Annales Academiæ Scientiarum Fennicæ
Mathematica
Volumen 29, 2004, 295–306

LEBESGUE POINTS IN VARIABLE EXPONENT SPACES

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Abstract. In this paper we prove that the concept of Lebesgue points generalizes naturally to the setting of variable exponent Lebesgue and Sobolev spaces. We assume that the variable exponent is log-Hölder continuous, which, although restrictive, is a common assumption in variable exponent spaces.

1. Introduction

In recent years there has been a great upswing of interest and research in variable exponent Lebesgue and Sobolev spaces. Due to these efforts many classical questions are now understood quite well also in the variable exponent case, for instance potential and maximal operators and Sobolev and Poincaré inequalities cf. [7], [8], [9], [10], [13], [14], [15], [19], [26]. In parallel with the study of the spaces there has also been increasing interest in studying related differential equations under generalized regularity conditions cf. [1], [2], [3], [11]. Both of these issues are also related to the modeling of electro-rheological fluids, cf. [24].

Despite impressive advances, some classical questions have remained completely unstudied in variable exponent spaces, escaping without even a mention. The topic of this paper, Lebesgue points, belongs to this category. Lebesgue points are important since they allow us to move beyond average estimates to pointwise estimates of Lebesgue and Sobolev functions.

Lebesgue points in Lebesgue spaces, the topic of Section 3, are quite simple to handle and require no in-depth knowledge of variable exponent spaces. We show that if the exponent is bounded then almost every point is a Lebesgue point. In Section 4 we study Lebesgue points in Sobolev spaces. In order to say anything useful about these we need some sort of capacity. A suitable variable exponent Sobolev type capacity was introduced only recently by Harjulehto, Hästö, Koskenoja and Varonen [15]. This is one of the reasons that Lebesgue points have not been previously studied in variable exponent spaces. Another important reason is the lack of tools for approaching this question in a local manner. In this paper we will adapt methods from a likewise very recent paper by Kinnunen

2000 Mathematics Subject Classification: Primary 46E35.

and Latvala, [18]. We prove in Theorems 4.6 and 4.12 that Sobolev functions behave pointwise as we would expect from classical theory, provided the exponent is log-Hölder continuous, i.e. we show that

$$\lim_{r\to 0} \fint_{B(x.r)} |u(y) - u^*(x)|^{p^*(y)}\, dy = 0$$

quasieverywhere, where u^* is the quasicontinuous representative of $u \in W^{1,p(\,\cdot\,)}(\mathbf{R}^n)$ and p^* is the pointwise Sobolev conjugate exponent of p. We start by giving the necessary definitions in Section 2.

2. Notation and definitions

We denote by $\mathbf{R}^n$ the Euclidean space of dimension $n \geqslant 2$. For $x \in \mathbf{R}^n$ and $r > 0$ we denote by $B(x,r)$ the open ball with center x and radius r. For $u \in L^1(\mathbf{R}^n)$ and $E \subset \mathbf{R}^n$ of positive measure we denote

$$u_E = \fint_E |u(x)|\, dx = \frac{1}{|E|}\int_E |u(x)|\, dx.$$

We will next introduce variable exponent Lebesgue and Sobolev spaces in $\mathbf{R}^n$; note that we nevertheless use the standard definitions of the spaces $L^p(\Omega)$ and $W^{1,p}(\Omega)$ for fixed exponent $p \geqslant 1$ and open $\Omega \subset \mathbf{R}^n$.

