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ON HARMONIC QUASICONFORMAL SELF-MAPPINGS OF THE UNIT BALL

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Abstract. It is proved that any family of harmonic K-quasiconformal mappings $\{u = P[f], u(0) = 0\}$ of the unit ball onto itself is a uniformly Lipschitz family providing that $f \in C^{1,\alpha}$. Moreover, the Lipschitz constant tends to 1 as $K \to 1$.

1. Introduction and auxiliary results

A twice differentiable function u defined in an open subset Ω of the Euclidean space $\mathbf{R}^n$ will be called *harmonic* if it satisfies the differential equation

$$\Delta u(x) = D_{11}u(x) + D_{22}u(x) + \cdots + D_{nn}u(x) = 0.$$

In this paper B^n denotes the unit ball in $\mathbf{R}^n$, and S^{n-1} denotes the unit sphere. We will consider the vector norm $\|x\| = (\sum_{i=1}^n x_i^2)^{1/2}$ and two matrix norms, $\|A\|_2 = (\sum_{i,j=1}^n a_{i,j}^2)^{1/2}$ and $\|A\| = \sup\{\|Ax\| : \|x\| = 1\}$.

A homeomorphism $u\colon \Omega \to \Omega'$ between two open subsets Ω and Ω' of the Euclidean space $\mathbf{R}^n$ will be called a *K-quasiconformal* ($K \geq 1$) or, shortly, a q.c. mapping if

(i) u is an absolutely continuous function in every segment parallel to some of the coordinate axes and there exist the partial derivatives which are locally L^n integrable functions on Ω. We will write $u \in ACL^n$, and
(ii) u satisfies the condition $\|u'(x)\|^n/K \leq J_u(x) \leq Kl(u'(x))^n$ at x almost everywhere on Ω where $l(u'(x)) := \inf\{\|u'(x)\zeta\| : \|\zeta\| = 1\}$ and $J_u(x)$ is the Jacobian determinant of u (see [9]).

Note that the condition $u \in ACL^n$ guarantees the existence of the first derivative of u almost everywhere (see [9]). The condition (i) is equivalent with the fact that u is continuous and belongs to the Sobolev space $W^1_{n,\mathrm{loc}}(\Omega)$.

Let $f\colon S^{n-1} \to \mathbf{R}^n$ be a bounded integrable function on the unit sphere S^{n-1}. Let P be a Poisson kernel, i.e., the function

$$P(x, \eta) = \frac{1 - \|x\|^2}{\|x - \eta\|^n}.$$

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Then

$$u(x) = P[f](x) = \int_{S^{n-1}} P(x,\eta) f(\eta)\, d\sigma(\eta) \tag{1.1}$$

is a harmonic mapping defined in the unit ball B^n. Here $d\sigma$ is the Lebesgue $n-1$-dimensional measure of the Euclidean sphere satisfying the condition $P[1](x) \equiv 1$. It is well known that if f is continuous, then the mapping $u = P[f]$ has a continuous extension $\tilde{u}$ to the boundary and $\tilde{u} = f$ on S^{n-1}.

If k is a nonnegative integer and $\alpha \in (0,1]$, then $C^{k,\alpha}(\bar{\Omega})$ is defined to be the set of k times continuously differentiable functions in an open set Ω such that

$$\|u\|_{k,\alpha} := \sum_{|\beta|\le k} \|D^\beta u\| + \sum_{|\beta|=k} \sup_{x,y\in\Omega} \|D^\beta u(x) - D^\beta u(y)\| \cdot \|x-y\|^{-\alpha}$$

is finite. If f is a function defined in the unit sphere S^{n-1}, then we will say that $f \in C^{k,\alpha}(S^{n-1})$ if the function defined by $u(x) = f(x/\|x\|)$ is in $C^{k,\alpha}(A(1/2,2))$, where $A(1/2,2) := \{x : 1/2 \le \|x\| \le 2\}$. Gilbarg and Hörmander among the other results in [3] prove the following proposition (see [3,Theorem 6.1]).

