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A GENERALIZED NEVANLINNA THEOREM FOR SUPERTEMPERATURES

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Abstract. A new proof of Nevanlinna's first fundamental theorem for supertemperatures enables us to generalize the result to cover the case of the mean values of supertemperatures over level surfaces of Green functions generally. Generalizations in another direction are also obtained.

1. Introduction

In [11], an analogue for supertemperatures of Nevanlinna's First Fundamental Theorem for superharmonic functions was presented. A new and much simpler proof has now been discovered, which enables us to generalize the theorem to include the general case of the mean values of supertemperatures over level surfaces of Green functions considered in [10]. This is the result of Theorem 1 below, and it is followed by an application to thermic majorization in Theorem 2. Subsequently, all the theorems in [11] are generalized in this way, and those which are concerned with supertemperatures on lower half-spaces $\mathbf{R}^n \times]-\infty, a[$ are extended to results on sets of the form $\Lambda(p_0, D)$ which appear in the statement of the strong minimum principle, where D is any open set that is Dirichlet regular for the adjoint heat operator $\sum_{i=1}^n D_i^2 + D_t$.

Generalizations in another direction are also obtained. Instead of the quotients of surface mean values that appear in [11, Theorem 2], we use quotients of differences of the generalized mean values below. This renders the condition "$v(p_0) = \infty$" redundant, and the corresponding infinity conditions of [11, Theorems 3 and 4] are also absent from the generalizations below. One consequence of this is that, whereas [11, Theorem 4] gave Hausdorff measure estimates of certain polar sets, in the corresponding result below the sets need not be polar.

We work in $\mathbf{R}^{n+1}$, a typical point of which we usually denote by p or q, and rarely by (x, t) with $x \in \mathbf{R}^n$ and $t \in \mathbf{R}$. We denote by D an open subset of $\mathbf{R}^{n+1}$ that is Dirichlet regular for the adjoint heat operator, and by G_D its Green function for the heat operator $\sum_{i=1}^n D_i^2 - D_t$ (but in the case $D = \mathbf{R}^{n+1}$ we omit the subscript). All our positive measures are locally finite Borel measures, and our signed measures are differences of pairs of positive measures. The terms 'positive'

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and 'increasing' are used in the wide sense. Given a positive measure μ on D, its Green potential $G_D\mu$ is defined by

$$G_D\mu(p) = \int_D G_D(p,q)\,d\mu(q)$$

for all $p \in D$. A temperature is a solution of the heat equation, and a supertemperature is the corresponding analogue of a superharmonic function. A thermic minorant of a supertemperature w is a temperature u such that $u \le w$. We denote by E an open subset of $\mathbf{R}^{n+1}$ (that is not necessarily Dirichlet regular). If w is a supertemperature on E, then the Riesz Decomposition Theorem associates with w a positive measure on E, which we call the Riesz measure for w. (See [6] and [7], or [3], for details.) If v is also a supertemperature on E, and $u = v - w$ whenever the difference is defined (hence on E less a polar set), then u is called a δ-subtemperature on E. If ν and ω are the Riesz measures for v and w respectively, then $\nu - \omega$ will be called the Riesz (signed) measure for u. The Riesz measure for u is uniquely determined [11].

For all $c > 0$, we put $\tau(c) = (4\pi c)^{-n/2}$. Let $p_0 \in D$. There is a positive, bounded solution h of the adjoint heat equation on D such that $G_D(p_0,\cdot\,) = G(p_0,\cdot\,) - h$, so that $G_D(p_0,\cdot\,)$ is infinitely differentiable on $D\backslash\{p_0\}$. It therefore follows from Sard's Theorem ([4, p. 45]) that, for almost every $c > 0$, the set $\{p \in D : G_D(p_0,p) = \tau(c)\}$ is a smooth regular n-dimensional manifold. We call such a value of c a regular value. We put

$$\Omega_D(p_0,c) = \big\{p \in D : G_D(p_0,p) > \tau(c)\big\}$$

and

$$\overline{\Omega}'_D(p_0,c) = \overline{\Omega}_D(p_0,c)\backslash\{p_0\},$$

omitting the subscript if $D = \mathbf{R}^{n+1}$ (in which case $\Omega(p_0,c)$ is the heat ball with centre p_0 and radius c). Because $G_D(p_0,\cdot\,)$ is a solution of the adjoint equation on $D\backslash\{p_0\}$, each set $\Omega_D(p_0,c)$ is Dirichlet regular for that equation. For any regular value of c,

$$\partial\Omega_D(p_0,c) = \big\{p \in D : G_D(p_0,p) = \tau(c)\big\} \cup \{p_0\}.$$

Since $G_D(p_0,\cdot\,)$ is lower semicontinuous on D, each $\Omega_D(p_0,c)$ is an open set. The assumption that D is Dirichlet regular for the adjoint operator implies that $G_D(p_0,\cdot\,)$ can be continuously extended to zero on ∂D, so that $\overline{\Omega}_D(p_0,c) \subseteq D$ for all $c > 0$. Furthermore, $\Omega_D(p_0,c)$ is bounded and connected ([10, p. 167]). If c is a regular value, then the outward unit normal $\nu = (\nu_x,\nu_t)$ to $\partial\Omega_D(p_0,c)$ is given by the standard formula $\nu = -\nabla G_D(p_0,\cdot\,)\|\nabla G_D(p_0,\cdot\,)\|^{-1}$. Therefore, if $\nabla_x u = (D_1u,\ldots,D_nu)$ denotes the gradient in the spatial variables only, we have

$$\langle \nabla_x G_D(p_0,\cdot\,),\nu_x\rangle = -\|\nabla_x G_D(p_0,\cdot\,)\|^2\|\nabla G_D(p_0,\cdot\,)\|^{-1}.$$

This function is bounded on $\partial\Omega_D(p_0,c)\backslash\{p_0\}$ (see [10, pp. 167–8]). The surface mean value over $\partial\Omega_D(p_0,c)$ is defined by

$$\mathscr{M}_D(u,p_0,c)=\int_{\partial\Omega_D(p_0,c)}-\langle\nabla_x G_D(p_0,\cdot\,),\nu_x\rangle u\,d\sigma$$

whenever the integral exists; here σ denotes surface area measure.

2. The generalized Nevanlinna theorem

In this section we present our generalization of [11, Theorem 1], along with some immediate consequences.

