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ACL AND DIFFERENTIABILITY OF Q-HOMEOMORPHISMS

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Abstract. It is established that a Q-homeomorphism in $\mathbf{R}^n$, $n \geq 2$, is absolute continuous on lines, furthermore, in $W^{1,1}_{\mathrm{loc}}$ and differentiable a.e. whenever $Q \in L^1_{\mathrm{loc}}$.

1. Introduction

Let G and G' be domains in $\mathbf{R}^n$, $n \geq 2$, and let $Q\colon G \to [1,\infty]$ be a measurable function. A homeomorphism $f\colon G \to G'$ is called a *Q-homeomorphism* if

$$M(f\Gamma) \leq \int_G Q(x) \cdot \varrho^n(x)\, dx \tag{1.1}$$

for every family Γ of paths in G and every admissible function ϱ for Γ. Here the notation m refers to the Lebesgue measure in $\mathbf{R}^n$. This conception is a natural generalization of the geometric definition of a quasiconformal mapping, see 13.1 and 34.6 in [Va].

Recall that, given a family of paths Γ in $\mathbf{R}^n$, a Borel function $\varrho\colon \mathbf{R}^n \to [0,\infty]$ is called *admissible* for Γ, abbr. $\varrho \in \operatorname{adm}\Gamma$, if

$$\int_\gamma \varrho\, ds \ \geq\ 1 \tag{1.2}$$

for all $\gamma \in \Gamma$. The (conformal) *modulus* of Γ is the quantity

$$M(\Gamma) = \inf_{\varrho \in \operatorname{adm}\Gamma} \int_G \varrho^n(x)\, dx. \tag{1.3}$$

This class of Q-homeomorphisms was first introduced and studied in [MRSY$_1$]–[MRSY$_3$]. The main goal of the theory of Q-homeomorphisms is to clear up various interconnections between properties of the majorant $Q(x)$ and the corresponding properties of the mappings themselves. In particular, the problem of the local and boundary behavior of Q-homeomorphisms has been studied in $\mathbf{R}^n$ first in the case $Q \in BMO$ (bounded mean oscillation) in the papers [MRSY$_1$]–[MRSY$_3$] and

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[RSY$_1$]–[RSY$_2$], and then in the case of $Q \in FMO$ (finite mean oscillation) and other cases in the papers [IR$_1$]–[IR$_2$], [RS] and [RSY$_3$]–[RSY$_6$].

In what follows, if A, B and C are sets in $\mathbf{R}^n$, then $\Delta(A,B,C)$ denotes a collection of all continuous curves $\gamma\colon [a,b] \to \mathbf{R}^n$ joining A and B in C, i.e. $\gamma(a) \in A$, $\gamma(b) \in B$ and $\gamma(t) \in C$, $t \in (a,b)$.

Here a *condenser* is a pair $E = (A,C)$ where $A \subset \mathbf{R}^n$ is open and C is non-empty compact set contained in A. E is a *ringlike condenser* if $B = A \setminus C$ is a ring, i.e., if B is a domain whose complement $\overline{\mathbf{R}^n} \setminus B$ has exactly two components where $\overline{\mathbf{R}^n} = \mathbf{R}^n \cup \{\infty\}$ is the one point compactification of $\mathbf{R}^n$.

2. On the ACL property of Q-homeomorphisms

Theorem 2.1. *Let G and G' be domains in $\mathbf{R}^n$, $n \geq 2$, and $f\colon G \to G'$ be Q-homeomorphism with $Q \in L^1_{\mathrm{loc}}$. Then $f \in ACL$.*

Proof. Let $I = \{x \in \mathbf{R}^n\colon a_i < x_i < b_i,\ i = 1, \ldots, n\}$ be an n-dimensional interval in $\mathbf{R}^n$ such that $\overline{I} \subset G$. Then $I = I_0 \times J$ where I_0 is an $(n-1)$-dimensional interval in $\mathbf{R}^{n-1}$ and J is an open segment of the axis x_n, $J = (a,b)$. Next we identify $\mathbf{R}^{n-1} \times \mathbf{R}$ with $\mathbf{R}^n$. We prove that for almost everywhere segments $J_z = \{z\} \times J$, $z \in I_0$, the mapping $f|_{J_z}$ is absolutely continuous.