Let $p\colon \mathbf{R}^n \to [1,\infty)$ be a measurable function (called the *variable exponent* on $\mathbf{R}^n$). Throughout this paper the function p always denotes a variable exponent; also, we define $p^+ = \operatorname{ess\,sup}_{x\in\mathbf{R}^n} p(x)$ and $p^- = \operatorname{ess\,inf}_{x\in\mathbf{R}^n} p(x)$. We define the *variable exponent Lebesgue space* $L^{p(\,\cdot\,)}(\mathbf{R}^n)$ to consist of all measurable functions $u\colon \mathbf{R}^n \to \mathbf{R}$ such that $\varrho_{p(\,\cdot\,)}(\lambda u) = \int_{\mathbf{R}^n} |\lambda u(x)|^{p(x)}\, dx < \infty$ for some $\lambda > 0$. The function $\varrho_{p(\,\cdot\,)}\colon L^{p(\,\cdot\,)}(\mathbf{R}^n) \to [0,\infty)$ is called the *modular* of the space $L^{p(\,\cdot\,)}(\mathbf{R}^n)$. We define a norm, the so-called *Luxemburg norm*, on this space by the formula

$$\|u\|_{p(\,\cdot\,)} = \inf\bigl\{\lambda > 0 : \varrho_{p(\,\cdot\,)}(u/\lambda) \leqslant 1\bigr\}.$$

The *variable exponent Sobolev space* $W^{1,p(\,\cdot\,)}(\mathbf{R}^n)$ is the subspace of functions $u \in L^{p(\,\cdot\,)}(\mathbf{R}^n)$ whose distributional gradient exists almost everywhere and satisfies $|\nabla u| \in L^{p(\,\cdot\,)}(\mathbf{R}^n)$. The function $\varrho_{1,p(\,\cdot\,)}\colon W^{1,p(\,\cdot\,)}(\mathbf{R}^n) \to [0,\infty)$ is defined by $\varrho_{1,p(\,\cdot\,)}(u) = \varrho_{p(\,\cdot\,)}(u) + \varrho_{p(\,\cdot\,)}(|\nabla u|)$. The norm $\|u\|_{1,p(\,\cdot\,)} = \|u\|_{p(\,\cdot\,)} + \|\nabla u\|_{p(\,\cdot\,)}$ makes $W^{1,p(\,\cdot\,)}(\mathbf{R}^n)$ a Banach space. For more details on the variable exponent spaces see [20].

In [15] Harjulehto, Hästö, Koskenoja and Varonen introduced a Sobolev capacity in the variable exponent Sobolev space, which is defined as follows. Suppose that E is an arbitrary subset of $\mathbf{R}^n$. We denote

$$S_{p(\,\cdot\,)}(E) = \bigl\{u \in W^{1,p(\,\cdot\,)}(\mathbf{R}^n) : u \geqslant 1 \text{ in an open set containing } E\bigr\}.$$

The *Sobolev $p(\cdot)$-capacity* of E is defined by

$$C_{p(\cdot)}(E) = \inf_{u \in S_{p(\cdot)}(E)} \int_{\mathbf{R}^n} \big(|u(x)|^{p(x)} + |\nabla u(x)|^{p(x)}\big)\, dx.$$

In case $S_{p(\cdot)}(E) = \emptyset$, we set $C_{p(\cdot)}(E) = \infty$. If $1 < p^- \le p^+ < \infty$, then the Sobolev $p(\cdot)$-capacity is an outer measure and Choquet capacity [15, Corollaries 3.3 and 3.4]. As in the fixed exponent case the capacity is a finer measure than the n-dimensional Lebesgue measure, cf. [15, Section 4]. We say that a claim holds *quasieverywhere* if it holds except in a set of capacity zero. A function $u\colon \Omega \to \mathbf{R}$ is said to be *quasicontinuous* if for every $\varepsilon > 0$ there exists an open set $U \subset \Omega$ with $C_{p(\cdot)}(U) < \varepsilon$ such that u is continuous in $\Omega \setminus U$.

3. Lebesgue spaces

Although functions in L^p are not in general continuous, they do possess the following mean-continuity property: for $u \in L^p_{\mathrm{loc}}(\mathbf{R}^n)$ we have

$$\lim_{r \to 0} \fint_{B(x,r)} |u(y) - u(x)|^p \, dy = 0$$

for almost every x. The points x at which this property holds are called *Lebesgue points.*

The next theorem generalizes the concept of Lebesgue points to the variable exponent Lebesgue spaces. Our proof is standard and is based on the following fact: in L^1 almost every point is a Lebesgue point.