Theorem 1. *Let E be an open set, let D be an open superset of E that is Dirichlet regular for the adjoint heat operator, let $p_0\in E$, and let c and d be regular values such that $0<c\le d$ and $\overline{\Omega}_D(p_0,d)\subseteq E$. If w is a supertemperature on E with Riesz measure μ, then*

$$\mathscr{M}_D(w,p_0,c)=\mathscr{M}_D(w,p_0,d)-\int_c^d\tau'(\gamma)\mu\big(\overline{\Omega}'_D(p_0,\gamma)\big)\,d\gamma \tag{1}$$

and

$$w(p_0)=\mathscr{M}_D(w,p_0,d)-\int_0^d\tau'(\gamma)\mu\big(\overline{\Omega}'_D(p_0,\gamma)\big)\,d\gamma. \tag{2}$$

Proof. Let V be a bounded open set such that $\overline{\Omega}_D(p_0,d)\subseteq V$ and $\overline{V}\subseteq E$. By adding a constant if necessary, we may suppose that $w\ge 0$ on V. Then w can be written in the form $w=G\mu_V+u$ on V, where μ_V is the restriction of μ to V extended by zero to $\mathbf{R}^{n+1}$. Since μ_V is finite, $G\mu_V$ is a supertemperature on $\mathbf{R}^{n+1}$. By [7, Theorem 19], there is a temperature v on V such that $G\mu_V=G_D\mu_V+v$ on D, so that $w=G_D\mu_V+h$ on V, where $h=u+v$. Now observe that the means in (1) are finite (by [10, Theorem 2], or [1], [2]), and that $\mathscr{M}_D(h,p_0,c)=h(p_0)$ (by [10, Theorem 1]). It follows that

$$\begin{aligned}\mathscr{M}_D(w,p_0,c)-\mathscr{M}_D(w,p_0,d)&=\mathscr{M}_D(G_D\mu_V,p_0,c)-\mathscr{M}_D(G_D\mu_V,p_0,d)\\&=\int_V\Big(\mathscr{M}_D\big(G_D(\,\cdot\,,q),p_0,c\big)\\&\quad-\mathscr{M}_D\big(G_D(\,\cdot\,,q),p_0,d\big)\Big)\,d\mu(q)\\&=\int_V\Big(\big(\tau(c)\wedge G_D(p_0,q)\big)-\big(\tau(d)\wedge G_D(p_0,q)\big)\Big)\,d\mu(q)\end{aligned}$$

in view of [2, Theorem 2 Corollary], and [1]. By definition of $\Omega_D(p_0,c)$, we have $\tau(c)\wedge G_D(p_0,q)=\tau(c)$ if and only if $q\in\overline{\Omega}'_D(p_0,c)$, so that

$$\begin{aligned}\big(\tau(c)\wedge G_D(p_0,q)\big)&-\big(\tau(d)\wedge G_D(p_0,q)\big)\\ &=\begin{cases}\tau(c)-\tau(d) & \text{if } q\in\overline{\Omega}'_D(p_0,c),\\ G_D(p_0,q)-\tau(d) & \text{if } q\in\overline{\Omega}_D(p_0,d)\backslash\overline{\Omega}_D(p_0,c),\\ 0 & \text{if } q\notin\overline{\Omega}'_D(p_0,d).\end{cases}\end{aligned}$$

Hence

$$\mathscr{M}_D(w,p_0,c)-\mathscr{M}_D(w,p_0,d)=\int_{\Omega_D(p_0,d)}\Big(\big(\tau(c)\wedge G_D(p_0,q)\big)-\tau(d)\Big)\,d\mu(q).$$

If we now put $\lambda(\gamma)=\mu\big(\overline{\Omega}'_D(p_0,\gamma)\big)$ whenever $0<\gamma\le d$, we have

$$\begin{aligned}\mathscr{M}_D(w,p_0,c)-\mathscr{M}_D(w,p_0,d)&=\int_0^d\Big(\big(\tau(c)\wedge\tau(\gamma)\big)-\tau(d)\Big)\,d\lambda(\gamma)\\ &=\Big(\big(\tau(c)\wedge\tau(\gamma)\big)-\tau(d)\Big)\lambda(\gamma)\,\Big|_0^d-\int_c^d\tau'(\gamma)\lambda(\gamma)\,d\gamma\\ &=-\int_c^d\tau'(\gamma)\mu\big(\overline{\Omega}'_D(p_0,\gamma)\big)\,d\gamma,\end{aligned}$$

which proves (1). Making $c\to 0$ in (1), we obtain (2).

The corollaries of [11, Theorem 1] can also be extended to the present situation.

Corollary 1. *Let E be an open set, let D be an open superset of E that is Dirichlet regular for the adjoint heat operator, and let $p_0\in E$. If w is a supertemperature on E, then $\mathscr{M}_D(w,p_0,c)=o\big(\tau(c)\big)$ as $c\to 0$ through regular values.*

The proof is similar to that of the case $D=\mathbf{R}^{n+1}$ in [11].

Corollary 2. *Let D be an open set which is Dirichlet regular for the adjoint heat operator, let $p_0\in D$, let w be a positive supertemperature on an open superset of $\Lambda(p_0,D)\cup\{p_0\}$, and let μ be the Riesz measure for w. Then*

$$\tau(c)\mu\big(\overline{\Omega}'_D(p_0,c)\big)\le\mathscr{M}_D(w,p_0,c)$$

for all regular values of c.

Proof. Note that $G_D(p_0, q) > 0$ if and only if $q \in \Lambda(p_0, D)$ (see [7, Theorem 14], and [3, p. 300]). Therefore $\overline{\Omega}_D(p_0, c) \subseteq \Lambda(p_0, D) \cup \{p_0\}$ for all $c > 0$. It now follows from Theorem 1 that, if c and d are regular values with $c < d$, then

$$\begin{aligned}\mathscr{M}_D(w, p_0, c) &\geq -\int_c^d \tau'(\gamma)\mu\big(\overline{\Omega}'_D(p_0, \gamma)\big)\, d\gamma \\ &\geq \mu\big(\overline{\Omega}'_D(p_0, c)\big)\big(\tau(c) - \tau(d)\big) \\ &\to \tau(c)\mu\big(\overline{\Omega}'_D(p_0, c)\big)\end{aligned}$$

as $d \to \infty$.

Corollary 3. *Let D be an open set which is Dirichlet regular for the adjoint heat operator, let $p_0 \in D$, and let μ be the Riesz measure for a positive supertemperature w on $E \in \{D, \Lambda(p_0 D)\}$. Then*

$$\lim_{c \to \infty} \tau(c)\mu\big(\overline{\Omega}'_D(p, c)\big) = 0$$

for all $p \in E$.

Proof. By a result in [9], the domain $\Lambda(p_0, D)$ is Dirichlet regular for the adjoint heat operator. Furthermore, the Green function for $\Lambda(p_0, D)$ is the restriction of G_D to $\Lambda(p_0, D) \times \Lambda(p_0, D)$ (by [7, Theorem 14], and [3, p. 300]), so that $\Omega_{\Lambda(p_0,D)}(p, c) = \Omega_D(p, c)$ for all $p \in \Lambda(p_0, D)$. It therefore suffices to prove the result with $E = D$. The greatest thermic minorant u of w is given by

$$u(p) = \lim_{c \to \infty} \mathscr{M}_D(w, p, c)$$

for all $p \in D$ (see [10, Theorem 7], and [8]). Since μ is also the Riesz measure for $w - u$, it therefore follows from Corollary 2 and [10, Theorem 1] that

$$\tau(c)\mu\big(\overline{\Omega}'_D(p_0, c)\big) \leq \mathscr{M}_D(w - u, p, c) = \mathscr{M}_D(w, p, c) - u(p) \to 0$$

as $c \to \infty$ through regular values. Hence, given $\varepsilon > 0$ we can find K such that $\mu\big(\overline{\Omega}'_D(p_0, c)\big) \leq \varepsilon\tau(c)^{-1}$ for all regular values of $c > K$, and hence for all $c > K$ because $\mu\big(\overline{\Omega}'_D(p_0, \cdot\,)\big)$ is an increasing function of c, and τ is continuous.

