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GROMOV HYPERBOLICITY OF CERTAIN CONFORMAL INVARIANT METRICS

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Abstract. The unit ball $\mathbf{B}^n$ is shown to be Gromov hyperbolic with respect to the Ferrand metric $\lambda^*_{\mathbf{B}^n}$ and the modulus metric $\mu_{\mathbf{B}^n}$, and dimension dependent upper bounds for the Gromov delta are obtained. In the two-dimensional case Gromov hyperbolicity is proved for all simply connected domains G. For λ^*_G also the case $G = \mathbf{R}^n \setminus \{0\}$ is studied.

1. Introduction

In the eighties Gromov introduced a notion of hyperbolic space, which has thereafter been studied and further developed by many authors. For a long time the research was centered at hyperbolic group theory, but lately also many researchers in geometric function theory have developed an interest towards the theory of Gromov hyperbolic spaces. For a general overview of the topic, the books of [CoDePa] and [BuBuIv] can be mentioned.

One of the primary questions is of course which metric spaces (X, d) meet the Gromov hyperbolicity condition and which ones do not? In geometric function theory, there are nowadays many metrics around which are of "hyperbolic type" in the sense that they are defined in domains, the boundary geometry of which—more or less completely—determine the behavior of the metric. They are also negatively curved if the curvature can be determined, and in case it can not, still bilipschitz equivalence to a negatively curved metric can mostly be established. A fundamental result is that the hyperbolic metric is also hyperbolic in the sense of Gromov, in all domains where it is defined. A similar result has been proved also for the well-known *quasihyperbolic metric* [BoHeKo], only here it is required that we restrict to domains which are uniform in the sense of Martio and Sarvas, [MaSa].

Typically, if we can prove or disprove Gromov hyperbolicity for a metric in some domain, the result immediately follows for a related metric or a different domain with the same metric, if we find a suitable quasi-isometry between the spaces. However, in general this method works only when the metrics involved are geodesic, or at least intrinsic, and many metrics which are defined for instance by point-pair functions, fail to meet the requirement of intrinsity. Such metrics are the

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distance ratio metrics (j-metrics), the *Apollonian metric* and *Seittenranta's metric* (see [Se]). The non-geodesic case has been considered by eg. Bonk and Schramm in [BoSc] and by Väisälä [Vä2].

As many of the strongest results in the theory of Gromov hyperbolic spaces rely on geodesity, for non-geodesic metrics this means that even natural and easy questions can get quite complicated. For instance, one would expect that a metric of the above described "hyperbolic type" is also hyperbolic in the sense of Gromov, as is the case with the hyperbolic and quasihyperbolic metrics. The question regarding the Gromov hyperbolicity of the distance ratio metrics has recently been answered by Hästö in [Hä], and as his results show, the situation might be very delicate; in [Hä] two metrics j_G and $\tilde{j}_G$ are studied, and it turns out that one is Gromov hyperbolic in any domain, whereas the other one fails to be so in all domains except the complement of a point. Still the two metrics are very closely connected, in that they are bilipschitz equivalent and both have the quasihyperbolic metric as their inner metric. Also for the metrics studied in this article it seems like it is more or less "a coincidence" that the proof can be carried out in the special cases proved. One is tempted to conjecture Gromov hyperbolicity to hold for the μ_G and λ_G^* metrics also in the general case when $n \geq 3$, but when our knowledge on the relation with certain special functions is taken away, the problem becomes hard to grasp.

In this article I study two special non-intrinsic metrics, known as the *modulus metric* and *Ferrand's metric*. Both are defined using the concept of conformal modulus, and consequently they are examples of conformally invariant metrics. The main results in the article are the proofs of the fact that both of the above metrics are Gromov hyperbolic in the n-ball, and that the Ferrand metric is Gromov hyperbolic also in punctured n-space. We also derive upper estimates for the Gromov constants. The method is to verify the inequality (2.5) by means of inequalities for the special functions connected to the modulus and Ferrand metrics, especially the Teichmüller and Grötzsch capacity functions.

2. Preliminaries and definitions

The domains considered in this article are of the type $G \subsetneq \bar{\mathbf{R}}^n$, that is, proper subsets of the compactified n-space. We start by defining the *modulus metric* μ_G introduced by Gál in [Gá], and its "dual quantity" λ_G, which was introduced by Lelong-Ferrand in [Le].