Proposition 1.1. *The Dirichlet problem*

$$\Delta u = g \text{ in } B^n, \quad u = f \text{ on } S^{n-1}$$

has a unique solution $u \in C^{k,\alpha}(\bar{B}^n)$, $k \ge 1$, for every $g \in C^{k-2,\alpha}(\bar{B}^n)$ and for every $f \in C^{k,\alpha}(S^{n-1})$.

Using the previous proposition we obtain:

Proposition 1.2. *If the function $f \in C^{1,\mu}$, where $0 < \mu \le 1$, then the function u has a $C^{1,\mu}$ extension to the boundary and the relations*

$$\lim_{r\to 1} \frac{\partial u \circ S}{\partial \theta_i}(r, \theta_1, \dots, \theta_{n-1}) = \frac{\partial f \circ T}{\partial \theta_i}(\theta_1, \dots, \theta_{n-1})$$

hold for all $i \in \{1, \dots, n-1\}$, where S and T are spherical coordinates

$$(S(r, \theta_1, \dots, \theta_{n-1}) = rT(\theta_1, \dots, \theta_{n-1})),$$

and $\theta_{n-1} = \varphi$. See the proof of Lemma 1.6 below for details of the definition of T.

Proposition 1.3. [2] *If f is a K-quasiconformal self-mapping of the unit ball B^n with $f(0) = 0$, then there exists a constant $M_1(n,K)$, satisfying the condition $M_1(n,K) \to 1$ as $K \to 1$, such that*

$$\|f(x) - f(y)\| \le M_1(n,K)\|x-y\|^{K^{1/(1-n)}}. \tag{1.2}$$

See also [1] *for some constant that is not asymptotically sharp.*

Proposition 1.4. *Let A be a nonsingular matrix. Let $\lambda_1^2 \ge \cdots \ge \lambda_n^2$ be the eigenvalues of the matrix $A^T A$. Let $\tilde{A} = \det A \cdot A^{-1}$ and let $k(A) := \frac{\lambda_1}{\lambda_n}$. Then*

$$\|A\|_2 = (\mathrm{Trace}\, A^T A)^{1/2} = \left(\sum_{k=1}^n \lambda_k^2\right)^{1/2}, \quad |\det A| = \prod_{k=1}^n \lambda_k \tag{1.3}$$

and

$$k(A^{-1}) = k(\tilde{A}) = k(A). \tag{1.4}$$

If in addition A is a K-quasiconformal mapping, then

$$\|A\| \leq \frac{k(A)}{\sqrt{n-1+k(A)^2}}\|A\|_2 \tag{1.5}$$

and

$$k(A) \leq K + \sqrt{K^2-1}. \tag{1.6}$$

For the proof of the last assertion, see [11].

Lemma 1.5. *Let u be a harmonic mapping defined in the unit ball having a C^1 extension to the boundary S^{n-1}. Then*

$$\sup_{\|x\|\leq 1} \|u'(x)\|_2 = \sup_{\|\eta\|=1} \|u'(\eta)\|_2.$$

Proof. Let $u = (u_1, \dots u_n)$. For all (i,j) the function $u_{i,j} = \frac{\partial u_i}{\partial x_j}$ is bounded harmonic. Hence there exists a bounded integrable function $g_{i,j}$ defined on the unit sphere such that $u_{i,j} = P[g_{i,j}]$. Then

$$u'(x) = \int_{S^{n-1}} g(\eta)P(x,\eta)\, d\sigma(\eta),$$

where $g(\eta)$ is $n \times n$ dimensional matrix $(g_{i,j}(\eta))_{i,j=1}^n$ and it coincides with $u'(\eta)$. By definition we have

$$\|u'(x)\|_2 = \left\| \lim_{m\to\infty} \sum_{k=1}^m u'(\eta_k)P(x,\eta_k)\sigma(S_k) \right\|_2,$$

where $S^{n-1} = \sum_{k=1}^m S_k$ and $\delta_k = \max\limits_{1\leq k\leq m} (\mathrm{Diam}\,(S_k)) \to 0$ as $m \to \infty$. Hence,

$$\|u'(x)\|_2 \leq \lim_{m\to\infty} \sum_{k=1}^m \|u'(\eta_k)\|_2 P(x,\eta_k)\sigma(S_k) = \sup_{\|\eta\|=1} \|u'(\eta)\|_2 P[1] = \max_{\|\eta\|=1} \|u'(\eta)\|_2.$$

The proof is completed. □

Lemma 1.6. *The integral*

$$I = \int_{S^{n-1}} \|a-\eta\|^{\gamma}\, d\sigma(\eta),$$

$a \in S^{n-1}$ converges if and only if $\gamma > 1-n$. If $\gamma = 2-n$, then $I = 1$.