Theorem 1 can also be used to extend [10, Theorem 7].

Theorem 2. *Let D be an open set which is Dirichlet regular for the adjoint heat operator, and let w be a supertemperature on D. Then the following statements are equivalent*:

(i) *w has a thermic minorant on D.*

(ii) *There is a sequence $\{p_j\}$ in D such that $D = \bigcup_{j=1}^\infty \Lambda(p_j, D)$ and for each j the function $\mathscr{M}_D(w, p_j, \cdot\,)$ is bounded below on the set of all regular values.*

(iii) *There is a sequence $\{p_j\}$ in D such that $D = \bigcup_{j=1}^\infty \Lambda(p_j, D)$ and*

$$\int_1^\infty \gamma^{-(n+2)/2}\mu\big(\overline{\Omega}'_D(p_j, \gamma)\big)\, d\gamma < \infty \tag{3}$$

for all j, where μ is the Riesz measure for w.

Proof. The equivalence of (i) and (ii) is [10, Theorem 7]. To prove that (ii) and (iii) are equivalent, let $\{p_j\}$ be a sequence in D such that $D = \bigcup_{j=1}^{\infty} \Lambda(p_j, D)$. Given j, if c and d are regular values and $c < d$, then

$$\mathscr{M}_D(w, p_j, c) = \mathscr{M}_D(w, p_j, d) - \tau'(1) \int_c^d \gamma^{-(n+2)/2} \mu\big(\overline{\Omega}'_D(p_j, \gamma)\big)\, d\gamma$$

by Theorem 1. Furthermore, the means are finite valued ([10, Theorem 2], or [2]). Therefore, if we fix c and make $d \to \infty$, we see that $\mathscr{M}_D(w, p_j, \cdot)$ is bounded below if and only if (3) holds.

3. The behaviour of the means for small regular values

Theorem 3 below generalizes [11, Theorem 2] in two directions. First it removes the restriction $v(p_0) = \infty$, and second it replaces $\mathscr{M}$ by $\mathscr{M}_D$.

We need some notation. Let E be an open set, and let D be an open superset of E that is Dirichlet regular for the adjoint heat operator. If $\overline{\Omega}_D(p_0, d) \subseteq E$, ν is a positive measure on E, and $0 \le b < c \le d$, we put

$$I_{\nu,D}(p_0; b, c) = -\int_b^c \tau'(\gamma) \nu\big(\overline{\Omega}'_D(p_0, \gamma)\big)\, d\gamma = \kappa_n \int_b^c \gamma^{-(n+2)/2} \nu\big(\overline{\Omega}'_D(p_0, \gamma)\big)\, d\gamma,$$

where $\kappa_n = -\tau'(1) = n2^{-n-1}\pi^{-n/2}$.

We include for completeness the definition of

$$\limsup_{0<b<c\to 0} f(b,c),$$

although it is the natural one. Those of the corresponding *lim inf* and *lim* are then obvious.

Definition. Suppose that $f(b,c)$ is defined as an extended-real number for Lebesgue almost all b and c such that $0 < b < c < d$, and that $l \in \mathbf{R}$. We write

$$\limsup_{0<b<c\to 0} f(b,c) = l$$

if to each $\varepsilon > 0$ there corresponds $\delta > 0$ such that $f(b,c) < l + \varepsilon$ whenever $f(b,c)$ is defined with $0 < b < c < \delta$, and there is a sequence $\{(b_k, c_k)\}$ such that $0 < b_k < c_k \to 0$ and $f(b_k, c_k) \to l$ as $k \to \infty$. We also write

$$\limsup_{0<b<c\to 0} f(b,c) = \infty$$

if there is a sequence $\{(b_k, c_k)\}$ such that $0 < b_k < c_k \to 0$ and $f(b_k, c_k) \to \infty$. Finally, we write

$$\limsup_{0<b<c\to 0} f(b,c) = -\infty$$

if to each $A \in \mathbf{R}$ there corresponds $\delta > 0$ such that $f(b,c) < A$ whenever $f(b,c)$ is defined with $0 < b < c < \delta$.

Theorem 3. *Let E be an open set, let D be an open superset of E that is Dirichlet regular for the adjoint heat operator, let u be a δ-subtemperature on E with Riesz measure μ, and let ν be a positive measure on E. Then*

$$\limsup_{0<b<c\to 0} \frac{\mathscr{M}_D(u,p,b)-\mathscr{M}_D(u,p,c)}{I_{\nu,D}(p;b,c)} \le \limsup_{d\to 0} \frac{\mu\big(\overline{\Omega}'_D(p,d)\big)}{\nu\big(\overline{\Omega}'_D(p,d)\big)} \tag{4}$$

whenever the latter exists. Furthermore, if $I_{\nu,D}(p;0,c)<\infty$ for all sufficiently small values of c, and $u(p)$ is defined and finite, then

$$\limsup_{c\to 0} \frac{u(p)-\mathscr{M}_D(u,p,c)}{I_{\nu,D}(p;0,c)} \le \limsup_{d\to 0} \frac{\mu\big(\overline{\Omega}'_D(p,d)\big)}{\nu\big(\overline{\Omega}'_D(p,d)\big)}. \tag{5}$$

Proof. Suppose that the upper limit on the right-hand side of (4) exists, and denote it by l. If $l=\infty$ there is nothing to prove. Otherwise, given a real number $A>l$, we can find $\delta>0$ such that

$$\frac{\mu\big(\overline{\Omega}'_D(p,d)\big)}{\nu\big(\overline{\Omega}'_D(p,d)\big)} < A \qquad \text{whenever } 0<d<\delta. \tag{6}$$

If $\nu\big(\overline{\Omega}'_D(p,d)\big)=0$ for all $d<\eta(\le\delta)$, then (6) can hold only if $\mu\big(\overline{\Omega}'_D(p,d)\big)<0$ for all $d<\eta$. Then $I_{\nu,D}(p;b,c)=0$ whenever $c<\eta$, and (1) shows that

$$\mathscr{M}_D(u,p,b)-\mathscr{M}_D(u,p,c)<0$$

for all regular values such that $0<b<c<\eta$, so that (4) holds with both sides $-\infty$. On the other hand, if $\nu\big(\overline{\Omega}'_D(p,d)\big)>0$ for all d, then by (1)

$$\begin{aligned}\frac{\mathscr{M}_D(u,p,b)-\mathscr{M}_D(u,p,c)}{I_{\nu,D}(p;b,c)} &= \frac{-1}{I_{\nu,D}(p;b,c)}\int_b^c \tau'(\gamma)\nu\big(\overline{\Omega}'_D(p,\gamma)\big)\frac{\mu\big(\overline{\Omega}'_D(p,\gamma)\big)}{\nu\big(\overline{\Omega}'_D(p,\gamma)\big)}\,d\gamma \\ &< A\end{aligned}$$

for all regular values such that $0<b<c<\delta$, and again (4) holds.