3.1. Theorem. *Let $p^+ < \infty$. If $u \in L^{p(\cdot)}(\mathbf{R}^n)$, then*

$$\lim_{r \to 0} \fint_{B(x,r)} |u(y) - u(x)|^{p(y)} \, dy = 0$$

for almost every $x \in \mathbf{R}^n$.

Proof. Let $\{r_i\}_{i=1}^\infty$ be a countable dense subset of $\mathbf{R}$. Since $p^+ < \infty$, we conclude that $|u(\cdot) - r_i|^{p(\cdot)} \in L^1_{\mathrm{loc}}(\mathbf{R}^n)$. Thus for every i there exists $E_i \subset \mathbf{R}^n$ of measure zero such that

$$\lim_{r \to 0} \fint_{B(x,r)} |u(y) - r_i|^{p(y)} \, dy = |u(x) - r_i|^{p(x)} \tag{3.2}$$

for every $x \in \mathbf{R}^n \setminus E_i$. Denote $E = \bigcup_{i=1}^\infty E_i$ and note that $|E| = 0$. Then (3.2) holds for every $x \in \mathbf{R}^n \setminus E$ and every i.

Let $0 < \varepsilon < 1$ and $x \in \mathbf{R}^n \setminus E$. We choose r_i so that $|u(x) - r_i| < \varepsilon/2^{p^+ +1}$ and obtain

$$\begin{aligned}
\limsup_{r\to 0} & \fint_{B(x,r)} |u(y) - u(x)|^{p(y)}\,dy \\
&\le 2^{p^+}\Big(\limsup_{r\to 0} \fint_{B(x,r)} |u(y) - r_i|^{p(y)}\,dy + \fint_{B(x,r)} |r_i - u(x)|^{p(y)}\,dy\Big) \\
&\le 2^{p^+}\big(|u(x) - r_i|^{p(x)} + |u(x) - r_i|\big) \\
&\le 2^{p^+ +1}|u(x) - r_i| < \varepsilon,
\end{aligned}$$

and so x is a Lebesgue point. □

3.3. Remark. Since being a Lebesgue point is a local property, it suffices to assume that $u \in L^{p(\cdot)}_{\mathrm{loc}}$ and that $\operatorname{ess\,sup}_{x\in K} p(x) < \infty$ for compact $K \subset \mathbf{R}^n$ in the previous theorem.

4. Sobolev spaces

In this section we consider Lebesgue points of functions in Sobolev spaces. We proceed as follows: First we note, using a result of Kinnunen [17], that the Hardy–Littlewood maximal function of a Sobolev function is a Sobolev function. This yields a capacity weak type estimate of the Hardy–Littlewood maximal function. Using these results we prove that

$$\lim_{r\to 0} \fint_{B(x.r)} u(y)\,dy = u^*(x)$$

exists quasieverywhere and u^* is the quasicontinuous representative of u. Finally we show that

$$\lim_{r\to 0} \fint_{B(x.r)} |u(y) - u^*(x)|^{p^*(y)}\,dy = 0$$

quasieverywhere in $\{x \in \mathbf{R}^n : p(x) < n\}$. Here p^* is the pointwise Sobolev conjugate exponent. To use these methods we need to make some assumptions on the exponent and therefore we start by defining some conditions.

4.1. Definition. We say that the variable exponent p is log*-Hölder continuous* if there exists a constant $C > 0$ such that

$$|p(x) - p(y)| \le \frac{C}{-\log|x-y|}$$

for every $x, y \in \mathbf{R}^n$, $|x - y| \le \frac{1}{2}$.

Note that log-Hölder continuous functions are sometimes called weak Lipschitz or Dini–Lipschitz continuous functions. However, this terminology obscures the clear relationship to Hölder continuity and will not be used in this paper.