Let Γ be a family of curves in $\bar{\mathbf{R}}^n$. By $\mathscr{F}(\Gamma)$ we denote the family of *admissible functions*, that is, non-negative Borel-measurable functions $\rho\colon \bar{\mathbf{R}}^n \to \bar{\mathbf{R}}$ such that

$$\int_\gamma \rho \, ds \geq 1$$

for each locally rectifiable curve $\gamma \in \Gamma$. For $p \geq 1$ the *p-modulus* of Γ is defined by

$$\mathsf{M}_p(\Gamma) = \inf_{\rho \in \mathscr{F}(\Gamma)} \int_{\mathbf{R}^n} \rho^p \, dm,$$

where m is the n-dimensional Lebesgue measure. The modulus is an outer measure in the space of curve families in $\bar{\mathbf{R}}^n$. If $p = n$ the modulus $\mathsf{M}(\Gamma) = \mathsf{M}_n(\Gamma)$ is called the *conformal modulus*, and it is then a conformal invariant, i.e. if $f\colon G \to G'$ is a conformal mapping and Γ is a curve family in G, then $\mathsf{M}(\Gamma) = \mathsf{M}(f\Gamma)$. For basic properties of moduli, see [Vä1].

For $E, F, G \subset \bar{\mathbf{R}}^n$ we denote by $\Delta(E, F; G)$ the family of all closed non-constant curves joining E and F in G, that is, $\gamma\colon [a, b] \to \bar{\mathbf{R}}^n$ belongs to $\Delta(E, F; G)$ if one of $\gamma(a), \gamma(b)$ belongs to E and the other to F, and furthermore $\gamma(t) \in G$ for all $a < t < b$.

Recall also that for the pair (A, C), $C \subset A \subset \mathbf{R}^n$ where A is open and C compact, we can define the *conformal capacity* of (A, C) by

$$\operatorname{cap}(A, C) = \inf_u \int_{\mathbf{R}^n} |\nabla u|^n \, dm,$$

where the infimum is taken over all non-negative ACL^n functions with compact support in A such that $u(x) \geq 1$ for $x \in C$. It is widely known that

$$\operatorname{cap}(A, C) = \mathsf{M}\big(\Delta(C, \partial A; A)\big).$$

We say that a compact set $E \subset \mathbf{R}^n$ is *of capacity zero*, denoted $\operatorname{cap} E = 0$, if there exists a bounded open set A with $E \subset A$ and $\operatorname{cap}(A, E) = 0$. A compact set $E \subset \bar{\mathbf{R}}^n$ is of capacity zero if it can be mapped by a Möbius transformation onto a bounded set of capacity zero. A set E which is not of capacity zero is said to have *positive capacity* and this is denoted $\operatorname{cap} E > 0$.

For $x, y \in G \subsetneq \bar{\mathbf{R}}^n$ λ_G is defined by

$$\lambda_G(x, y) = \inf_{C_x, C_y} \mathsf{M}\big(\Delta(C_x, C_y; G)\big),$$

where $C_z = \gamma_z[0, 1)$ and $\gamma_z\colon [0, 1] \to G$ is a curve such that $z \in C_z$ and $\gamma_z(t) \to \partial G$ when $t \to 1$ and $z = x, y$. Correspondingly,

$$\mu_G(x, y) = \inf_{C_{xy}} \mathsf{M}\big(\Delta(C_{xy}, \partial G; G)\big),$$

where C_{xy} is such that $C_{xy} = \gamma[0, 1]$ and γ is a curve with $\gamma(0) = x$ and $\gamma(1) = y$.

It is not difficult to show that both quantities μ_G and λ_G are conformal invariants, and that μ_G is a metric when $\operatorname{cap} \partial G > 0$. In a general domain G it is not known whether the values of $\mu_G(x, y)$ and $\lambda_G(x, y)$ can be expressed in explicit form, eg. in terms of some special functions. For $G = \mathbf{B}^n$, however, we have the formulas

$$(2.1) \qquad \mu_{\mathbf{B}^n}(x, y) = 2^{n-1}\tau_n\Big(\frac{1}{\sinh^2 \frac{1}{2}\rho(x, y)}\Big) = \gamma_n\Big(\frac{1}{\tanh \frac{1}{2}\rho(x, y)}\Big),$$