Proof. Because of symmetry, it is enough to take $a = (1, 0, \dots, 0)$. Let $T = (x_1, x_2, ..., x_n)\colon K^{n-1} \to S^{n-1}$ be spherical coordinates:

$$\begin{aligned} x_1 &= \cos\theta_1, \\ x_2 &= \sin\theta_1 \cos\theta_2, \\ &\vdots \\ x_{n-1} &= \sin\theta_1 \sin\theta_2 \cdots \sin\theta_{n-2} \cos\varphi, \\ x_n &= \sin\theta_1 \sin\theta_2 \cdots \sin\theta_{n-2} \sin\varphi. \end{aligned}$$

Here the cube $K^{n-1} = [0, \pi] \times \cdots \times [0, \pi] \times [0, 2\pi]$ is $n-1$-dimensional. Then

$$D_T(\theta_1, \dots, \theta_{n-2}, \varphi) = \sin^{n-2}\theta_1 \cdots \sin\theta_{n-2}. \tag{1.7}$$

It follows that

$$\begin{aligned} I &= \int_{S^{n-1}} \|a - \eta\|^\gamma \, d\sigma(\eta) \\ &= \frac{2\pi}{\omega_{n-1}} \int_0^\pi 2^{\gamma+n-2} \cos^{n-2}(\theta_1/2) \sin^{n-2}(\theta_1/2) \sin^\gamma(\theta_1/2) \, d\theta_1 \\ &\quad \cdot \int_0^\pi \sin^{n-3}\theta_2 d\theta_2 \cdots \int_0^\pi \sin\theta_{n-2} \, d\theta_{n-2} \\ &= 2^{\gamma+n-2} \frac{\int_0^\pi \cos^{n-2}(\theta_1/2) \sin^{n+\gamma-2}(\theta_1/2) \, d\theta_1}{\int_0^\pi \sin^{n-2}\theta_1 \, d\theta_1}. \end{aligned}$$

Hence the integral converges if and only if $n+\gamma-2 > -1$, i.e., $\gamma > 1-n$. Moreover,

$$\int_0^\pi \cos^{n-2}(\theta_1/2) \, d\theta_1 = \frac{\sqrt{\pi}\,\Gamma(\frac{n-1}{2})}{\Gamma(\frac{n}{2})} = \int_0^\pi \sin^{n-2}\theta_1 \, d\theta_1.$$

Hence $I = 1$ for $\gamma = 2 - n$. □

Lemma 1.7. *For an arbitrary set of real numbers $\{x_1, x_2, \dots, x_n\}$ there holds the inequality*

$$x_1^2 x_2^2 \cdots x_{n-1}^2 + x_1^2 \cdots x_{n-2}^2 x_n^2 + \cdots + x_2^2 \cdots x_{n-1}^2 x_n^2 \le \frac{(x_1^2 + \cdots + x_n^2)^{n-1}}{n^{n-2}}. \tag{1.8}$$

Proof. If $n = 2$, then we have nothing to prove. Assume $n \ge 3$. We will consider two cases.

Case 1. There exists i such that $x_i = 0$. Say $x_n = 0$. Then the inequality (1.8) is equivalent to

$$x_1^2 x_2^2 \cdots x_{n-1}^2 \le \frac{(x_1^2 + \cdots + x_{n-1}^2)^{n-1}}{n^{n-2}},$$

which follows directly by using the arithmetic-geometric mean inequality and the inequality $(1 + \frac{1}{n-1})^{n-1} \le n$.