The inequality (5) can be proved in a similar way, using (2) instead of (1).

Corollary. *If u is a δ-subtemperature with Riesz measure μ on an open set E, then*

$$\lim_{0<b<c\to 0} \frac{\mathscr{M}(u,p,b)-\mathscr{M}(u,p,c)}{c-b} = \kappa_n \lim_{d\to 0}\frac{\mu\big(\overline{\Omega}'(p,d)\big)}{d^{(n+2)/2}} \tag{7}$$

whenever the latter exists. Furthermore, if $u(p)$ is defined and finite, then

$$\lim_{c\to 0}\frac{\mathscr{M}(u,p,c)-u(p)}{c} = -\kappa_n \lim_{d\to 0}\frac{\mu\big(\overline{\Omega}'(p,d)\big)}{d^{(n+2)/2}}$$

whenever the latter exists.

Proof. We take $D=\mathbf{R}^{n+1}$ and ν to be $(n+1)$-dimensional Lebesgue measure in Theorem 3, noting that the reverse inequalities hold for lower limits. A routine calculation shows that $\nu\big(\overline{\Omega}'(p,\gamma)\big)=\lambda_n\,\gamma^{(n+2)/2}$, where

$$\lambda_n = 2^{n+1}(n\pi)^{n/2}(n+2)^{-(n+2)/2}.$$

Therefore, if $0\le b<c$ and $\overline{\Omega}(p,c)\subseteq E$, then

$$I_\nu(p;b,c)=\kappa_n\lambda_n\int_b^c d\gamma=\left(\frac{n}{n+2}\right)^{(n+2)/2}(c-b).$$

It now follows from (4) and (5), together with their duals for lower limits, that

$$\lim_{0<b<c\to 0}\frac{\mathscr{M}(u,p,b)-\mathscr{M}(u,p,c)}{\kappa_n\lambda_n(c-b)}=\lim_{d\to 0}\frac{\mu\big(\overline{\Omega}'(p,d)\big)}{\lambda_n d^{(n+2)/2}}$$

and

$$\lim_{c\to 0}\frac{u(p)-\mathscr{M}(u,p,c)}{\kappa_n\lambda_n c}=\lim_{d\to 0}\frac{\mu\big(\overline{\Omega}'(p,d)\big)}{\lambda_n d^{(n+2)/2}}$$

whenever the last limit exists.

It is not immediately apparent that Theorem 3 is a generalization of [11, Theorem 2]. To demonstrate that it is, we first write it in a different form, then deduce the earlier result as the case $D=\mathbf{R}^{n+1}$ of the subsequent corollary.

Theorem 4. *Let E be an open set, and let D be an open superset of E that is Dirichlet regular for the adjoint operator. Let u be a δ-subtemperature with Riesz measure μ, and v a supertemperature with Riesz measure ν, on E. Then*

$$\limsup_{0<b<c\to 0}\frac{\mathscr{M}_D(u,p,b)-\mathscr{M}_D(u,p,c)}{\mathscr{M}_D(v,p,b)-\mathscr{M}_D(v,p,c)}\le\limsup_{d\to 0}\frac{\mu\big(\overline{\Omega}_D'(p,d)\big)}{\nu\big(\overline{\Omega}_D'(p,d)\big)}\tag{8}$$

whenever the latter exists. Furthermore, if $u(p)$ is defined and finite, and $v(p)<\infty$, then

$$\limsup_{c\to 0}\frac{u(p)-\mathscr{M}_D(u,p,c)}{v(p)-\mathscr{M}_D(v,p,c)}\le\limsup_{d\to 0}\frac{\mu\big(\overline{\Omega}_D'(p,d)\big)}{\nu\big(\overline{\Omega}_D'(p,d)\big)}.$$

Proof. In view of Theorem 1 and the finiteness of the mean values, the result follows from Theorem 3.

Corollary. *If the hypotheses of Theorem 4 are satisfied and* $v(p) = \infty$, *then*

$$\limsup_{c\to 0} \frac{\mathscr{M}_D(u,p,c)}{\mathscr{M}_D(v,p,c)} \le \limsup_{c\to 0} \frac{\mu\big(\overline{\Omega}'_D(p,c)\big)}{\nu\big(\overline{\Omega}'_D(p,c)\big)}. \tag{9}$$

Proof. Let l denote the left-hand side of (8). We prove that

$$\limsup_{c\to 0} \frac{\mathscr{M}_D(u,p,c)}{\mathscr{M}_D(v,p,c)} \le l, \tag{10}$$

which implies that (9) holds, in view of (8). We may assume that $l < \infty$. Given a real number $A > l$, we choose $\delta > 0$ such that

$$\frac{\mathscr{M}_D(u,p,b) - \mathscr{M}_D(u,p,c)}{\mathscr{M}_D(v,p,b) - \mathscr{M}_D(v,p,c)} < A$$

whenever b and c are regular values such that $0 < b < c < \delta$. By [10, Theorem 2], $\mathscr{M}_D(v,p,d) \to v(p)$ as $d \to 0$ (through regular values), so that our hypothesis $v(p) = \infty$ means we may suppose that $\mathscr{M}_D(v,p,d) > 0$ for all regular values of $d < \delta$. With this assumption, we fix a regular value of $c < \delta$. Given $\varepsilon > 0$, we choose $\eta < c$ such that both

$$\frac{\mathscr{M}_D(v,p,c)}{\mathscr{M}_D(v,p,b)} < \varepsilon \qquad \text{and} \qquad \frac{\mathscr{M}_D(u,p,c)}{\mathscr{M}_D(v,p,b)} < \varepsilon$$

for all regular values of $b < \eta$. Then

$$\begin{aligned}\frac{\mathscr{M}_D(u,p,b)}{\mathscr{M}_D(v,p,b)} &= \frac{\mathscr{M}_D(u,p,b) - \mathscr{M}_D(u,p,c)}{\mathscr{M}_D(v,p,b) - \mathscr{M}_D(v,p,c)}\left(1 - \frac{\mathscr{M}_D(v,p,c)}{\mathscr{M}_D(v,p,b)}\right) + \frac{\mathscr{M}_D(u,p,c)}{\mathscr{M}_D(v,p,b)} \\ &< \max\big\{A, (1-\varepsilon)A\big\} + \varepsilon\end{aligned}$$

if $b < \eta$. This proves (10), and (9) follows.

We can also generalize [11, Theorem 3] in a similar way.

