv Q_{\tilde{z}}(\alpha\varrho, \alpha^{2m}s)$ for $\alpha > 0$. Then $10Q \Subset Q_2$. From (24) we conclude, that the assumptions (13) and (14) of Lemma 13 are fulfilled with $(r_{\tilde{z}}^{-\alpha}\lambda, \hat{c}^p)$ instead of (λ, κ), i.e. (note that $\hat{c} \geq 1$)

$$\left(r_{\tilde{z}}^{-\alpha}\lambda\right)^p \leq \fint_Q g^p \, dz \quad \text{and} \quad \fint_{10Q} g^p \, dz \leq \hat{c}^p \left(r_{\tilde{z}}^{-\alpha}\lambda\right)^p. \tag{25}$$

The application of the lemma then yields the following reverse-Hölder inequality

$$\fint_{10Q} |D^m u|^p \, dz \leq c \left(\fint_{2Q} |D^m u|^q \, dz \right)^{\frac{p}{q}} + c \fint_{2Q} b^p \, dz, \tag{26}$$

where $q = \max\{p-1, \frac{np}{n+2m}\}$ and c depends on $n, N, m, p, L/\nu$ (and κ). Since $\varrho \le \frac{r_{\tilde z}}{20}$ and $\gamma \le 1$ we have for all $z \in 10Q$ that $\mathrm{d}_{\mathscr P}(z) \le \min\{r_{\tilde z}+10\varrho,\ \sqrt[2m]{r_{\tilde z}^{2m} + \gamma(10\varrho)^{2m}}\} \le \frac{3}{2} r_{\tilde z}$ and consequently

$$f = \hat c^{-1}\ \mathrm{d}_{\mathscr P}^{\alpha}\ g \le c(n,m)\ r_{\tilde z}^{\alpha}\, g \quad \text{on } 10Q. \tag{27}$$

Moreover, from $\mathrm{d}_{\mathscr P}(z) \ge \min\{r_{\tilde z} - 10\varrho,\ \sqrt[2m]{r_{\tilde z}^{2m} - \gamma(10\varrho)^{2m}}\} \ge \frac{1}{2} r_{\tilde z}$ we infer that

$$g = \hat c\ d_{\mathscr P}^{-\alpha}\ f \le c(n,m,p)\ r_{\tilde z}^{-\alpha} f \quad \text{on } 10Q. \tag{28}$$

We now define $k \equiv \mathrm{d}_{\mathscr P}^{\alpha}\ b$ and show that there exists a constant $c = c(n, N, m, p, L/\nu)$ such that

$$c^{-1}\ \lambda^p \overset{(a)}{\le} ⨍_{10Q} f^p\, dz \overset{(b)}{\le} c \left(⨍_{2Q} f^q\, dz \right)^{\frac{p}{q}} + c ⨍_{2Q} k^p\, dz \overset{(c)}{\le} c^2\ \lambda^p. \tag{29}$$

The bound (a) follows from (25), the fact that $|10Q|/|Q| = 10^{n+2m}$ and $r_{\tilde z}^{\alpha} g \le c\, f$ on $10Q$ by (28):

$$\lambda^p \le ⨍_Q (r_{\tilde z}^{\alpha} g)^p\, dz \le c ⨍_Q f^p\, dz \le 10^{n+2m} c ⨍_{10Q} f^p\, dz = c(n,m,p) ⨍_{10Q} f^p\, dz.$$

We infer the second bound, i.e. (b) from (27), (25), the fact that $|10Q|/|Q| = 10^{n+2m}$, (26) and (28):

$$\begin{aligned} ⨍_{10Q} f^p\, dz &\le c ⨍_{10Q} (r_{\tilde z}^{\alpha} g)^p\, dz \le c\ \lambda^p \le c ⨍_Q (r_{\tilde z}^{\alpha} g)^p\, dz = c\ r_{\tilde z}^{\alpha p} ⨍_Q \big(|D^m u| + b\big)^p\, dz \\ &\le c\ r_{\tilde z}^{\alpha p} \left(\left(⨍_{2Q} |D^m u|^q\, dz \right)^{\frac{p}{q}} + ⨍_{2Q} b^p\, dz \right) \le c \left(⨍_{2Q} f^q\, dz \right)^{\frac{p}{q}} + c ⨍_{2Q} k^p\, dz, \end{aligned}$$

where $c = c(n, N, m, p, L/\nu)$. Finally, (c) follows from Hölder's inequality, (27), the fact that $|10Q|/|2Q| = 5^{n+2m}$ and (25):

$$\left(⨍_{2Q} f^q\, dz \right)^{\frac{p}{q}} + ⨍_{2Q} k^p\, dz \le ⨍_{2Q} \big(f^p + k^p\big)\, dz \le c ⨍_{10Q} \big(r_{\tilde z}^{\alpha} g\big)^p\, dz \le c(n,m,p)\ \lambda^p.$$

Hence, (29) holds.