Case 2. For every i, $x_i \ne 0$. We will prove the inequality by solving the following extremal problem: $L := x_1^2 x_2^2 \cdots x_{n-1}^2 + x_1^2 \cdots x_{n-2}^2 x_n^2 + \cdots + x_2^2 \cdots x_{n-1}^2 x_n^2 \to \text{ext}$ for $D := x_1^2 + \cdots + x_n^2 = r^2$, for some real number r. The Lagrangean of the corresponding problem is $L-\lambda D$ and the critical points of it satisfy the following system: $L_{x_i}=2x_i\lambda$, $i=1,\dots,n$, and $D=r^2$. Denote by P the product $x_1^2\cdots x_n^2$ and by S the sum $\frac{1}{x_1^2}+\cdots+\frac{1}{x_n^2}$. Then $L=PS$. Using the fact that $P_{x_i}=\frac{2P}{x_i}$, $i=1,\dots,n$, we obtain $\frac{2PS}{x_i^2}-\frac{2P}{x_i^4}=2\lambda$ for $i=1,\dots,n$. It follows that

$$\left(\frac{1}{x_i^2}-\frac{1}{x_j^2}\right)\left(S-\frac{1}{x_i^2}-\frac{1}{x_j^2}\right)=0 \quad \text{for every } i\neq j. \tag{1.9}$$

From (1.9) it follows that $x_i^2=x_j^2$ for every i and j. Since the set $\{x : D=r^2\}$ is compact, it follows that the points x satisfying the equalities $x_i^2=x_j^2$ for all i and j are points of the maximum of the function L. From the last fact it follows (1.8) at once. □

Lemma 1.8. *Let $A\colon \mathbf{R}^n\to\mathbf{R}^n$ be a linear operator such that $A=[a_{ij}]_{i,j=1,\dots,n}$.*

a) *There holds the inequality*

$$\|Ax_1\times\cdots\times Ax_{n-1}\|\le\frac{1}{\sqrt{(n-1)^{n-1}}}\|A\|_2^{n-1}\|x_1\times\cdots\times x_{n-1}\|. \tag{1.10}$$

b) *If A is K-quasiconformal, then*

$$\|Ax_1\times\cdots\times Ax_{n-1}\|\le L(K,n)\|A\|_2^{n-1}\|x_1\times\cdots\times x_{n-1}\|, \tag{1.11}$$

where

$$L(K,n)=\min\left\{\frac{K+\sqrt{K^2-1}}{\sqrt{n^{n-2}(n-1+(K+\sqrt{K^2-1})^2)}},\frac{1}{\sqrt{(n-1)^{n-1}}}\right\}. \tag{1.12}$$

The inequalities (1.10) *and* (1.11) *are sharp.*

Observe that $\lim_{K\to 1}L(K,n)=n^{\frac{1-n}{2}}$.

Proof. a) If $n=2$, then we can easily check that (1.10) holds. Assume $n\ge 3$. If $x_1,\dots,x_{n-1}$ are linearly dependent vectors, then the inequality follows from the fact that

$$Ax_1\times\cdots\times Ax_{n-1}=\tilde{A}^T x_1\times\cdots\times x_{n-1}=0.$$

Otherwise, applying the Gram–Schmidt algorithm, we construct a sequence of vectors f_i, $i=1,\dots,n$, such that $\langle f_i,f_i\rangle=1$, $\langle f_i,f_j\rangle=0$ for $i\neq j$, and $\mathscr{L}(x_1,\dots,x_i)=\mathscr{L}(f_1,\dots,f_i)$ for $i=1,\dots,n-1$, where $\mathscr{L}(z_1,\dots,z_m)$ is the linear space generated by $\{z_1,\dots,z_m\}$.

Let $F=(f_{i,j})$ be an $n\times n$ matrix defined such that $f_j=\sum_{i=1}^n f_{ij}e_i$, where $e_1^T=(1,0\dots,0)$, …, $e_n^T=(0,0,\dots,1)$. Then,

$$\|AF\|_2=\|A\|_2. \tag{1.13}$$

Let us prove this fact. By definition,

$$\|AF\|_2^2=\sum_{i,j=1}^n\langle A^Te_i,Fe_j\rangle^2=\sum_{i,j=1}^n\langle A^Te_i,f_j\rangle^2.$$