Theorem 5. *Let* E *be an open set, let* D *be an open superset of* E *that is Dirichlet regular for the adjoint operator, and let* u *be a* δ*-subtemperature with Riesz measure* μ *on* E. *Let* $\alpha > 0$, *let* f *be a positive, increasing, absolutely continuous function on* $[0,\alpha]$, *and let*

$$\hat{f}(b,c) = \kappa_n \int_b^c \gamma^{-(n+2)/2} f(\gamma)\, d\gamma$$

whenever $0 \le b < c \le \alpha$. *Then*

$$\limsup_{0<b<c\to 0} \frac{\mathscr{M}_D(u,p,b) - \mathscr{M}_D(u,p,c)}{\hat{f}(b,c)} \le \limsup_{d\to 0} \frac{\mu\big(\overline{\Omega}'_D(p,d)\big)}{f(d)} \tag{11}$$

for every $p \in E$. *Furthermore, if* $u(p)$ *is defined and finite, and* $\hat{f}(0,c) < \infty$ *for all sufficiently small values of* c, *then*

$$\limsup_{c\to 0} \frac{u(p) - \mathscr{M}_D(u,p,c)}{\hat{f}(0,c)} \le \limsup_{d\to 0} \frac{\mu\big(\overline{\Omega}'_D(p,d)\big)}{f(d)}. \tag{12}$$

Proof. If $f(0) \ne 0$, then $\mu\big(\overline{\Omega}'_D(p,d)\big) = o\big(f(d)\big)$ as $d \to 0$, so we have to prove that

$$\limsup_{0<b<c\to 0} \frac{\mathscr{M}_D(u,p,b) - \mathscr{M}_D(u,p,c)}{\hat{f}(b,c)} = 0.$$

(Note that in this case $\hat{f}(0,c) = \infty$, so that the conditions for (12) are not satisfied.) If $0 < b < c < \alpha$, then

$$\hat{f}(b,c) \ge f(0) \int_b^c -\tau'(\gamma)\,d\gamma = f(0)\big(\tau(b) - \tau(c)\big),$$

so that

$$0 \le \frac{\tau(b) - \tau(c)}{\hat{f}(b,c)} \le \frac{1}{f(0)}.$$

Also, by Theorem 1,

$$\frac{\mathscr{M}_D(u,p,b) - \mathscr{M}_D(u,p,c)}{\tau(b) - \tau(c)} = \frac{1}{\tau(b) - \tau(c)} \int_b^c -\tau'(\gamma)\mu\big(\overline{\Omega}'_D(p,\gamma)\big)\,d\gamma,$$

which is $o(1)$ as $0 < b < c \to 0$ because $\mu\big(\overline{\Omega}'_D(p,\gamma)\big) = o(1)$ as $\gamma \to 0$. It follows that

$$\frac{\mathscr{M}_D(u,p,b) - \mathscr{M}_D(u,p,c)}{\hat{f}(b,c)} = \left(\frac{\mathscr{M}_D(u,p,b) - \mathscr{M}_D(u,p,c)}{\tau(b) - \tau(c)}\right)\left(\frac{\tau(b) - \tau(c)}{\hat{f}(b,c)}\right) = o(1)$$

as $0 < b < c \to 0$ through regular values.

Now consider the case where $f(0) = 0$. We choose $d \le \alpha$ such that $\overline{\Omega}_D(p,d)$ is contained in E, and define a measure ν on E by putting

$$d\nu = -\|\nabla_x G_D(p,\cdot\,)\|^2 \left(\frac{f'}{\tau'}\right)\left(\frac{G_D(p,\cdot\,)^{-2/n}}{4\pi}\right)\chi_{\Omega_D(p,d)}\,d\lambda,$$

where χ_A denotes the characteristic function of a set A, and λ denotes $(n+1)$-dimensional Lebesgue measure. If $0 < c < d$, it follows from results in [10, pp. 167–70] that

$$\begin{aligned}
\nu\big(\overline{\Omega}'_D(p,c)\big) &= -\int_{\Omega_D(p,c)} \|\nabla_x G_D(p,\cdot\,)\|^2 \left(\frac{f'}{\tau'}\right)\left(\frac{G_D(p,\cdot\,)^{-2/n}}{4\pi}\right) d\lambda \\
&= \int_0^c \left(\int_{\partial\Omega_D(p,\gamma)} \frac{\|\nabla_x G_D(p,\cdot\,)\|^2}{\|\nabla G_D(p,\cdot\,)\|} f'(\gamma)\, d\sigma\right) d\gamma \\
&= \int_0^c \mathscr{M}_D(1,p,\gamma) f'(\gamma)\, d\gamma \\
&= \int_0^c f'(\gamma)\, d\gamma = f(c),
\end{aligned}$$

so that

$$I_{\nu,D}(p;b,c) = -\int_b^c \tau'(\gamma) f(\gamma)\, d\gamma = \hat{f}(b,c)$$

whenever $0 \le b < c \le d$. The inequalities (11) and (12) now follow from Theorem 3.

The special case $D = \mathbf{R}^{n+1}$ of the following corollary is [11, Theorem 3].

Corollary 1. *If the hypotheses of Theorem* 5 *are satisfied and* $\hat{f}(0,\alpha) = \infty$, *then*

$$\limsup_{b\to 0} \frac{\mathscr{M}_D(u,p,b)}{\hat{f}(b,\alpha)} \le \limsup_{d\to 0} \frac{\mu\big(\overline{\Omega}'_D(p,d)\big)}{f(d)}$$

for each $p \in E$.

Proof. Given $p \in E$, let l denote the left-hand side of (11). In view of (11), it is more than enough to prove that

$$\limsup_{b\to 0} \frac{\mathscr{M}_D(u,p,b)}{\hat{f}(b,\alpha)} \le l. \tag{13}$$

We may assume that $l < \infty$. Given any real number $A > l$, we can find $\delta > 0$ such that

$$\frac{\mathscr{M}_D(u,p,b) - \mathscr{M}_D(u,p,c)}{\hat{f}(b,c)} < A$$

whenever b and c are regular values such that $0 < b < c < \delta$. Fix $c < \delta$. Given $\varepsilon > 0$, we choose $\eta < c$ such that both

$$\frac{\mathscr{M}_D(u,p,c)}{\hat{f}(b,\alpha)} < \varepsilon \qquad \text{and} \qquad \frac{\hat{f}(c,\alpha)}{\hat{f}(b,\alpha)} < \varepsilon$$

whenever $0 < b < \eta$. Then

$$\frac{\mathscr{M}_D(u,p,b)}{\hat{f}(b,\alpha)} = \frac{\mathscr{M}_D(u,p,b) - \mathscr{M}_D(u,p,c)}{\hat{f}(b,c)}\left(1 - \frac{\hat{f}(c,\alpha)}{\hat{f}(b,\alpha)}\right) + \frac{\mathscr{M}_D(u,p,c)}{\hat{f}(b,\alpha)}$$
$$< \max\bigl\{A, (1-\varepsilon)A\bigr\} + \varepsilon$$

for every regular value of $b < \eta$. The inequality (13) follows.

The extra generality of Theorem 5 over its first corollary enables us to generalize [11, Theorem 3 Corollary], and remove its restriction on the range of values of its parameter β, as follows.