$$(2.2) \qquad \lambda_{\mathbf{B}^n}(x, y) = \tfrac{1}{2}\tau_n\big(\sinh^2 \tfrac{1}{2}\rho(x, y)\big),$$

where γ_n and τ_n are the capacity functions of the Grötzsch and Teichmüller condensers, respectively, and $\rho(x, y) = \rho_{\mathbf{B}^n}(x, y)$ is the hyperbolic distance between the points x and y. The capacity functions are defined by

$$\begin{cases}\gamma_n(s)=\mathrm{cap}(\mathbf{R}^n\setminus\{te_1: t\ge s\},\bar{\mathbf{B}}^n),\ s>1\\ \tau_n(s)=\mathrm{cap}(\mathbf{R}^n\setminus\{te_1: t\ge s\},[-e_1,0]),\ s>0.\end{cases}$$

Many properties of γ_n and τ_n can be found in [AnVaVu, Chapter 11] and [Vu1, Section 5]. For instance, both functions are decreasing homeomorphisms, and we have inequalities

$$\begin{cases}\omega_{n-1}(\log\lambda_n s)^{1-n}\le\gamma_n(s)\le\omega_{n-1}(\log s)^{1-n}\\ \omega_{n-1}\big(\log(\lambda_n^2 s)\big)^{1-n}\le\tau_n(s-1)\le\omega_{n-1}(\log s)^{1-n}\end{cases}\tag{2.3}$$

for $s>1$, where λ_n denotes the Grötzsch constant (cf. [Vu1, 7.21]), and ω_{n-1} is the $(n-1)$-dimensional area of the unit sphere S^{n-1}. The Grötzsch capacity function also satisfies the inequality

$$2^{n-1}c_n\log\Big(\frac{s+1}{s-1}\Big)\le\gamma_n(s)\le 2^{n-1}c_n\log\Big(4\,\frac{s+1}{s-1}\Big),\tag{2.4}$$

(cf. [AnVaVu, (5.3),11.20]), where c_n is the *spherical cap inequality constant* defined by

$$c_2=\frac{2}{\pi},\quad c_n=2^{1-n}\omega_{n-2}\Big(\int_0^{\pi/2}\sin^{\frac{2-n}{n-1}}t\,dt\Big)^{1-n},\ \ n\ge 3.$$

For any metric space (X,d), we define the *Gromov product* of two points $x,y\in X$ with respect to a base point $w\in X$ by

$$(x|y)_w=\frac{1}{2}\big(d(x,w)+d(y,w)-d(x,y)\big).$$

Using the notation $a\vee b=\max(a,b)$ and $a\wedge b=\min(a,b)$, the space (X,d) is said to be *Gromov δ-hyperbolic* if for every triple $x,y,z\in X$ and a fixed $w\in X$ the inequality

$$(x|z)_w\ge(x|y)_w\wedge(y|z)_w-\delta$$

is satisfied. Here we will mostly use the equivalent inequality

$$d(x,z)+d(y,w)\le\big(d(x,w)+d(y,z)\big)\vee\big(d(x,y)+d(z,w)\big)+2\delta.\tag{2.5}$$

3. The Ferrand metric $\lambda^*_{\mathbf{B}^n}$

We start by defining the conformally invariant metric λ^*_G, often referred to as the *Ferrand metric* by setting

$$\lambda^*_G(x,y)=\lambda_G(x,y)^{1/(1-n)}.\tag{3.1}$$

The part that λ^*_G is a conformal invariant is clear. In [Fe] it is shown that it is a metric for all $G\subset\bar{\mathbf{R}}^n$ with $\mathrm{card}(\bar{\mathbf{R}}^n\setminus G)\ge 2$, but there are only few cases for which we have explicit formulas for the Ferrand metric. However, in the cases $G=\mathbf{B}^n$, and $G=\mathbf{R}^2\setminus\{0\}$ such a formula can be found. In this section we prove that the metric space $(\mathbf{B}^n,\lambda^*_{\mathbf{B}^n})$ is Gromov hyperbolic, and derive an upper bound for the Gromov delta. We first record the following lemma:

Lemma 3.2. *Let $f\colon [0,\infty) \to [0,\infty)$ be increasing with $f(0) = 0$, and let $s,t \geq 0$. Then, if $f(t)/t$ is decreasing on $(0,\infty)$, we have that*

$$f(s+t) \leq f(s) + f(t)$$

and if $f(t)/t$ is increasing on $(0,\infty)$, we have that

$$f(s+t) \geq f(s) + f(t). \qquad \square$$

The first part of the lemma can be proved as a special case of [AnVaVu, 1.24], and the second follows a similar reasoning. Using this we can prove the following inequality:

Lemma 3.3. *Let $f(x) = \log(\cosh^2 x)$. Then, for all $x, y \geq 0$ we have that*

$$f(x) + f(y) \leq f(x+y) \leq f(x) + f(y) + \log 4.$$

Proof. The first inequality follows directly from Lemma 3.2, by checking that $f(t)/t$ is increasing. The second inequality says that

$$\begin{aligned}\log\big(\cosh^2(x+y)\big) &\leq \log(\cosh^2 x) + \log(\cosh^2 y) + \log(2^2)\\ &= \log\big(2^2(\cosh^2 x)(\cosh^2 y)\big).\end{aligned}$$

But this is true, since $\cosh(x+y) \leq 2\cosh x \ \cosh y$. $\square$

Theorem 3.4. *The metric space $(\mathbf{B}^n, \lambda^*_{\mathbf{B}^n})$ is Gromov δ-hyperbolic, with Gromov constant*

$$\begin{aligned}\delta &\leq \tfrac{1}{2}\left(\tfrac{\omega_{n-1}}{2}\right)^{1/(1-n)}\left(\log\tfrac{400}{9} + 4\log\lambda_n\right)\\ &\leq \left(\tfrac{\omega_{n-1}}{2}\right)^{1/(1-n)}\left(2n + \tfrac{2}{n} - 3 + \log\frac{80}{3}\right),\end{aligned}$$

where ω_{n-1} denotes the $(n-1)$-dimensional surface area of S^{n-1} and λ_n is the Grötzsch constant.

Proof. Let $x, y, z, w \in \mathbf{B}^n$. We may now rewrite the second inequality in (2.3) as

$$\omega_{n-1}^{1/(1-n)}\log(1+s) \leq \tau_n(s)^{1/(1-n)} \leq \omega_{n-1}^{1/(1-n)}\big(2\log\lambda_n + \log(1+s)\big). \tag{3.5}$$

Then, by the definition (2.2), the second inequality in (3.5) and the first inequality in Lemma 3.3, we see that

$$\begin{aligned}
&\lambda^*_{\mathbf{B}^n}(x,z)+\lambda^*_{\mathbf{B}^n}(y,w)\\
&=\lambda_{\mathbf{B}^n}(x,z)^{1/(1-n)}+\lambda_{\mathbf{B}^n}(y,w)^{1/(1-n)}\\
&=\big(\tfrac12\big)^{1/(1-n)}\Big(\tau_n(\sinh^2\tfrac12\rho(x,z))^{1/(1-n)}+\tau_n(\sinh^2\tfrac12\rho(y,w))^{1/(1-n)}\Big)\\
&\le\big(\frac{\omega_{n-1}}{2}\big)^{1/(1-n)}\Big(\log(\cosh^2\tfrac12\rho(x,z))+\log(\cosh^2\tfrac12\rho(y,w))\Big)\\
&\quad+4\big(\frac{\omega_{n-1}}{2}\big)^{1/(1-n)}\log\lambda_n\\
&\le\big(\frac{\omega_{n-1}}{2}\big)^{1/(1-n)}\log\Big(\cosh^2\big(\tfrac12\rho(x,z)+\tfrac12\rho(y,w)\big)\Big)+4\big(\frac{\omega_{n-1}}{2}\big)^{1/(1-n)}\log\lambda_n.
\end{aligned}$$

Now we make use of the fact that the hyperbolic metric of $\mathbf{B}^n$ is Gromov hyperbolic with $\delta_0=\log 3$ (see [CoDePa, 4.3] Prop. 4.3 in Chapter 1). Then we know that

$$\tfrac12\rho(x,z)+\tfrac12\rho(y,w)\le\big(\tfrac12\rho(x,w)+\tfrac12\rho(y,z)\big)\vee\big(\tfrac12\rho(x,y)+\tfrac12\rho(z,w)\big)+\log 3.$$