Let $A^Te_i = \sum_{i,j} b_{ij} f_j$. Multiplying by f_k, we obtain that $\langle A^Te_i, f_k\rangle = b_{ik}$. Hence,

$$A^Te_i = \sum_{j=1}^{n} \langle A^Te_i, f_j\rangle f_j,$$

and, consequently,

$$\|A^Te_i\|^2 = \sum_{j=1}^{n} \langle A^Te_i, f_j\rangle^2.$$

Combining these, we obtain that

$$\|A\|_2^2 = \sum_{i=1}^{n} \|A^Te_i\|^2 = \sum_{i,j=1}^{n} \langle A^Te_i, f_j\rangle^2 = \|AF\|_2^2.$$

Let $x_i = \sum_{j=1}^n x_{ij} f_j$, $i = 1, \dots, n-1$. Then

$$Ax_1 \times \cdots \times Ax_{n-1} = \sum_{\sigma} \varepsilon_\sigma x_{1,\sigma_1} \dots x_{n-1\sigma_{n-1}} Af_1 \times \cdots \times Af_{n-1}.$$

It follows that

$$\begin{aligned}
\|Ax_1 \times \cdots \times Ax_{n-1}\|^2 &= \left\| \sum_{\sigma} \varepsilon_\sigma x_{1,\sigma_1} \dots x_{n-1\sigma_{n-1}} Af_1 \times \dots Af_{n-1} \right\|^2 \\
&= \|Af_1 \times \dots Af_{n-1}\|^2 \|x_1 \times \cdots \times x_{n-1}\|^2 \\
&\le \frac{1}{(n-1)^{n-1}} \left(\sum_{i=1}^{n} \|Af_i\|^2 \right)^{n-1} \|x_1 \times \cdots \times x_{n-1}\|^2 \\
&= \frac{1}{(n-1)^{n-1}} \|AF\|_2^{2(n-1)} \|x_1 \times \cdots \times x_{n-1}\|^2.
\end{aligned}$$

If $A = (a_{ij})$ such that $a_{ii} = 1$, $i = 1, \dots, n-1$, and $a_{ij} = 0$ otherwise and $x_i = e_i$, then the equality of the theorem holds.

b) If $n = 2$, then (1.11) follows from (1.5) and (1.6). Assume $n \ge 3$. From $Ax_1 \times \cdots \times Ax_{n-1} = \tilde{A}^T x_1 \times \cdots \times x_{n-1}$ it follows that

$$\|Ax_1 \times \cdots \times Ax_{n-1}\| \le \|\tilde{A}\| \|x_1 \times \cdots \times x_{n-1}\|.$$

Let $\tilde\lambda_1^2 \ge \cdots \ge \tilde\lambda_n^2$ be the eigenvalues of the matrix $\tilde{A}^T\tilde{A}$. According to Proposition 1.4,

$$\frac{\|\tilde{A}\|_2^2}{\|\tilde{A}\|^2} = \sum_{k=1}^{n} \frac{\tilde\lambda_k^2}{\tilde\lambda_1^2} \ge 1 + \frac{n-1}{k^2(\tilde{A})} \ge \frac{(K + \sqrt{K^2-1})^2 + n - 1}{(K + \sqrt{K^2-1})^2}.$$

It follows that

$$\|\tilde{A}\| \le \frac{K + \sqrt{K^2-1}}{\sqrt{n - 1 + (K + \sqrt{K^2-1})^2}} \|\tilde{A}\|_2. \tag{1.14}$$

On the other hand,

$$\|\tilde{A}\|_2 = \sqrt{\sum_{k=1}^{n} \|\tilde{A}e_k\|^2} = \sqrt{\sum_{k=1}^{n} \|Ae_1 \times \cdots \times Ae_{k-1} \times Ae_{k+1} \times \cdots \times Ae_n\|^2}.$$

Using the fact that

$$\|Ae_1 \times \cdots \times Ae_{k-1} \times Ae_{k+1} \times \cdots \times Ae_n\| \le \|Ae_1\| \cdots \|Ae_{k-1}\| \cdot \|Ae_{k+1}\| \cdots \|Ae_n\|$$

and denoting by x_i', $i = 1, \ldots, n$, the real numbers $\|Ae_i\|$ after applying the inequality (1.8), we obtain

$$\|\tilde{A}\|_2 \le \frac{n}{\sqrt{n^n}} \|A\|_2^{n-1}. \tag{1.15}$$

From (1.14) and (1.15) we obtain the desired inequality. If A is the unit matrix (or, more generally, if A is any orthogonal transformation), then A is $K = 1$-quasiconformal and the equality holds. Thus the inequality is sharp. □

2. The main result

Martio [8] was the first who considered harmonic quasiconformal mappings on the complex plane. Recent papers [4]–[6] and [10] bring much light on the topic of quasiconformal harmonic mappings on the plane.