4.2. Definition. We say that p satisfies *condition* $\mathscr{M}$ if $1 < p^- \le p^+ < \infty$, p is log-Hölder continuous and there exists a constant $C > 0$ such that

$$|p(x) - p(y)| \le \frac{C}{\log(e + |x|)}$$

for every $x, y \in \mathbf{R}^n$, $|y| \ge |x|$.

Condition $\mathscr{M}$ and log-Hölder continuity have appeared in several places in the study of variable exponent spaces. Cruz-Uribe, Fiorenze and Neugebauer showed, following the work of Diening [6] and Nekvinda [21], that the condition $\mathscr{M}$ is sufficient for the Hardy–Littlewood maximal operator to be bounded from $L^{p(\cdot)}(\mathbf{R}^n)$ to itself [4, Theorem 1.5] (see also [5]). log-Hölder continuity is somehow crucial for the boundedness of the Hardy–Littlewood maximal operator, as was shown by Pick and Růžička [23], whereas Nekvinda gave an example showing that the decay condition is not necessary [22]. Samko [25, Theorem 3] and Fan and Zhao [11, Theorem 3.2] proved, independently, that $C_0^\infty(\mathbf{R}^n)$ is dense in $W^{1,p(\cdot)}(\mathbf{R}^n)$ provided p is log-Hölder continuous.

For $G \subset \mathbf{R}^n$ we define $p_G^- = \operatorname{ess\,inf}_{x\in G} p(x)$ and $p_G^+ = \operatorname{ess\,sup}_{x\in G} p(x)$. Using these quantities, Diening gave the following geometric interpretation of log-Hölder continuity:

4.3. Lemma ([6, Lemma 3.2]). *Let* $p\colon \mathbf{R}^n \to [1, \infty)$. *The following conditions are equivalent*:

(1) p *is* log*-Hölder continuous.*

(2) *There exists a constant* c *such that* $|B|^{p_B^- - p_B^+} \le c$ *for all open balls* B.

The following proposition is an adaptation to the variable exponent case of results of J. Kinnunen from [17]. The proof follows easily from the fixed exponent case.

4.4. Proposition. *Suppose* p *satisfies condition* $\mathscr{M}$. *If* $u \in W^{1,p(\cdot)}(\mathbf{R}^n)$, *then* $\mathscr{M}u \in W^{1,p(\cdot)}(\mathbf{R}^n)$ *and* $|\nabla \mathscr{M}u(x)| \le \mathscr{M}|\nabla u(x)|$ *for almost every* $x \in \mathbf{R}^n$.

Proof. Since $u \in W^{1,1}_{\mathrm{loc}}(\mathbf{R}^n)$, it follows from [17] that $|\nabla \mathscr{M}u(x)| \le \mathscr{M}|\nabla u(x)|$ for almost every $x \in \mathbf{R}^n$. Since $|\nabla u| \in L^{p(\cdot)}(\mathbf{R}^n)$, it follows by [4, Theorem 1.5] that $\mathscr{M}|\nabla u| \in L^{p(x)}(\mathbf{R}^n)$. Since $|\nabla \mathscr{M}u| \le \mathscr{M}|\nabla u|$ pointwise a.e., this implies that $|\nabla \mathscr{M}u| \in L^{p(x)}(\mathbf{R}^n)$, as well. It follows from [4, Theorem 1.5] that $\mathscr{M}u \in L^{p(x)}(\mathbf{R}^n)$ and thus $\mathscr{M}u \in W^{1,p(x)}(\mathbf{R}^n)$. □

In the remaining part of this article we will adapt the proof of [18, Theorem 4.5] by J. Kinnunen and V. Latvala to variable exponent spaces. For simplicity of exposition, we split their result into two parts, Theorems 4.6 and 4.12. The proof of the first of these is nearly the same as in the fixed exponent case.