Now, let $f(x)=\log(\cosh^2 x)$. Then, for any positive numbers a,b,c,d, we know by the fact that f is a positive increasing function, and by applying Lemma 3.3 twice, that

$$\begin{aligned}
f\big((a+b)\vee(c+d)+\delta_0\big)&=f(a+b+\delta_0)\vee f(c+d+\delta_0)\\
&\le\big(f(a)+f(b)\big)\vee\big(f(c)+f(d)\big)+f(\delta_0)+2\log 4.
\end{aligned}$$

But then, by the above calculation and the inequality (3.5)

$$\begin{aligned}
&\lambda^*_{\mathbf{B}^n}(x,z)+\lambda^*_{\mathbf{B}^n}(y,w)\\
&\le\big(\frac{\omega_{n-1}}{2}\big)^{1/(1-n)}\Big(\big(f(\tfrac12\rho(x,w))+f(\tfrac12\rho(y,z))\big)\vee\big(f(\tfrac12\rho(x,y))+f(\tfrac12\rho(z,w))\big)\Big)\\
&\quad+\ \big(\frac{\omega_{n-1}}{2}\big)^{1/(1-n)}\big(\log\tfrac{25}{9}+2\log 4+4\log\lambda_n\big)\\
&\le\big(\lambda^*_{\mathbf{B}^n}(x,w)+\lambda^*_{\mathbf{B}^n}(y,z)\big)\vee\big(\lambda^*_{\mathbf{B}^n}(x,y)+\lambda^*_{\mathbf{B}^n}(z,w)\big)\\
&\quad+\ \big(\frac{\omega_{n-1}}{2}\big)^{1/(1-n)}\big(\log\tfrac{400}{9}+4\log\lambda_n\big).
\end{aligned}$$

This proves the first inequality in the statement. The estimate for δ not involving the Grötzsch constant follows from the inequality $\lambda_n\le\exp(n+\frac1n-\frac32+\log 2)$, see [AnVaVu, 12.21]. □

Then, by conformal invariance, the Riemann mapping theorem immediately yields the following.

Corollary 3.6. *A simply connected proper subdomain $G\subsetneq\mathbf{R}^2$ is Gromov δ-hyperbolic with respect to the metric λ^*_G, where*

$$\delta\le\frac{\log 5462}{2\pi}\approx 1.3696.$$

□

In estimating the Gromov delta we have used the fact that $\lambda_2 = 4$ (see [LeVi, p. 61, (2.10)]).

Also in the case of the punctured n-space $\mathbf{R}^n_* = \mathbf{R}^n \setminus \{0\}$, it is possible to establish Gromov hyperbolicity of Ferrand's metric. In the case $n \geq 3$ there is no explicit formula, but the natural way to prove Gromov hyperbolicity is to use a double-sided estimate by means of the Teichmüller capacity function, proved by Vuorinen in [Vu2];

$$\tau_n\left(\frac{|x-y|}{|x| \wedge |y|}\right) \leq \lambda_{\mathbf{R}^n_*}(x,y) \leq \tau_n\left(\frac{|x-y| + \big||x|-|y|\big|}{2(|x| \wedge |y|)}\right).$$

This is valid for $n \geq 2$, and immediately yields the inequality

$$\tau_n\left(\frac{|x-y|}{|x| \wedge |y|}\right)^{1/(1-n)} \geq \lambda^*_{\mathbf{R}^n_*}(x,y) \geq \tau_n\left(\frac{|x-y|}{2(|x| \wedge |y|)}\right)^{1/(1-n)}. \tag{3.7}$$

We will also use the fact that the *distance ratio metric* j_G defined by

$$j_G(x,y) = \log\left(1 + \frac{|x-y|}{\operatorname{dist}(x,\partial G) \wedge \operatorname{dist}(y,\partial G)}\right) \tag{3.8}$$

is Gromov hyperbolic in $G = \mathbf{R}^n \setminus \{0\}$. This was recently shown by Hästö in [Hä]:

Lemma 3.9. *Let $G \subsetneq \mathbf{R}^n$ be an open set. Then j_G is Gromov hyperbolic if and only if G has a single boundary point. In this case the Gromov delta satisfies $\delta \leq \log 3$.* □

Note that since the domains in this article are generally equipped with the Möbius space topology, we are actually studying $G = \bar{\mathbf{R}}^n \setminus \{0, \infty\}$. Of course, otherwise λ^*_G wouldn't even be a metric. However, this gives us no limitations regarding the use of Lemma 3.9. Now we can prove the following:

Theorem 3.10. *The metric space $(\mathbf{R}^n_*, \lambda^*_{\mathbf{R}^n_*})$ is Gromov hyperbolic, with*

$$\delta \leq 2\omega_{n-1}^{1/(1-n)} \log 6\lambda_n^2 \leq 2\omega_{n-1}^{1/(1-n)}\big(2n + \tfrac{2}{n} - 3 + \log 24\big).$$

Proof. By the definition of $\lambda^*_{\mathbf{R}^n_*}$ and by the inequalities (3.7) and (2.3) we have that

$$\begin{aligned}
&\lambda^*_{\mathbf{R}^n_*}(x,z) + \lambda^*_{\mathbf{R}^n_*}(y,w) \\
&\leq \tau_n\left(\frac{|x-z|}{|x| \wedge |z|}\right)^{1/(1-n)} + \tau_n\left(\frac{|y-w|}{|y| \wedge |w|}\right)^{1/(1-n)} \\
&\leq \omega_{n-1}^{1/(1-n)}\left(\log\left(1 + \frac{|x-z|}{|x| \wedge |z|}\right) + \log\left(1 + \frac{|y-w|}{|y| \wedge |w|}\right)\right) + 4\,\omega_{n-1}^{1/(1-n)} \log \lambda_n \\
&= \omega_{n-1}^{1/(1-n)}\Big(j_{\mathbf{R}^n_*}(x,z) + j_{\mathbf{R}^n_*}(y,w)\Big) + 4\,\omega_{n-1}^{1/(1-n)} \log \lambda_n.
\end{aligned}$$

By Lemma 3.9, and the inequalities (3.7) and

$$\log(1+x) \le \log(2+x) = \log(1+\tfrac{1}{2}x) + \log(2), \tag{3.11}$$

we now see that

$$\begin{aligned}
&j_{\mathbf{R}^n_*}(x,z) + j_{\mathbf{R}^n_*}(y,w) \\
&\le \big(j_{\mathbf{R}^n_*}(x,w) + j_{\mathbf{R}^n_*}(y,z)\big) \vee \big(j_{\mathbf{R}^n_*}(x,y) + j_{\mathbf{R}^n_*}(z,w)\big) + 2\log 3 \\
&\le \left(\log\left(1 + \frac{|x-w|}{2(|x| \wedge |w|)}\right) + \log\left(1 + \frac{|y-z|}{2(|y| \wedge |w|)}\right)\right) \\
&\quad \vee \left(\log\left(1 + \frac{|x-y|}{2(|x| \wedge |y|)}\right) + \log\left(1 + \frac{|w-z|}{2(|z| \wedge |w|)}\right)\right) \\
&\quad + \ 2\log 3 + 2\log 2.
\end{aligned}$$

But from this, and the above computation it follows by using (2.3) and (3.7) again, that we have Gromov hyperbolicity with constant

$$\delta \le 2\ \omega_{n-1}^{1/(1-n)} \log 6 + 4\omega_{n-1}^{1/(1-n)} \log \lambda_n,$$

which gives the constant stated. □

Remark 3.12. It is unclear what the role of the exponent $1/(1-n)$ in the definition of λ_G^* is. It can be said though, that for this approach in proving the above results the use of this particular exponent is necessary, as it allows us to use the inequalities (2.3). However, for $n = 2$ and simply connected domains, actually any power $p \in (0,1]$ makes λ_G^{-p} a metric, and for the unit ball $\mathbf{B}^n$ the same is true for $p \in (0, 1/(n-1)]$, see [AnVaVu, 16.1,16.2]. Also, in [Le] it was shown that $\lambda_G^{-1/n}$ is a metric for any proper subdomain $G \subsetneq \mathbf{R}^n$. The question of Gromov hyperbolicity in these other cases remains unsolved.

4. The modulus metric $\mu_{\mathbf{B}^n}$

Also for the modulus metric, we have an explicit formula in the case of the unit ball, and here a similar proof as for the Ferrand metric can be carried out to show Gromov hyperbolicity. In this case, however, we are allowed to work in slightly more general domains, since in general removing a set of capacity zero does not affect the modulus metric.