Consider the set function $\Phi(B) = m(f(B \times J))$ defined over the algebra of all the Borel sets B in I_0. Note that by the Lebesgue theorem on differentiability for non-negative sub-additive locally finite set functions, see e.g. III.2.4 in [RR], there exists a finite limit for a.e. $z \in I_0$

$$\varphi(z) = \lim_{r \to 0} \frac{\Phi\left(B^{n-1}(z,r)\right)}{\Omega_{n-1} r^{n-1}} \tag{2.2}$$

where $B^{n-1}(z,r)$ is a ball in $I_0 \subset \mathbf{R}^{n-1}$ centered at $z \in I_0$ of the radius $r > 0$.

Let Δ_i, $i = 1, 2, \ldots$, be some enumeration S of all intervals in J such that $\overline{\Delta_i} \subset J$ and the ends of Δ_i are the rational numbers. Set

$$\varphi_i(z) := \int\limits_{\Delta_i} Q(z, x_n)\, dx_n .$$

Then by the Fubini theorem, see e.g. III. 8.1 in [Sa], the functions $\varphi_i(z)$ are a.e. finite and integrable in $z \in I_0$. In addition, by the Lebesgue theorem on differentiability of the indefinite integral there is a.e. a finite limit

$$\lim_{r \to 0} \frac{\Phi_i(B^{n-1}(z,r))}{\Omega_{n-1} r^{n-1}} = \varphi_i(z) \tag{2.3}$$

where Φ_i for a fixed $i = 1, 2, \ldots$ is the set function

$$\Phi_i(B) = \int\limits_B \varphi_i(\zeta)\, d\zeta$$

given over the algebra of all the Borel sets B in I_0.

Let us show that the mapping f is absolutely continuous on each segment J_z, $z \in I_0$, where the finite limits (2.2) and (2.3) exist. Fix one of such a point z. We have to prove that the sum of diameters of the images of an arbitrary finite collection of mutually disjoint segments in $J_z = \{z\} \times J$ tends to zero with the total length of the segments. In view of the continuity of the mapping f, it is sufficient to verify this fact only for mutually disjoint segments with rational ends in J_z. So, let $\Delta_i^* = \{z\} \times \overline{\Delta_i} \subset J_z$ where $\Delta_i \in S$, $i = 1, \ldots, k$ under the corresponding re-enumeration of S, are mutually disjoint intervals. Without loss of generality, we may assume that $\overline{\Delta_i}$, $i = 1, \ldots, k$ are also mutually disjoint.

Let $\delta > 0$ be an arbitrary rational number which is less than of half the minimum of the distances between Δ_i^*, $i = 1, \ldots, k$, and also less than their distances to the end-points of the interval J_z. Let $\Delta_i^* = \{z\} \times [\alpha_i, \beta_i]$ and $A_i = A_i(r) = B^{n-1}(z, r) \times (\alpha_i - \delta, \beta_i + \delta)$, $i = 1, \ldots, k$, where $B^{n-1}(z, r)$ is an open ball in $I_0 \subset \mathbf{R}^{n-1}$ centered at the point z of the radius $r > 0$. For small $r > 0$, (A_i, Δ_i^*), $i = 1, \ldots, k$, are ringlike condensers in I and hence $(fA_i, f\Delta_i^*)$, $i = 1, \ldots, k$, are also ringlike condensers in G'.

According to [Ge], see also [He] and [Sh],

$$\operatorname{cap}(fA_i, f\Delta_i^*) = M\left(\triangle\left(\partial fA_i, f\Delta_i^*; fA_i\right)\right)$$

and, in view of homeomorphism of f,

$$\triangle\left(\partial fA_i, f\Delta_i^*; fA_i\right) = f\left(\triangle\left(\partial A_i, \Delta_i^*; A_i\right)\right).$$

Thus, since f is a Q-homeomorphism we obtain that

$$\operatorname{cap}(fA_i, f\Delta_i^*) \le \int\limits_G Q(x) \cdot \rho^n(x)\, dx$$

for every function $\rho \in \operatorname{adm} \triangle(\partial A_i, \Delta_i^*; A_i)$. In particular, the function

$$\rho(x) = \begin{cases} \frac{1}{r}, & x \in A_i, \\ 0, & x \in \mathbf{R}^n \setminus A_i, \end{cases}$$

is admissible under $r < \delta$ and, thus,

$$\operatorname{cap}(fA_i, f\Delta_i^*) \le \frac{1}{r^n} \int\limits_{A_i} Q(x)\, dx. \tag{2.4}$$