Thus, for a.e. $\tilde z \in Q_2$ with $f(\tilde z) > \lambda$ there exists a parabolic cylinder Q with center $\tilde z$ such that (29) holds. Therefore we can apply Lemma 15 with $(c, 2Q, 10Q)$ instead of $(\kappa, Q, 5Q)$ to infer that there exists $\varepsilon_0 = \varepsilon_0(n, N, m, p, L/\nu) \in (0, \sigma - p)$ such that $f \in L^{p+\varepsilon_0}_{\mathrm{loc}}(Q_2)$ and there holds

$$\int_{Q_2} f^{p+\varepsilon}\, dz \le c\ \lambda_1^{\varepsilon} \int_{Q_2} f^p\, dz + c \int_{Q_2} k^{p+\varepsilon}\, dz \quad \text{for all } \varepsilon \in (0, \varepsilon_0], \tag{30}$$

where $c = c(n, N, m, p, L/\nu)$. Using in turn that $|D^m u| \le g \le \hat c\, f$ on Q_1 (since $\mathrm{d}_{\mathscr P}(z) \ge \min\{1, \sqrt[2m]{2^{2m} - 1}\} \ge 1$ for $z \in Q_1$), $|Q_2|/|Q_1| = 2^{n+2m}$, $f \le 2^{\alpha} \hat c^{-1} g$ on Q_2

(since $\mathrm{d}_{\mathscr{P}}(z) \le 2$ for $z \in Q_2$) and that $\lambda_1 \le 2^\alpha(1+\lambda_0^2)^{\frac12} = 2^\alpha(1+\fint_{Q_2} g^p\,dz)^{\frac12}$ by the definition of λ_1 and λ_0, we obtain

$$\begin{aligned}
\fint_{Q_1} |D^m u|^{p+\varepsilon}\,dz &\le c \fint_{Q_1} f^{p+\varepsilon}\,dz \le c\,\lambda_1^\varepsilon \fint_{Q_2} f^p\,dz + c\fint_{Q_2} k^{p+\varepsilon}\,dz\\
&\le c\,\lambda_1^\varepsilon \fint_{Q_2} g^p\,dz + c\fint_{Q_2} b^{p+\varepsilon}\,dz\\
&\le c\left(1+\fint_{Q_2} g^p\,dz\right)^{\frac{\varepsilon}{2}} \fint_{Q_2} g^p\,dz + c\fint_{Q_2} b^{\,p+\varepsilon}\,dz\\
&\le c + c\left(\fint_{Q_2} g^p\,dz\right)^{1+\frac{\varepsilon}{2}} + c\fint_{Q_2} b^{\,p+\varepsilon}\,dz\\
&= c\left(\fint_{Q_2} \big(|D^m u| + b\big)^p\,dz\right)^{1+\frac{\varepsilon}{2}} + c\fint_{Q_2} \big(b^{\,p+\varepsilon}+1\big)\,dz,
\end{aligned}$$

where $c = c(n,N,m,p,L/\nu)$. This shows the assertion in the case $p \ge 2$.

In the case $\max\{1, \frac{2n}{n+2m}\} < p < 2$ the proof is very much similar to the one for the case $p \ge 2$. But now, we have to choose the exponents in a different way, since the scaling factor of the parabolic cylinders γ—defined below—is larger then 1. We set

$$f \equiv \hat{c}^{-1}\ \mathrm{d}_{\mathscr{P}}^\alpha\ g, \quad \text{with } \alpha \equiv \frac{2m(n+2m)}{p(n+2m)-2n},$$

where $\hat{c} \ge 1$ is a constant which will be chosen properly later and the parabolic distance $\mathrm{d}_{\mathscr{P}}$ to the boundary of Q_2 was defined in (23). Moreover, we set

$$\lambda_1 \equiv \max\{\lambda_0, 2^\alpha\}, \quad \text{with } \lambda_0 \equiv \left(\fint_{Q_2} g^p\,dz\right)^{\frac{1}{d}},$$

where $d = p - \frac{n(2-p)}{2m}$ was defined in the statement of the Theorem. We now consider $\lambda \ge \lambda_1$. Suppose that $\tilde{z}$ is a point in Q_2 with $f(\tilde{z}) > \lambda$. By $r_{\tilde{z}} \equiv \mathrm{d}_{\mathscr{P}}(\tilde{z})$ we denote the parabolic distance of $\tilde{z}$ to the boundary of Q_2. Furthermore, as scaling factor for the parabolic cylinders we choose (note that $r_{\tilde{z}}^{-\alpha}\lambda \ge 2^{-\alpha}\cdot 2^\alpha = 1$)

$$\gamma \equiv \gamma(\tilde{z}) \equiv \big(r_{\tilde{z}}^{-\alpha}\lambda\big)^{2-p} \ge 1.$$

Now, once again we have to find a suitable cylinder around $\tilde{z}$ for which the conditions (13) and (14) of Lemma 13 are fulfilled and which is contained in Q_2.