4.5. Proposition. *Suppose p satisfies condition $\mathscr{M}$. Then for every $\lambda > 0$ and every $u \in W^{1,p(\cdot)}(\mathbf{R}^n)$ we have*

$$C_{p(\cdot)}\big(\{x \in \mathbf{R}^n : \mathscr{M}u(x) > \lambda\}\big) \le c \max\left\{ \left\| \frac{u}{\lambda} \right\|_{1,p(\cdot)}, \left\| \frac{u}{\lambda} \right\|_{1,p(\cdot)}^{p^+} \right\}.$$

Proof. Since $\mathscr{M}u$ is lower semi-continuous, the set $\{x \in \mathbf{R}^n : \mathscr{M}u(x) > \lambda\}$ is open for every $\lambda > 0$. By Proposition 4.4 we can use $\mathscr{M}u/\lambda = \mathscr{M}u/\lambda$ as a test function for the capacity. This yields, by [12, Theorem 1.3],

$$\begin{aligned} C_{p(\cdot)}\big(\{x \in \mathbf{R}^n : \mathscr{M}u(x) > \lambda\}\big) &\le \varrho_{1,p(\cdot)}\left(\mathscr{M}\frac{u}{\lambda}\right) \\ &\le \max\left\{ \left\| \mathscr{M}\frac{u}{\lambda} \right\|_{1,p(\cdot)}, \left\| \mathscr{M}\frac{u}{\lambda} \right\|_{1,p(\cdot)}^{p^+} \right\}. \end{aligned}$$

Now the claim follows by Proposition 4.4 and [4, Theorem 1.5]. □

4.6. Theorem. *Suppose p satisfies condition $\mathscr{M}$ and let $u \in W^{1,p(\cdot)}(\mathbf{R}^n)$. Then there exists a set $E \subset \mathbf{R}^n$ of zero $p(\cdot)$-capacity such that*

$$u^*(x) = \lim_{r\to 0} \fint_{B(x,r)} u(y)\,dy$$

exists for every $x \in \mathbf{R}^n \setminus E$. The function u^* is the $p(\cdot)$-quasicontinuous representative of u.

Proof. Since smooth functions are dense in $W^{1,p(\cdot)}(\mathbf{R}^n)$ [25, Theorem 3], we can choose a sequence $\{u_i\}$ of continuous functions in $W^{1,p(\cdot)}(\mathbf{R}^n)$ with $\|u - u_i\|_{p(\cdot)} \le 2^{-2i}$. For $i = 1, 2, \dots$ denote

$$A_i = \big\{x \in \mathbf{R}^n : \mathscr{M}(u - u_i)(x) > 2^{-i}\big\}, \quad B_i = \bigcup_{j=i}^{\infty} A_j \quad \text{and} \quad E = \bigcap_{j=1}^{\infty} B_j.$$

Proposition 4.5 implies that $C_{p(\cdot)}(A_i) \le c2^{-i}$, the subadditivity of $C_{p(\cdot)}$ implies that $C_{p(\cdot)}(B_i) \le c2^{1-i}$ and [15, Theorem 3.2(vi)] implies that $C_{p(\cdot)}(E) = 0$.

We next consider the relationship between u and u_i outside these sets. We have

$$|u_i(x) - u_{B(x,r)}| \le \fint_{B(x,r)} |u_i(x) - u_i(y)|\,dy + \fint_{B(x,r)} |u_i(y) - u(y)|\,dy.$$

Since u_i is continuous, the first term in the upper bound goes to zero with r and so we get

$$\limsup_{r\to 0} |u_i(x) - u_{B(x,r)}| \le \mathscr{M}(u_i - u)(x).$$

Thus we have $\limsup_{r\to 0}|u_i(x)-u_{B(x,r)}|\le 2^{-i}$ for $x\in \mathbf{R}^n\setminus A_i$. It follows that $\{u_i\}$ converges uniformly on $\mathbf{R}^n\setminus B_j$ for every $j>0$. Denote the limit function, which is continuous in every B_j, by u^*. Then

$$\limsup_{r\to 0}|u^*(x)-u_{B(x,r)}|\le |u^*(x)-u_i(x)|+\limsup_{r\to 0}|u_i(x)-u_{B(x,r)}|.$$