Corollary 2. *Let E be an open set, let D be an open superset of E that is Dirichlet regular for the adjoint operator, let u be a δ-subtemperature with Riesz measure μ on E, and let $p \in E$. Then*

$$\left(\frac{n-\beta}{2}\right) \limsup_{0<b<c\to 0} \frac{\mathscr{M}_D(u,p,b) - \mathscr{M}_D(u,p,c)}{b^{-(n-\beta)/2} - c^{-(n-\beta)/2}} \le \kappa_n \limsup_{d\to 0} \frac{\mu\bigl(\overline{\Omega}'_D(p,d)\bigr)}{d^{\beta/2}}$$

if $0 \le \beta < n$,

$$\limsup_{0<b<c\to 0} \frac{\mathscr{M}_D(u,p,b) - \mathscr{M}_D(u,p,c)}{\log(c/b)} \le \kappa_n \limsup_{d\to 0} \frac{\mu\bigl(\overline{\Omega}'_D(p,d)\bigr)}{d^{n/2}},$$

and

$$\left(\frac{\beta-n}{2}\right) \limsup_{0<b<c\to 0} \frac{\mathscr{M}_D(u,p,b) - \mathscr{M}_D(u,p,c)}{c^{(\beta-n)/2} - b^{(\beta-n)/2}} \le \kappa_n \limsup_{d\to 0} \frac{\mu\bigl(\overline{\Omega}'_D(p,d)\bigr)}{d^{\beta/2}}$$

if $\beta > n$.

Proof. We take $f(d) = d^{\beta/2}$ for $\beta \ge 0$, in Theorem 5. Then

$$\hat{f}(b,c) = \kappa_n \int_b^c \gamma^{(\beta-n-2)/2}\, d\gamma,$$

which is equal to κ_n times

$$\begin{array}{ll} 2(b^{-(n-\beta)/2} - c^{-(n-\beta)/2})/(n-\beta) & \text{if } 0 \le \beta < n, \\ \log(c/b) & \text{if } \beta = n, \\ 2(c^{(\beta-n)/2} - b^{(\beta-n)/2})/(\beta-n) & \text{if } \beta > n, \end{array}$$

and the result follows.

4. The parabolic Hausdorff measures of certain sets

We use Theorem 5 to study the size of the sets of points p where

$$\frac{\mathscr{M}_D(u,p,b)-\mathscr{M}_D(u,p,c)}{\hat{f}(b,c)}$$

is unbounded as $0<b<c\to 0$, for a given function f and supertemperature u. The size is estimated in terms of the parabolic Hausdorff measures discussed in [5], which have the appropriate mixed homogeneity.

We recall the necessary definitions. Let h be an increasing function on $]0,\infty[$ such that $h(0+)=0$. Let $\mathscr{P}$ denote the class of all sets of the form

$$\left(\prod_{i=1}^{n}[a_i,a_i+r]\right)\times[a,a+r^2].$$

The set of this form which is centred at p is denoted by $P(p,r)$. For an arbitrary set S, the outer parabolic h-measure of S is defined by

$$\mathscr{P}-h-m_*(S)=\lim_{\delta\to 0+}\inf\left\{\sum_{i=1}^{\infty}h(\operatorname{diam}P_i):P_i\in\mathscr{P},\ E\subseteq\bigcup_{i=1}^{\infty}P_i,\ \operatorname{diam}P_i<\delta\right\}.$$

The associated measure, defined on a σ-field that contains the Borel sets, is denoted by $\mathscr{P}-h-m$. When $h(s)=s^{\alpha}$ for some $\alpha>0$, we write $\mathscr{P}-\Lambda^{\alpha}-m$ for $\mathscr{P}-h-m$.

Theorem 6. *Let E be an open set, let D be an open superset of E that is Dirichlet regular for the adjoint operator, and let u be a supertemperature on E. Let $0<\alpha<\infty$, and let h be an increasing function on $[0,\infty[$ that is absolutely continuous on $\left[0,\sqrt{\alpha}\,\right]$ and satisfies $h(0)=0$, $h(2s)\le Kh(s)$ for all $s>0$, where K is a constant. Put*

$$F(b,c)=\kappa_n\int_b^c\gamma^{-(n+2)/2}h\big(\sqrt{\gamma}\,\big)\,d\gamma$$

whenever $0<b<c\le\alpha$.

(i) *The set*

$$\left\{p\in E:\ \limsup_{0<b<c\to 0}\frac{\mathscr{M}_D(u,p,b)-\mathscr{M}_D(u,p,c)}{F(b,c)}=\infty\right\}\tag{14}$$

has $\mathscr{P}-h$-measure zero.

(ii) *If, in addition,*

$$\int_0^{\sqrt{\alpha}}s^{-n-1}h(s)\,ds=\infty,$$

then the set

$$\left\{p\in E:\limsup_{b\to 0}\frac{\mathscr{M}_D(u,p,b)}{F(b,\alpha)}=\infty\right\}\tag{15}$$

also has $\mathscr{P}-h$-measure zero.

Proof. The sets (14) and (15) are subsets of

$$\left\{p\in E:\limsup_{d\to 0}\frac{\mu\big(\overline{\Omega}'_D(p,d)\big)}{h\big(\sqrt{d}\,\big)}=\infty\right\}, \tag{16}$$

by Theorem 5 and its Corollary 1 (with $f(\gamma)=h\big(\sqrt{\gamma}\,\big)$). Furthermore, since $G_D\le G$ on D (by [7, Theorem 10]), we have $\Omega_D(p,d)\subseteq\Omega(p,d)$. Therefore, given $d>0$, if we choose $r=3\sqrt{nd/e}$ then $\overline{\Omega}'_D(p,d)\subseteq P(p,r)$. It follows that the set (16) is a subset of

$$\left\{p\in E:\limsup_{r\to 0}\frac{\mu\big(P(p,r)\big)}{h(\delta r)}=\infty\right\}, \tag{17}$$

where $\delta=\sqrt{e/9n}<1$. If i is chosen so that $2^i\delta>1$, then

$$h(\delta r)\ge K^{-i}h(2^i\delta r)\ge K^{-i}h(r).$$

Therefore the set (17) is a subset of

$$\left\{p\in E:\limsup_{r\to 0}\frac{\mu\big(P(p,r)\big)}{h(r)}=\infty\right\},$$

which has $\mathscr{P}-h$-measure zero by the lemma in [11].

Remark. The case $D=\mathbf{R}^{n+1}$ of Theorem 6(ii) was proved in [11].

Corollary. *Let E be an open set, let D be an open superset of E that is Dirichlet regular for the adjoint operator, and let u be a supertemperature on E.*

(i) *If $0<\beta<n$, then the set*

$$\left\{p\in E:\limsup_{0<b<c\to 0}\frac{\mathscr{M}_D(u,p,b)-\mathscr{M}_D(u,p,c)}{b^{-(n-\beta)/2}-c^{-(n-\beta)/2}}=\infty\right\}$$

has $\mathscr{P}-\Lambda^\beta$-measure zero.

(ii) *The set*

$$\left\{p\in E:\limsup_{0<b<c\to 0}\frac{\mathscr{M}_D(u,p,b)-\mathscr{M}_D(u,p,c)}{\log(c/b)}=\infty\right\}$$

has $\mathscr{P}-\Lambda^n$-measure zero.

(iii) *If $n<\beta\le n+2$, then the set*

$$\left\{p\in E:\limsup_{0<b<c\to 0}\frac{\mathscr{M}_D(u,p,b)-\mathscr{M}_D(u,p,c)}{c^{(\beta-n)/2}-b^{(\beta-n)/2}}=\infty\right\} \tag{18}$$

has $\mathscr{P}-\Lambda^\beta$-measure zero.