In this paper we consider the same problem in the space. The problem in the space is much more complicated because of lack of the technique of complex analysis.

In this paper we prove the following theorem:

Theorem 2.1. (The main result) *Let $K \ge 1$ be arbitrary and $n \in \mathbf{N}$. Then there exists a constant $M' = M'(n, K)$ such that if $u = P[f]$ is a K-quasiconformal harmonic self-mapping of the unit ball B^n with $u(0) = 0$ such that $f \in C^{1,\alpha}$ for some α, $0 < \alpha \le 1$, then*

$$\|u(x) - u(y)\| \le M' \|x - y\|, \quad x, y \in B^n. \tag{2.1}$$

Moreover, $M'(n, K) \to 1$ as $K \to 1$.

Proof. Let f be an arbitrary $C^{1,\alpha}$-injective function from the unit sphere S^{n-1} onto itself. Let S be the mapping defined on $K_0 = [0,1] \times [0,\pi] \times \cdots \times [0,\pi] \times [0,2\pi]$ by $S(r,\theta) = rT(\theta)$, where $\theta = (\theta_1, \ldots \theta_{n-2}, \varphi)$. Then the mapping x, $x(\theta) = f(T(\theta))$, defines the outer normal vector field $\mathbf{n}_x$ in S^{n-1} at the point $x(\theta) = f(T(\theta)) = (x_1, x_2, \ldots, x_n)$ by the formula

$$\mathbf{n}_x(x(\theta)) = x_{\theta_1} \times \cdots \times x_{\theta_{n-2}} \times x_\varphi = \begin{vmatrix} e_1 & e_2 & \ldots & e_n \\ x_{1\theta_1} & x_{2\theta_1} & \ldots & x_{n\theta_1} \\ \ldots & \ldots & \ldots & \ldots \\ x_{1\theta_{n-2}} & x_{2\theta_{n-2}} & \ldots & x_{n\theta_{n-2}} \\ x_{1\varphi} & x_{2\varphi} & \ldots & x_{n\varphi} \end{vmatrix}. \tag{2.2}$$

Let $u = P[f]$. Let $u(S(r,\theta)) = (y_1, y_1, \dots, y_n)$, where S are spherical coordinates. According to Proposition 1.2, we obtain that the following limit relations hold:

$$\lim_{r\to 1} y_{i\varphi}(r,\theta) = x_{i\varphi}(\theta), \quad i \in \{1,\dots,n\}, \tag{2.3}$$

$$\lim_{r\to 1} y_{i\theta_j}(r,\theta) = x_{i\theta_j}(\theta), \quad i \in \{1,\dots,n\},\ j \in \{1,\dots,n-2\}, \tag{2.4}$$

and

$$\lim_{r\to 1} y_{ir}(r,\theta) = \lim_{r\to 1} \frac{x_i(\theta) - y_i(r,\theta)}{1-r}, \quad i \in \{1,\dots,n\}. \tag{2.5}$$