Theorem 4.1. *Let $E \subset \mathbf{B}^n$ be a compact set with $\operatorname{cap} E = 0$, and let $G = \mathbf{B}^n \setminus E$. Then the metric space (G, μ_G) is Gromov δ-hyperbolic, with Gromov constant*

$$\delta \le 2^{n-1} c_n \log 12.$$

Especially, if $G \subsetneq \mathbf{R}^2$ is a domain of type $G = D \setminus E$, where D is simply connected and E is compact with $\operatorname{cap} E = 0$, then (G, μ_G) is Gromov hyperbolic with

$$\delta \le \frac{4 \log 12}{\pi} \approx 3.1639.$$

Proof. We first note that by the definition of the hyperbolic tangent we have

$$\log\left(\frac{\frac{1}{\tanh x}+1}{\frac{1}{\tanh x}-1}\right)=\log\left(\frac{1+\tanh x}{1-\tanh x}\right)=2x. \tag{4.2}$$

Using Definition (2.1) and the inequality (2.4) together with (4.2) we get

$$\begin{aligned}&\mu_{\mathbf{B}^n}(x,z)+\mu_{\mathbf{B}^n}(y,w)\\&=\gamma_n\Big(\frac{1}{\tanh\frac{1}{2}\rho(x,z)}\Big)+\gamma_n\Big(\frac{1}{\tanh\frac{1}{2}\rho(y,w)}\Big)\\&\le 2^n c_n\log 4+2^{n-1}c_n\big(\rho(x,z)+\rho(w,y)\big)\end{aligned}$$

Now, the proof may be carried out exactly as in the proof of Theorem 3.4, as the linear function $f\colon x\mapsto 2x$ clearly satisfies $f(x+y)=f(x)+f(y)$ for all $x,y\in\mathbf{R}$. Then we get

$$\begin{aligned}&\mu_{\mathbf{B}^n}(x,z)+\mu_{\mathbf{B}^n}(y,w)\\&\le\big(\mu_{\mathbf{B}^n}(x,w)+\mu_{\mathbf{B}^n}(y,z)\big)\vee\big(\mu_{\mathbf{B}^n}(x,y)+\mu_{\mathbf{B}^n}(w,z)\big)+2^n c_n\log 12.\end{aligned}$$

For $n=2$ we may again use the Riemann mapping theorem and conformal invariance to get Gromov hyperbolicity for all simply connected domains. The fact that $\mu_{\mathbf{B}^n}=\mu_G$, and in the two-dimensional case $\mu_D=\mu_G$ follows directly from the definitions of capacity and modulus, and from the fact that (zero) capacity is preserved in conformal mappings. □

Remark 4.3. Note that the upper bounds involving the Grötzsch constant λ_n obtained for the Gromov delta in the cases $\lambda^*_{\mathbf{B}^n}$ and $\lambda^*_{\mathbf{R}^n_*}$ grow without bound as $n\to\infty$. This follows from the fact that the function $n\mapsto\lambda_n$ grows without bound (see [AnVaVu, 12.37]), and the result that $\omega_{n-1}^{1/(1-n)}$ is strictly increasing. Namely, let $a<b$ be real numbers. Then clearly

$$a^{1/(1-x)}>b^{1/(1-x)},\quad\text{for all } x\ge 1.$$

It is known that ω_{n-1} increases for $2\le n\le 7$ and decreases for $n\ge 7$ ([AnVaVu, 2.28]), and that any function $a^{1/(1-x)}$ is strictly increasing in $[1,\infty)$. Thus, for $n\ge 8$ we get

$$\omega_{n-1}^{1/(1-n)}>\omega_{n-2}^{1/(1-n)}>\omega_{n-2}^{1/(1-(n-1))},$$

and thus $\omega_{n-1}^{1/(1-n)}$ is strictly increasing for $n\ge 7$. For values $2\le n\le 8$ this is also true, and can easily be checked by computing the values (see [AnVaVu, p. 44]).

Contrary to the Ferrand metric, the upper bound obtained for $\mu_{\mathbf{B}^n}$ in fact approaches zero as n grows, which follows from [AnVaVu, 2.34]. As the function $2^n c_n$ is decreasing, the constant $4\log 12/\pi$ is an upper bound for the Gromov delta of $(\mathbf{B}^n,\mu_{\mathbf{B}^n})$ for all $n\ge 3$, but of course a better constant can easily be calculated for the higher dimensions.

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