On the other hand, by Lemma 5.9 in [MRV]

$$\operatorname{cap}(fA_i, f\Delta_i^*) \ge \left(C_n \frac{d_i^n}{m_i}\right)^{\frac{1}{n-1}} \tag{2.5}$$

where d_i is a diameter of the set $f\Delta_i^*$ and m_i is a volume of the set fA_i and C_n is a constant depending only on n.

Combining (2.4) and (2.5), we have the inequalities

$$\left(\frac{d_i^n}{m_i}\right)^{\frac{1}{n-1}} \leq \frac{c_n}{r^n}\int\limits_{A_i} Q(x)\,dm(x), \quad i=1,\dots,k, \tag{2.6}$$

where the constant c_n depends only on n.

By the discrete Hölder inequality we obtain

$$\sum_{i=1}^{k} d_i \leq \left(\sum_{i=1}^{k}\left(\frac{d_i^n}{m_i}\right)^{\frac{1}{n-1}}\right)^{\frac{n-1}{n}}\left(\sum_{i=1}^{k} m_i\right)^{\frac{1}{n}}, \tag{2.7}$$

i.e.,

$$\left(\sum_{i=1}^{k} d_i\right)^{n} \leq \left(\sum_{i=1}^{k}\left(\frac{d_i^n}{m_i}\right)^{\frac{1}{n-1}}\right)^{n-1}\Phi(B(z,r)), \tag{2.8}$$

and in view of (2.6)

$$\left(\sum_{i=1}^{k} d_i\right)^{n} \leq \gamma_n\,\frac{\Phi(B^{n-1}(z,r))}{\Omega_{n-1}r^{n-1}}\left(\sum_{i=1}^{k}\frac{\int\limits_{A_i} Q(x)\,dx}{\Omega_{n-1}r^{n-1}}\right)^{n-1}, \tag{2.9}$$

where γ_n depends only on n. Letting here first $r \to 0$ and then $\delta \to 0$, we get by Lebesgue's theorem

$$\left(\sum_{i=1}^{k} d_i\right)^{n} \leq \gamma_n\varphi(z)\left(\sum_{i=1}^{k}\varphi_i(z)\right)^{n-1}. \tag{2.10}$$

Finally, in view of (2.10), the absolute continuity of the indefinite integral of Q over the segment J_z implies the absolute continuity of the mapping f over the same segment. Hence $f \in ACL$. □

3. On a.e. differentiability of Q-homeomorphisms

Here we extend the method developed in [Go] to Q-homeomorphisms with $Q \in L^1_{\mathrm{loc}}$, cf. also [BRZ], [Ch] and [VI].

Theorem 3.1. *Let G and G' be domains in $\mathbf{R}^n$, $n \geq 2$, and $f\colon G \to G'$ be a Q-homeomorphism with $Q \in L^1_{\mathrm{loc}}$. Then f is differentiable a.e. in G.*

Proof. Let us consider the set function $\Phi(B) = m(f(B))$ defined over the algebra of all the Borel sets B in G. Recall that by the Lebesgue theorem on the differentiability of non-negative sub-additive locally finite set functions, see III.2.4 in [RR] or 23.5 in [Va], there exists a finite limit for a.e. $z \in G$

$$\varphi(x) = \lim_{\varepsilon\to 0}\frac{\Phi(B(x,\varepsilon))}{\Omega_n\varepsilon^n} \tag{3.2}$$

where $B(x,\varepsilon)$ is a ball in $\mathbf{R}^n$ centered at $x \in G$ with the radius $\varepsilon > 0$.