Initially, we show (14) for radii $R \in [\frac{1}{20}\gamma^{-\frac{1}{2m}} r_{\tilde{z}}, \frac12\gamma^{-\frac{1}{2m}} r_{\tilde{z}}]$. Then, we have $Q_{\tilde{z}}(R, \gamma R^{2m}) \subset Q_2$. Since $\gamma R^{2m} \le (\frac{r_{\tilde{z}}}{2})^{2m} \le 1$, by the definitions of λ, γ, α, d (particularly that $n+2m = \alpha d$ and $(2-p)\frac{n}{2m} + d = p$) and due to the fact that

$R^{-1} \le 20\gamma^{\frac{1}{2m}} r_{\tilde z}^{-1}$ we obtain

$$\fint_{Q_{\tilde z}(R,\gamma R^{2m})} g^p\,dz \le \frac{|Q_2|}{|Q(R,\gamma R^{2m})|}\fint_{Q_2} g^p\,dz = \frac{2^{n+2m}}{\gamma R^{n+2m}}\lambda_0^d \le \frac{2^{n+2m}}{\gamma\,(\frac{1}{20}\gamma^{-\frac{1}{2m}} r_{\tilde z})^{n+2m}}\lambda^d$$
$$= c\,\gamma^{\frac{n}{2m}} r_{\tilde z}^{-(n+2m)}\lambda^d = c\left(r_{\tilde z}^{-\alpha}\lambda\right)^{(2-p)\frac{n}{2m}}(r_{\tilde z}^{-\alpha}\lambda)^d = \hat c^p\left(r_{\tilde z}^{-\alpha}\lambda\right)^p.$$

This fixes $\hat c = \hat c(n,m,p)$. In order to show (13) we infer from the Lebesgue differentiation theorem for a.e. $\tilde z \in Q_2$ with $f(\tilde z) > \lambda$

$$\lim_{r\to 0}\fint_{Q_{\tilde z}(r,\gamma r^{2m})} g^p\,dz = g(\tilde z)^p = \left(\hat c\ \mathrm{d}_{\mathscr P}(\tilde z)^{-\alpha} f(\tilde z)\right)^p > \hat c^p\left(r_{\tilde z}^{-\alpha}\lambda\right)^p,$$

where we have used that $\mathrm{d}_{\mathscr P}(\tilde z) = r_{\tilde z}$ and $f(\tilde z) > \lambda$. By the continuous dependence of the integral on the domain of integration there exists $\varrho = \varrho(\tilde z)$ with $0 < \varrho \le \frac{1}{20}\gamma^{-\frac{1}{2m}} r_{\tilde z}$, such that

$$\fint_{Q_{\tilde z}(\varrho,\gamma\varrho^{2m})} g^p\,dz = \hat c^p\left(r_{\tilde z}^{-\alpha}\lambda\right)^p \quad\text{and}\quad \fint_{Q_{\tilde z}(R,\gamma R^{2m})} g^p\,dz \le \hat c^p\left(r_{\tilde z}^{-\alpha}\lambda\right)^p$$

for all R with $\varrho \le R \le \frac{1}{2}\gamma^{-\frac{1}{2m}} r_{\tilde z}$. These estimates correspond with (24) and therefore we are now in completely the same situation as in the case $p \ge 2$. Proceeding as we did there we finally conclude the estimate (30), i.e. there exists $\varepsilon_0 = \varepsilon_0(n,N,m,p,L/\nu) \in (0,\sigma-p)$ such that $f \in L^{p+\varepsilon_0}_{\mathrm{loc}}(Q_2)$ and there holds

$$\fint_{Q_2} f^{p+\varepsilon}\,dz \le c\,\lambda_1^{\varepsilon}\fint_{Q_2} f^p\,dz + c\fint_{Q_2} k^{p+\varepsilon}\,dz \quad \text{for all } \varepsilon \in (0,\varepsilon_0].$$

From the definitions of f, g and λ_1 we infer with the same calculation as in the case $p \ge 2$, that

$$\fint_{Q_1} |D^m u|^{p+\varepsilon}\,dz \le c\left(\fint_{Q_2}\left(|D^m u|^p + b^p\right)dz\right)^{1+\frac{\varepsilon}{d}} + c\fint_{Q_2}\left(b^{p+\varepsilon}+1\right)dz,$$

where $c = c(n,N,m,p,L/\nu)$. This finishes the proof of Theorem 1. □

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Received 18 June 2007