From (2.3), (2.4), (2.5) and (1.1) we obtain

$$\begin{aligned}
\lim_{r\to 1} J_{u\circ S}(r,\theta) &= \lim_{r\to 1} \begin{vmatrix} \frac{x_1-y_1}{1-r} & \frac{x_2-y_2}{1-r} & \dots & \frac{x_n-y_n}{1-r} \\ x_{1\theta_1} & x_{2\theta_1} & \dots & x_{n\theta_1} \\ \dots & \dots & \dots & \dots \\ x_{1\theta_{n-2}} & x_{2\theta_{n-2}} & \dots & x_{n\theta_{n-2}} \\ x_{1\varphi} & x_{2\varphi} & \dots & x_{n\varphi} \end{vmatrix} \\
&= \lim_{r\to 1} \int_{S^{n-1}} \frac{1+r}{\|\eta - x\|^n} \begin{vmatrix} x_1 - f_1(\eta) & \dots & x_n - f_n(\eta) \\ x_{1\theta_1} & \dots & x_{n\theta_1} \\ \dots & \dots & \dots \\ x_{1\theta_{n-2}} & \dots & x_{n\theta_{n-2}} \\ x_{1\varphi} & \dots & x_{n\varphi} \end{vmatrix} d\sigma(\eta) \\
&= \lim_{r\to 1} \int_{S^{n-1}} \frac{1+r}{\|\eta - S(r,\theta)\|^n} \langle f(T(\theta)) - f(\eta), \mathbf{n}_{f\circ T}(T(\theta))\rangle \, d\sigma(\eta) \\
&= \lim_{r\to 1} D_x(\theta) \int_{S^{n-1}} \frac{1+r}{\|\eta - S(r,\theta)\|^n} \langle f(T(\theta)) - f(\eta), f(T(\theta))\rangle \, d\sigma(\eta) \\
&= \lim_{r\to 1} \frac{1+r}{2} D_x(\theta) \int_{S^{n-1}} \frac{\|f(T(\theta)) - f(\eta)\|^2}{\|\eta - S(r,\theta)\|^n} \, d\sigma(\eta).
\end{aligned}$$

Hence we have

$$\lim_{r\to 1} J_{u\circ S}(r,\theta) = D_x(\theta) \int_{S^{n-1}} \frac{\|f(T(\theta)) - f(\eta)\|^2}{\|\eta - T(\theta)\|^n} \, d\sigma(\eta), \tag{2.6}$$

where $x = f(T(\theta))$. Now from

$$\|u'(S(r,\theta))\|_2^n \le K n^{n/2} J_u(S(r,\theta)),$$

using the formula $J_{u\circ S}(1,\theta) = J_u(S(1,\theta)) \cdot D_T(\theta)$, we obtain

$$\lim_{r\to 1} \|u'(S(r,\theta))\|_2^n \le \lim_{r\to 1} K n^{n/2} J_u(S(r,\theta)) = \frac{K n^{n/2}}{D_T(\theta)} \lim_{r\to 1} J_{u\circ S}(r,\theta). \tag{2.7}$$

From Proposition 1.2 we deduce that

$$\begin{aligned}
&\lim_{r\to 1}\frac{\partial u\circ S}{\partial\theta_1}(r,\theta)\times\cdots\times\frac{\partial u\circ S}{\partial\theta_{n-2}}(r,\theta)\times\frac{\partial u\circ S}{\partial\varphi}(r,\theta)\\
&=\frac{\partial f\circ T}{\partial\theta_1}(\theta)\times\cdots\times\frac{\partial f\circ T}{\partial\theta_{n-2}}(\theta)\times\frac{\partial f\circ T}{\partial\varphi}(\theta).
\end{aligned}$$

Since

$$\frac{\partial u\circ S}{\partial\theta_i}(r,\theta)=ru'(S(r,\theta))\frac{\partial T}{\partial\theta_i},$$

using (1.11), we obtain that

$$D_x(\theta)\le L(K,n)\lim_{r\to 1}\|u'(S(r,\theta))\|_2^{n-1}D_T(\theta). \tag{2.8}$$

From (2.6)–(2.8) we obtain

$$\begin{aligned}
&\lim_{r\to 1}\|u'(S(r,\theta))\|_2^{n}\\
&\le L(K,n)\cdot Kn^{n/2}\lim_{r\to 1}\|u'(S(r,\theta))\|_2^{n-1}\int_{S^{n-1}}\frac{\|f(T(\theta))-f(\eta)\|^2}{\|\eta-T(\theta)\|^n}\,d\sigma(\eta),
\end{aligned}$$

i.e.,

$$\lim_{r\to 1}\|u'(S(r,\theta))\|_2\le L(K,n)\cdot Kn^{n/2}\int_{S^{n-1}}\frac{\|f(T(\theta))-f(\eta)\|^2}{\|\eta-T(\theta)\|^n}\,d\sigma(\eta). \tag{2.9}$$