Consider also the spherical ring $R_\varepsilon(x) = \{y\colon \varepsilon < |x-y| < 2\varepsilon\}$, $x \in G$, with $\varepsilon > 0$ such that $R_\varepsilon(x) \subset G$. Since $\left(fB\left(y,2\varepsilon\right), \overline{fB\left(y,\varepsilon\right)}\right)$ are ringlike condensers in G', according to [Ge], see also [He] and [Sh],

$$\operatorname{cap}(fB(x,2\varepsilon), \overline{fB(x,\varepsilon)}) = M(\triangle(\partial fB(x,2\varepsilon), \partial fB(x,\varepsilon); fR_\varepsilon(x)))$$

and, in view of homeomorphism of f,

$$\triangle\left(\partial fB\left(x,2\varepsilon\right), \partial fB\left(x,\varepsilon\right); fR_\varepsilon(x)\right) = f\left(\triangle\left(\partial B(x,2\varepsilon), \partial B(x,\varepsilon); R_\varepsilon(x)\right)\right).$$

Thus, since f is Q-homeomorphism, we obtain that

$$\operatorname{cap}(fB(x,2\varepsilon), \overline{fB(x,\varepsilon)}) \le \int\limits_G Q(x)\cdot\rho^n(x)\,dx$$

for every admissible function ρ for $\triangle(\partial B(x,2\varepsilon), \partial B(x,\varepsilon); R_\varepsilon(x))$. The function

$$\rho(x) = \begin{cases} \frac{1}{\varepsilon}, & \text{if } x \in R_\varepsilon(x), \\ 0, & \text{if } x \in G \setminus R_\varepsilon(x), \end{cases}$$

is admissible and, thus,

$$\operatorname{cap}(fB(x,2\varepsilon), \overline{fB(x,\varepsilon)}) \le \frac{2^n\Omega_n}{m(B(x,2\varepsilon))}\int\limits_{B(x,2\varepsilon)} Q(y)\,dy. \tag{3.3}$$

On the other hand, by Lemma 5.9 in [MRV] we have that

$$\operatorname{cap}(fB(x,2\varepsilon), \overline{fB(x,\varepsilon)}) \ge \left(C_n\frac{d^n(fB(x,\varepsilon))}{m(fB(x,2\varepsilon))}\right)^{\frac{1}{n-1}} \tag{3.4}$$

where C_n is a constant depending only on n, $d(A)$ and $m(A)$ denote the diameter and the Lebesgue measure of a set A in $\mathbf{R}^n$.

Combining (3.3) and (3.4), we obtain that

$$\frac{d(fB(x,\varepsilon))}{\varepsilon} \le \gamma_n\left(\frac{m(fB(x,2\varepsilon))}{m(B(x,2\varepsilon))}\right)^{1/n}\left(\frac{1}{m(B(x,2\varepsilon))}\int\limits_{B(x,2\varepsilon)} Q(y)\,dy\right)^{(n-1)/n}$$

and hence

$$L(x,f) \le \limsup_{\varepsilon\to 0}\frac{d(fB(x,\varepsilon))}{\varepsilon} \le \gamma_n\varphi^{1/n}(x)Q^{(n-1)/n}(x)$$

where

$$L(x,f) = \limsup_{y\to x}\frac{|f(y)-f(x)|}{|y-x|}\,. \tag{3.5}$$

Thus, $L(x,f) < \infty$ a.e. in G. Finally, applying the Rademacher–Stepanov theorem, see e.g. [Sa], p. 311, we conclude that f is differentiable a.e. in G. $\square$

Corollary 3.6. *Let G and G' be domains in $\mathbf{R}^n$, $n \ge 2$, and let $f\colon G \to G'$ be a Q-homeomorphism with $Q \in L^1_{\mathrm{loc}}$. Then f belongs to $W^{1,1}_{\mathrm{loc}}$.*

Proof. For $L(x, f)$ given by (3.5) and a Borel set $V \subset G$, we have that

$$\int_V L(x, f)\, dx \le \gamma_n \int_V \varphi^{1/n}(x) Q^{(n-1)/n}(x)\, dx$$

and applying the Hölder inequality we obtain

$$\int_V \varphi^{1/n}(x) Q^{(n-1)/n}(x)\, dx \le \left(\int_V \varphi(x)\, dx \right)^{1/n} \left(\int_V Q(x)\ dx \right)^{(n-1)/n}.$$

Finally, in view of $Q \in L^1_{loc}$, by the Lebesgue theorem we see that

$$\int_V L(x, f)\, dx \le \gamma_n\ (mV)^{1/n} \left(\int_V Q(x)\, dx \right)^{(n-1)/n} < \infty$$

and the conclusion follows by Theorem 2.1, see also [Maz]. □

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