Proof. The result is obtained by taking $h(s) = s^\beta$ for $\beta \in]0, n+2]$ in Theorem 6. (Note that $\mathscr{P}-\Lambda^\beta-m(S)=0$ for all subsets S of $\mathbf{R}^{n+1}$ if $\beta > n+2$.)

Remark. In Theorem 6(ii), the extra condition on h ensures that $F(b,\alpha) \to \infty$ as $b \to 0$, so that $u(p)=\infty$ for all p in the set (15); thus the set is polar. In Theorem 6(i), polarity is not so readily determined, and whether or not the set (14) is polar depends upon h. By [5, Theorem 1], if a set is not polar then its $\mathscr{P}-\Lambda^n$-measure must be strictly positive, and so the sets considered in parts (i) and (ii) of the corollary are polar. The set in part (iii), however, may not be polar, as the following example shows.

Example. Given β such that $n < \beta \le n+2$, put $\alpha = \frac{1}{3}(n+3-\beta)$, so that $\frac{1}{3} \le \alpha < 1$. Let $t_0 \in \mathbf{R}$, $Q = \prod_{i=1}^n]0,1[$, $S = Q \times]t_0-1, t_0[$, and

$$\mu(B) = \iint_{B\cap S} (t_0-t)^{-\alpha}\, dx\, dt$$

for every Borel subset B of $\mathbf{R}^{n+1}$. Then μ is a finite measure, so that $G\mu$ is a positive supertemperature on $\mathbf{R}^{n+1}$. We consider the case $D=E=\mathbf{R}^{n+1}$ of the above corollary. Given any $x_0 \in Q$, we put $p_0=(x_0,t_0)$ and can find $c_0>0$ such that $\Omega(p_0,c_0) \subseteq S$. Then, whenever $c < c_0$, we have

$$\mu\big(\overline{\Omega}'(p_0,c)\big) = \iint_{\Omega(p_0,c)} (t_0-t)^{-\alpha}\, dx\, dt = C_{n,\alpha} c^{(n+2-2\alpha)/2},$$

where $C_{n,\alpha} = 2^{n+1}(\pi n)^{n/2}(n+2-\alpha)^{-(n+2)/2}$. Therefore

$$c^{-\beta/2}\mu\big(\overline{\Omega}'(p_0,c)\big) = C_{n,\alpha} c^{(n+2-\beta-2\alpha)/2} \to \infty \qquad \text{as } c \to 0,$$

because

$$\frac{n+2-\beta-2\alpha}{2} = \frac{n-\beta}{6} < 0.$$

Therefore, by Theorem 5, Corollary 2,

$$\frac{\mathscr{M}(u,p_0,b)-\mathscr{M}(u,p_0,c)}{c^{(\beta-n)/2}-b^{(\beta-n)/2}} \to \infty$$

as $0<b<c\to 0$, so that the set (18) contains $Q\times\{t_0\}$, which is not polar.

5. The Riesz measures of supertemperatures on Dirichlet regular sets

In this section we generalize [11, Theorem 5] from the case of a lower half-space $\mathbf{R}^n\times]-\infty,a[$ to that of an arbitrary open set which is Dirichlet regular for the adjoint operator.

Theorem 7. *Let D be an open set which is Dirichlet regular for the adjoint heat operator, let $p_0\in D$, let $E\in\big\{D,\Lambda(p_0,D)\big\}$, and let μ be a positive measure on E.*

(i) *If μ is the Riesz measure of a supertemperature which has a thermic minorant on E, then*

$$\int_1^\infty \gamma^{-(n+2)/2}\mu\big(\overline{\Omega}'_D(p,\gamma)\big)\,d\gamma<\infty \tag{19}$$

for all $p\in E$.

(ii) *Conversely, if there is $p\in E$ such that (19) holds, then $G_D\mu$ is a supertemperature on $\Lambda(p,D)$. If, in addition,*

$$\int_0^1 \gamma^{-(n+2)/2}\mu\big(\overline{\Omega}'_D(p,\gamma)\big)\,d\gamma<\infty,$$

then $G_D\mu(p)<\infty$.

Proof. By a result in [9], the domain $\Lambda(p_0,D)$ is Dirichlet regular for the adjoint heat operator. Furthermore, the Green function for $\Lambda(p_0,D)$ is the restriction of G_D to $\Lambda(p_0,D)\times\Lambda(p_0,D)$ (by [7, Theorem 14], or [3, p. 300]), and $\Lambda\big(p,\Lambda(p_0,D)\big)=\Lambda(p,D)$ for any $p\in\Lambda(p_0,D)$. It therefore suffices to prove the result when $E=D$.

(i) Let w be a supertemperature which has a thermic minorant u on D, and whose Riesz measure is μ. Then μ is also the Riesz measure for $w-u$. Therefore, if $p\in D$ and c, d are regular values such that $c<d$, Theorem 1 shows that

$$\begin{aligned}\mathscr{M}_D(w-u,p,c)&=\mathscr{M}_D(w-u,p,d)+\kappa_n\int_c^d \gamma^{-(n+2)/2}\mu\big(\overline{\Omega}'_D(p,\gamma)\big)\,d\gamma\\&\geq\kappa_n\int_c^d \gamma^{-(n+2)/2}\mu\big(\overline{\Omega}'_D(p,\gamma)\big)\,d\gamma.\end{aligned}$$

Making $d\to\infty$ we obtain (19), because $\mathscr{M}_D(w-u,p,c)<\infty$.

(ii) Suppose that (19) holds for some $p=p_1\in D$. Let $\{k_j\}$ be an unbounded increasing sequence of regular values, and put

$$\begin{aligned}A_D(p_1;k_1,\infty)&=\Lambda(p_1,D)\backslash\overline{\Omega}'_D(p_1,k_1),\\A_D(p_1;k_1,k_j)&=\Omega_D(p_1,k_j)\backslash\overline{\Omega}'_D(p_1,k_1)\qquad\text{for } j>1.\end{aligned}$$

If $u = G_D\mu$, then for all $p \in D$ we put

$$\begin{aligned} u(p) &= \int_{\overline{\Omega}'_D(p_1,k_1)} G_D(p,q)\,d\mu(q) + \int_{A_D(p_1;k_1,\infty)} G_D(p,q)\,d\mu(q) \\ &= v_1(p) + v_2(p), \end{aligned}$$

say, and

$$u_j(p) = \int_{A_D(p_1;k_1,k_j)} G_D(p,q)\,d\mu(q)$$

for $j \geq 1$. Since μ is locally finite, v_1 and every u_j is a supertemperature on D. Since $\{u_j\}$ is increasing to the limit v_2, v_2 is a supertemperature on $\Lambda(p_1, D)$ if $v_2(p_1) < \infty$, by [6, Theorem 6]. Writing $\lambda(\gamma) = \mu\big(\overline{\Omega}'_D(p_1,\gamma)\big)$ for all $\gamma > 0$, we have

$$u_j(p_1) = \int_{k_1}^{k_j} \tau(\gamma)\,d\lambda(\gamma) = \big[\tau(\gamma)\lambda(\gamma)\big]_{k_1}^{k_j} - \int_{k_1}^{k_j} \tau'(\gamma)\lambda(\gamma)\,d\gamma.$$

Since (19) holds when $p = p_1$, we have

$$\lambda(c)\tau(c) = \lambda(c)\int_c^\infty -\tau'(\gamma)\,d\gamma \leq -\int_c^\infty \tau'(\gamma)\lambda(\gamma)\,d\gamma \to 0$$

as $c \to \infty$. Therefore

$$v_2(p_1) = \lim_{j\to\infty} u_j(p_1) = -\tau(k_1)\lambda(k_1) - \int_{k_1}^\infty \tau'(\gamma)\lambda(\gamma)\,d\gamma < \infty,$$

so that v_2, and hence u, is a supertemperature on $\Lambda(p_1, D)$.