As $i\to\infty$ the right-hand side of the previous equation tends to 0 for $x\in\mathbf{R}^n\setminus B_k$. Since the left-hand side does not depend on i, this means that it equals 0, so that $u^*(x)=\lim_{r\to 0}u_{B(x,r)}$ for all $x\in\mathbf{R}^n\setminus B_k$. Since this holds in the complement of every B_k, it holds in the complement of E as well. Since E has capacity zero, we are done with the existence part. Since u^* is continuous in every $\mathbf{R}^n\setminus B_k$, the claim regarding quasicontinuity is clear. □

To prove the other part of Theorem 4.5 from [18] we need some auxiliary lemmata. The idea of these lemmata is that the log-Hölder continuity implies that we can treat p as a constant locally, and this incurs a penalty of only a multiplicative constant.

4.7. Lemma. *Suppose that p is* log*-Hölder continuous. For $r\le 1$ we have*

$$C_{p(\,\cdot\,)}\big(B(x,r)\big)\le c\int_{B(x,r/5)} r^{-p(y)}\,dy,$$

where c depends on p and n.

Proof. Let u be a function which equals 1 on $B(x,r)$, $2-|y-x|/r$ on $B(x,2r)\setminus B(x,r)$ and 0 otherwise. Then u is a suitable test function for the capacity of $B(x,r)$. Using Lemma 4.3 for the last inequality, we find that

$$\begin{aligned}
C_{p(\,\cdot\,)}\big(B(x,r)\big) &\le \varrho_{1,p(\,\cdot\,)}(u)\le |B(x,2r)|+\int_{B(x,2r)} r^{-p(y)}\,dy\\
&\le 2\int_{B(x,2r)} r^{-p^+_{B(x,2r)}}\,dy = 2r^{-p^+_{B(x,2r)}}|B(x,2r)|\\
&\le 2\cdot 10^n r^{p^-_{B(x,2r)}-p^+_{B(x,2r)}}\int_{B(x,r/5)} r^{-p^-_{B(x,2r)}}\,dy\\
&\le C(p)10^n\int_{B(x,r/5)} r^{-p(y)}\,dy. \ \square
\end{aligned}$$

4.8. Lemma. *Suppose that p is* log*-Hölder continuous. Then there exists a constant $c\ge 1$ such that*

$$\frac{1}{c}\le \liminf_{r\to 0} r^{p(x)}\fint_{B(x,r)} r^{-p(y)}\,dy\le \limsup_{r\to 0} r^{p(x)}\fint_{B(x,r)} r^{-p(y)}\,dy\le c$$

for every $x\in\mathbf{R}^n$.

Proof. We have

$$\limsup_{r\to 0} r^{p(x)} \fint_{B(x,r)} r^{-p(y)}\,dy \le \limsup_{r\to 0} \sup_{y\in B(x,r)} r^{p(x)-p(y)} \le c,$$

where the second inequality follows from Lemma 4.3. The lower bound is derived similarly. □

The following lemma corresponds to Lemma 4.3 of [18]. The proof is also quite similar, although some extra work is needed to take care of the variability of the exponent.

4.9. Lemma. *Suppose that p is* log*-Hölder continuous and let $u \in W^{1,p(\,\cdot\,)}(\mathbf{R}^n)$. Then*

$$C_{p(\,\cdot\,)}\biggl(\biggl\{x\in\mathbf{R}^n : \limsup_{r\to 0} r^{p(x)} \fint_{B(x,r)} |\nabla u(y)|^{p(y)}\,dy > 0\biggr\}\biggr) = 0.$$

Proof. Let $\delta\in(0,1)$, $\varepsilon>0$ and

$$E_\varepsilon = \biggl\{x\in\mathbf{R}^n : \limsup_{r\to 0} r^{p(x)} \fint_{B(x,r)} |\nabla u(y)|^{p(y)}\,dy > \varepsilon\biggr\}.$$