Let $M=\max\{\|u'(x)\|_2:\|x\|=1\}=\lim_{r\to 1}\|u'(S(r,\theta_0))\|_2$ for some θ_0 and let $\mu=K^{1/(1-n)}$. It is clear that $0<\mu<1$. Let $\gamma=1-n+\mu^2$, and let $\nu=1-\mu$. According to Lemma 1.5,

$$\sup_{\|x\|\le 1}\|u'(x)\|_2=M.$$

Since

$$\|u(x)-u(y)\|\le\sup_{t\in B^n}\|u'(t)\|\cdot\|x-y\| \tag{2.10}$$

and according to Proposition 1.4,

$$\|u'(t)\|\le\frac{k(u'(t))}{\sqrt{n-1+k(u'(t))^2}}\|u'(t)\|_2,$$

it follows that

$$\|u(x)-u(y)\|\le M\sup_{t\in B^n}\frac{k(u'(t))}{\sqrt{n-1+k(u'(t))^2}}\|x-y\|. \tag{2.11}$$

From Proposition 1.4 we obtain

$$\frac{k(u'(t))}{\sqrt{n-1+k(u'(t))^2}}\le l:=\frac{K+\sqrt{K^2-1}}{\sqrt{n-1+(K+\sqrt{K^2-1})^2}}.$$

Now from (2.9), (1.2) and (2.11), we obtain

$$\begin{aligned} M &\le (Ml)^{\nu} L(K,n)\cdot Kn^{n/2}\int_{S^{n-1}}\|\eta - T(\theta_0)\|^{\gamma}\frac{\|f(T(\theta_0)) - f(\eta)\|^{2-\nu}}{\|T(\theta_0)-\eta\|^{\mu^2+\mu}}\,d\sigma(\eta)\\ &\le (Ml)^{\nu} L(K,n)\cdot Kn^{n/2} M_1(K,n)^{1+\mu}\int_{S^{n-1}}\|\eta - T(\theta_0)\|^{\gamma}\,d\sigma(\eta)\\ &= M_0(K,n)(Ml)^{\nu}. \end{aligned}$$

Hence we obtain

$$M \le l^{\nu/(1-\nu)}(M_0(K,n))^{1/(1-\nu)}. \tag{2.12}$$

The inequality (2.1) does hold for

$$M' = l\cdot l^{\nu/(1-\nu)}(M_0(K,n))^{1/(1-\nu)} = (l\cdot M_0(K,n))^{1/(1-\nu)}.$$

Using (1.2), Lemma 1.8 and Lemma 1.6, it follows that $\lim_{K\to 1} M'(K,n) = 1$. □

Corollary 2.2. *Let $K \ge 1$. Then there exists a constant $M' = M'(K)$ such that if $u = P[f]$ is a K-quasiconformal harmonic self-mapping of the unit disk D satisfying $u(0) = 0$, then*

$$\|u(z) - u(w)\| \le M'\|z - w\|, \quad z, w \in D. \tag{2.13}$$

Moreover, $M'(K) \to 1$ as $K \to 1$. See [10] *for some constant that is not asymptotically sharp.*

Proof. Let $D_m = u^{-1}(\{z : |z| < 1 - \frac{1}{m}\})$ and let $\varphi_m\colon D \mapsto D_m$ be a conformal mapping such that $\varphi_m(0) = 0$ and $\varphi_m'(0) > 0$. Then the mapping $u_m = \frac{m}{m-1}u\circ\varphi_m$ satisfies the conditions of Theorem 2.9. From (2.1) it follows that

$$\|u_m(z) - u_m(w)\| \le M'\|z - w\|, \quad z, w \in D.$$

Since $\lim_{m\to\infty} u_m(z) = u(z)$, the inequality (2.13) does hold for every quasiconformal harmonic mapping. □

2.1. Questions. a) How to eliminate the assumption $f \in C^{1,\alpha}$ in Theorem 2.1? b) Is every q.c. harmonic mapping of the unit ball onto itself, satisfying $u(0) = 0$, a bi-Lipschitz mapping? c) Does some q.c. harmonic mapping have critical points, i.e., the points in which the Jacobian is zero? Compare this problem with the plane version of the problem treated in [7].

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