For the last part, let $\{c_j\}$ be a decreasing null sequence of regular values (relative to p_1). Then

$$\begin{aligned} u(p_1) &= \lim_{j\to\infty} \int_{c_j}^\infty \tau(\gamma)\,d\lambda(\gamma) \\ &= \lim_{j\to\infty} \left(-\tau(c_j)\lambda(c_j) - \int_{c_j}^\infty \tau'(\gamma)\lambda(\gamma)\,d\gamma \right) \\ &\leq -\int_0^\infty \tau'(\gamma)\lambda(\gamma)\,d\gamma < \infty. \end{aligned}$$

6. Differences of positive supertemperatures on $\Lambda(p_0, D)$

Let u be a δ-subtemperature on an open set E, and let D be an open superset of E that is Dirichlet regular for the adjoint operator. If μ is the Riesz measure for u, then μ can be written minimally as a difference $\mu^+ - \mu^-$ of two positive measures on E. Whenever $\overline{\Omega}_D(p,c) \subseteq E$, we put

$$\lambda_D^+(p,c) = \mu^+\big(\overline{\Omega}'_D(p,c)\big), \qquad N_D^+(p,c) = -\int_0^c \tau'(\gamma)\lambda_D^+(p,\gamma)\,d\gamma,$$

and similarly for μ^-. We say that $u(p_0)$ is *finite* if $N_D^+(p_0,\cdot)$ and $N_D^-(p_0,\cdot)$ are both finite-valued, in which case it follows from (2) that u is the difference of two supertemperatures which are finite at p_0. If $u(p_0)$ is finite, we define the *characteristic* T_D of u at p_0 by

$$T_D(u,p_0,c) = \mathscr{M}_D(u^+,p_0,c) + N_D^+(p_0,c) - u(p_0)$$

for each regular value of c such that $\overline{\Omega}_D(p_0,c) \subseteq E$. We can use T_D to characterize those δ-subtemperatures on $\Lambda(p_0,D)$ or D that can be written as a difference of two positive supertemperatures, and thus generalize [11, Theorem 6].

Theorem 8. *Let D be an open set which is Dirichlet regular for the adjoint heat operator, let $p_0 \in D$, let $E \in \big\{D, \Lambda(p_0,D)\big\}$, and let u be a δ-subtemperature on E.*

(i) *If $u = u_1 - u_2$ is the difference of two positive supertemperatures on E, and $u(p_1)$ is finite, then $T_D(u,p_1,\cdot)$ is an increasing function such that $0 \le T_D(u,p_1,c) \le u_2(p_1)$ for all regular values of c, and there is a convex function ϕ such that $T_D(u,p_1,\cdot) = \phi\circ\tau$.*

(ii) *Conversely, if there is a sequence $\{p_j\}$ in E such that $E = \bigcup_{j=1}^\infty \Lambda(p_j,D)$, $u(p_j)$ is finite for all j, and $T_D(u,p_j,\cdot)$ is bounded above on the set of all regular values for each j, then u is the difference of two positive supertemperatures on E.*

Proof. (i) For $i \in \{1,2\}$, let μ_i be the Riesz measure for u_i, and put

$$\lambda_D^i(p_1,c) = \mu_i\big(\overline{\Omega}'_D(p_1,c)\big), \qquad N_D^i(p_1,c) = -\int_0^c \tau'(\gamma)\lambda_D^i(p_1,\gamma)\,d\gamma$$

for all $c > 0$. Since $u_1 \ge 0$, it follows from (2) that

$$0 = \mathscr{M}_D(u_1^-,p_1,c) = \mathscr{M}_D(u_1,p_1,c) + N_D^1(p_1,c) - u_1(p_1).$$

Since μ_1 and μ_2 are positive and $\mu = \mu_1 - \mu_2$, we have $\mu^+ \le \mu_1$ and $\mu^- \le \mu_2$, so that

$$N_D^+(p_1,c) \le N_D^1(p_1,c) = u_1(p_1) - \mathscr{M}_D(u_1,p_1,c).$$

Furthermore $u_1 \geq u^+$, so that $\mathscr{M}_D(u^+, p_1, c) \leq \mathscr{M}_D(u_1, p_1, c)$. Hence

$$T_D(u, p_1, c) \leq \mathscr{M}_D(u_1, p_1, c) + \big(u_1(p_1) - \mathscr{M}_D(u_1, p_1, c)\big) - u(p_1) = u_2(p_1).$$

Now let $v_2 = G_D\mu^-$ and $v_1 = u + v_2$. Applying (2) to each v_i and subtracting, we obtain

$$u(p_1) = \mathscr{M}_D(u, p_1, c) + N_D^+(p_1, c) - N_D^-(p_1, c),$$

so that

$$\begin{aligned} T_D(u, p_1, c) &= \mathscr{M}_D(u^+, p_1, c) + N_D^-(p_1, c) - \mathscr{M}_D(u, p_1, c) \\ &= \mathscr{M}_D(u^-, p_1, c) + N_D^-(p_1, c) \\ &= \mathscr{M}_D(u^-, p_1, c) + v_2(p_1) - \mathscr{M}_D(v_2, p_1, c) \\ &= v_2(p_1) - \mathscr{M}_D(v_2 - u^-, p_1, c). \end{aligned}$$

Let $p \in E$. If $u(p) \geq 0$, then $v_1(p) \geq v_2(p)$ and $v_2(p) - u^-(p) = v_2(p) = (v_1 \wedge v_2)(p)$. On the other hand, if $u(p) < 0$ then $v_1(p) < v_2(p)$ and $v_2(p) - u^-(p) = v_1(p) = (v_1 \wedge v_2)(p)$. Hence

$$T_D(u, p_1, c) = v_2(p_1) - \mathscr{M}_D(v_1 \wedge v_2, p_1, c).$$

Since $v_1 \wedge v_2$ is a supertemperature on E, the characteristic $T_D(u, p_1, \cdot\,)$ is increasing and real-valued on the set of regular values (by [10, Theorem 2]), there is a convex function ϕ such that $T_D(u, p_1, \cdot\,) = \phi \circ \tau$ (by [10, Theorem 3]) and $T_D(u, p_1, 0+) = v_2(p_1) - (v_1 \wedge v_2)(p_1) \geq 0$ (by [10, Theorem 2]).

(ii) The proof is similar to that of [11, Theorem 6(ii)].

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