For every $x\in E_\varepsilon$ there exists an arbitrarily small $r_x\in(0,\delta)$ such that

$$r_x^{p(x)} \fint_{B(x,r_x)} |\nabla u(y)|^{p(y)}\,dy > \varepsilon.$$

By choosing smaller r_x if necessary, we may, on account of Lemma 4.8 and the previous inequality, assume that

$$\int_{B(x,r_x)} |\nabla u(y)|^{p(y)}\,dy > \frac{\varepsilon}{c}\int_{B(x,r_x)} r^{-p(y)}\,dy, \tag{4.10}$$

where c does not depend on x or r_x.

By the Vitali covering theorem there exists a countable subfamily of pair-wise disjoint balls $B(x_i, r_{x_i})$ such that

$$E_\varepsilon \subset \bigcup_{i=1}^\infty B(x_i, 5r_{x_i}).$$

Denote $r_i = r_{x_i}$ and $B_i = B(x_i, r_i)$. By subadditivity and Lemma 4.7 we conclude that

$$C_{p(\,\cdot\,)}(E_\varepsilon) \le \sum_{i=1}^\infty C_{p(\,\cdot\,)}\bigl(B(x_i,5r_i)\bigr) \le c\sum_{i=1}^\infty \int_{B_i} r_i^{-p(y)}\,dy.$$

It follows from this and (4.10) that

$$C_{p(\,\cdot\,)}(E_\varepsilon) \le \frac{c}{\varepsilon}\sum_{i=1}^{\infty}\int_{B_i}|\nabla u(y)|^{p(y)}\,dy = \frac{c}{\varepsilon}\int_{\bigcup_{i=1}^{\infty}B_i}|\nabla u(y)|^{p(y)}\,dy. \tag{4.11}$$

As in [18] we then find, by the disjointness of the balls B_i, that

$$\Bigl|\bigcup_{i=1}^{\infty}B_i\Bigr| = \sum_{i=1}^{\infty}|B_i| < \sum_{i=1}^{\infty}\frac{r_i^{p(x_i)}}{\varepsilon}\int_{B_i}|\nabla u(y)|^{p(y)}\,dy \le \frac{\delta^{p^-}}{\varepsilon}\int_{\mathbf{R}^n}|\nabla u(y)|^{p(y)}\,dy.$$

Hence $\bigl|\bigcup_{i=1}^{\infty}B_i\bigr| \to 0$ as $\delta\to 0$, which by (4.11) implies that $C_{p(\,\cdot\,)}(E_\varepsilon) = 0$ for every $\varepsilon > 0$. Therefore it follows by subadditivity that $C_{p(\,\cdot\,)}(E_0) = C_{p(\,\cdot\,)}\bigl(\bigcup_{i\in\mathbf{N}}E_{1/i}\bigr) = 0$, which was to be shown. □

In the next theorem we denote by p^* the pointwise Sobolev conjugate of p, i.e. $p^*(x) = np(x)/\bigl(n-p(x)\bigr)$, for $p(x) < n$.

4.12. Theorem. *Suppose p satisfies condition $\mathscr{M}$ and let $u\in W^{1,p(\,\cdot\,)}(\mathbf{R}^n)$. Then there exists a set $E\subset\mathbf{R}^n$, $C_{p(\,\cdot\,)}(E)=0$, such that*

$$\lim_{r\to 0}\fint_{B(x,r)}|u(y)-u^*(x)|^{p^*(y)}\,dy = 0$$

for every $x\in\bigl\{x\in\mathbf{R}^n : p(x)<n\bigr\}\setminus E$.

Proof. Define

$$E = \Bigl\{x\in\mathbf{R}^n\colon \limsup_{r\to 0} r^{p(x)}\fint_{B(x,r)}|\nabla u(y)|^{p(y)}\,dy > 0\Bigr\}.$$

Then $C_{p(\,\cdot\,)}(E)=0$ by Lemma 4.9. We show that

$$\limsup_{r\to 0} r^{p(x)}\fint_{B(x,r)}|\nabla u(y)|^{p(y)}\,dy = 0$$

$$\Rightarrow$$

$$\limsup_{r\to 0}\fint_{B(x,r)}|u(y)-u_{B(x,r)}|^{p^*(y)}\,dy = 0$$

when $p(x)<n$, from which the claim clearly follows by Theorem 4.6.

Diening [7, Theorem 5.2] has shown that condition $\mathscr{M}$ implies the Sobolev inequality. Harjulehto and Hästö [14, Corollary 2.10] showed that condition $\mathscr{M}$

implies the Poincaré inequality. Combining these we get the Sobolev–Poincaré inequality

$$\|u-u_B\|_{L^{p^*(\cdot)}(B)} \le c\|u-u_B\|_{W^{1,p(\cdot)}(B)} \le c\|\nabla u\|_{L^{p(\cdot)}(B)},$$

where we denoted $B=B(x,r)$. From this and [12, Theorem 1.3] we conclude that

$$\varrho_{p^*(\cdot)}(u-u_B)^{1/p_B^{*-}} \le c\varrho_{p(\cdot)}(\nabla u)^{1/p_B^+}$$

(where the modulars are taken in B only). Hence

$$\begin{aligned}\fint_B |u(y)-u_B|^{p^*(y)}\,dy &= cr^{-n}\varrho_{p^*(\cdot)}(u-u_B)\\ &\le cr^{-n}\varrho_{p(\cdot)}(\nabla u)^{p_B^{*-}/p_B^+}\\ &= cr^{(n-p(x))p_B^{*-}/p_B^+-n}\left(r^{p(x)}\fint_B |\nabla u(y)|^{p(y)}\,dy\right)^{p_B^{*-}/p_B^+}.\end{aligned}$$

We see that it suffices to show that $r^{(n-p(x))p_B^{*-}/p_B^+-n} \le c$ as $r\to 0$. Since $p_B^{*-}=(p_B^-)^*$ we see that this is equivalent to

$$n\left(\frac{n-p(x)}{n-p_B^-}\frac{p_B^-}{p_B^+}-1\right)\log r \le c$$

at the same limit. We have

$$\frac{n-p(x)}{n-p_B^-}\frac{p_B^-}{p_B^+}-1 \ge \frac{n-p_B^+}{n-p_B^-}\frac{p_B^-}{p_B^+}-1 = \frac{n}{p_B^+(n-p_B^-)}(p_B^- - p_B^+).$$

Thus

$$\limsup_{r\to 0} n\left(\frac{n-p(x)}{n-p_B^-}\frac{p_B^-}{p_B^+}-1\right)\log r \le \frac{n^2}{p(x)\big(n-p(x)\big)}\limsup_{r\to 0}(p_B^- - p_B^+)\log r \le c,$$

where the last inequality is just Diening's condition from Lemma 4.3. □

4.13. Remark. It again suffices to assume that $u\in W^{1,p(\cdot)}_{\mathrm{loc}}(\mathbf{R}^n)$. It seems likely that we can also replace the assumptions on p by corresponding local ones, using the techniques of [16], but we will not get into that here.

4.14. Remark. If $p(x)>n$ then there exists $r_x>0$ such that

$$W^{1,p(\cdot)}\big(B(x,r_x)\big) \hookrightarrow W^{1,n+(p(x)-n)/2}\big(B(x,r_x)\big)$$

and hence u is continuous in a neighborhood of x, so that

$$\operatorname*{ess\,sup}_{y\in B(x,r)} |u(y)-u^*(x)| \to 0$$

as $r\to 0$. For $p(x)=n$ the theorem gives

$$\lim_{r\to 0}\fint_{B(x,r)} |u(y)-u^*(x)|^q\,dy = 0$$

outside the set E for any finite q. In this case we do not have zero supremum norm even in the fixed exponent case.

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Received 20 August 2003