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BOUNDARY LIMITS OF SPHERICAL MEANS FOR BLD AND MONOTONE BLD FUNCTIONS IN THE UNIT BALL

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Abstract. Our aim in this paper is to deal with the existence of boundary limits for BLD functions u on the unit ball $\mathbf{B}$ of $\mathbf{R}^n$ satisfying

$$\int_{\mathbf{B}} |\nabla u(x)|^p (1-|x|)^\alpha \, dx < \infty,$$

where ∇ denotes the gradient, $1 < p < \infty$ and $-1 < \alpha < p-1$. We consider the L^q-means over the spherical surfaces $S(0,r)$ centered at the origin with radius r, and show that

$$\liminf_{r \to 1} (1-r)^{(n-p+\alpha)/p-(n-1)/q} \left(\int_{S(0,r)} |u(x)|^q \, dS(x) \right)^{1/q} = 0$$

when $q > 0$ and $(n-p-1)/p(n-1) < 1/q < (n-p+\alpha)/p(n-1)$. If u is in addition monotone in $\mathbf{B}$ in the sense of Lebesgue, then u is shown to have weighted boundary limit zero.

1. Introduction

Let $\mathbf{R}^n$ denote the n-dimensional Euclidean space. We use the notation $B(x,r)$ to denote the open ball centered at x with radius $r > 0$, whose boundary is denoted by $S(x,r)$. Consider the L^q-means over $S(0,r)$ defined by

$$S_q(u,r) = \left(\frac{1}{|S(0,r)|} \int_{S(0,r)} |u(x)|^q \, dS(x) \right)^{1/q},$$

where $|S(0,r)|$ denotes the surface area, which is written as $|S(0,r)| = \sigma_n r^{n-1}$; in case $q = \infty$, $S_\infty(u,r)$ denotes the essential supremum of u over $S(0,r)$. We note by Hölder's inequality that $S_q(u,r)$ is nondecreasing for q.

Let u be a Green potential in the unit ball $\mathbf{B} = B(0,1)$. Gardiner [1, Theorem 2] showed that

$$\liminf_{r \to 1} (1-r)^{(n-1)(1-1/q)} S_q(u,r) = 0$$

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when $(n-3)/(n-1) < 1/q \leqq (n-2)/(n-1)$ and $q > 0$. This gives an extension of the result by Stoll [22] in the plane case, which states that

$$\liminf_{r\to 1}(1-r)S_\infty(u,r) = 0.$$

Recently Herron and Koskela [4, Theorem 7.3, Corollary 7.5] proved that

$$S_\infty(u,r) \leqq M\bigl[\log\bigl(2/(1-r)\bigr)\bigr]^{(n-1)/n}, \qquad 0 < r < 1,$$

with a positive constant M, when u is a monotone function on $\mathbf{B}$ with finite Dirichlet integral:

$$\int_{\mathbf{B}} |\nabla u(x)|^n \, dx < \infty;$$

see the next section for the definition of monotone functions. We here note that harmonic functions are monotone, $\mathscr{A}$-harmonic functions and hence coordinate functions of quasiregular mappings are monotone (see [3] and [18]). Thus the class of monotone functions is considerably wide.

Our main aim in this paper is to establish the analogue of these results for BLD and monotone BLD functions u on $\mathbf{B}$ satisfying

$$\int_{\mathbf{B}} |\nabla u(x)|^p \varrho(x)^\alpha \, dx < \infty, \tag{1}$$

where $\varrho(x) = 1 - |x|$, $1 < p < \infty$ and $-1 < \alpha < p-1$. We first study weighted boundary limits of spherical L^q-means for BLD functions satisfying (1), and establish a result corresponding to [16, Theorem 2.1] given in half spaces.

If u is a monotone BLD function on $B(x_0, 2r)$ and $p > n-1$, then the key for our results is the fact that

$$|u(x) - u(y)|^p \leqq M r^{p-n} \int_{B(x_0,2r)} |\nabla u(z)|^p \, dz \qquad \text{whenever } x,\ y \in B(x_0, r); \tag{2}$$

see e.g. [4, Lemma 7.1], [6, Remark, p. 9] and, for the case $p = n$, [26, Section 16]. If u is harmonic, then (2) holds for $p \geqq 1$ by the mean value property, so that the condition $p > n-1$ is not required for harmonic functions. Further we note that if $p > n$, then (2) holds for all BLD functions, on account of Sobolev's theorem. Thus, if we restrict ourselves to monotone functions, then we have only to consider the case $n-1 < p \leqq n$.

Related results are given by Gardiner [1], Stoll [22], [23], [24] and the first author [12], [13] and [16].

We wish to express our deepest appreciation to the referee for his useful suggestions.

2. Statement of results

If $1 < p < \infty$, G is an open set in $\mathbf{R}^n$ and $E \subset G$, then the relative p-capacity is defined by

$$C_p(E;G) = \inf \int_G f(y)^p \, dy,$$

where the infimum is taken over all nonnegative measurable functions f on G such that

$$\int_G |x-y|^{1-n} f(y)\, dy \geqq 1 \qquad \text{for every } x \in E;$$

see [8] and [15] for the basic properties of p-capacity.

Following Ziemer [28], we say that a locally integrable function u is p-precise in G if

(i) $\int_G |\nabla u(x)|^p \, dx < \infty$, where ∇ denotes the gradient;
(ii) for every $\varepsilon > 0$ there exists an open set ω such that $C_p(\omega, G) < \varepsilon$ and u is continuous as a function on $G - \omega$.

According to Ohtsuka [17], we say that a function u is locally p-precise in G if it is p-precise in every relatively compact open subset of G.

We note that if u is locally p-precise in G, then u is partially differentiable almost everywhere on G and its spherical means over $S(x,r)$ are well defined whenever $S(x,r) \subset G$, since a set of p-capacity zero has Hausdorff dimension at most $n-p$.

We first study the weighted boundary limits of spherical means for locally p-precise functions on $\mathbf{B}$ satisfying (1).

Theorem 1 (cf. [12, Theorem 2.1] and [16, Theorem 2.1]). *Let u be a locally p-precise function on $\mathbf{B}$ satisfying* (1) *with $-1 < \alpha < p-1$. If $p < q < \infty$ and*

$$\frac{n-p-1}{p(n-1)} < \frac{1}{q} < \frac{n-p+\alpha}{p(n-1)},$$

then

$$\liminf_{r \to 1} (1-r)^{(n-p+\alpha)/p-(n-1)/q} S_q(u,r) = 0.$$

The sharpness of the exponent will be discussed in the final section. For BLD functions in half spaces of $\mathbf{R}^n$, Theorem 1 was already given by the first author [16, Theorem 2.1]; for the reader's convenience, we give a proof of Theorem 1.

We say that a continuous function u is monotone in an open set G, in the sense of Lebesgue, if both

$$\max_{\overline{D}} u(x) = \max_{\partial D} u(x) \qquad \text{and} \qquad \min_{\overline{D}} u(x) = \min_{\partial D} u(x)$$

hold for every relatively compact open set D with the closure $\overline{D} \subset G$ (see [5]). Clearly, harmonic functions are monotone, and more generally, solutions of elliptic partial differential equations of second order and weak solutions for variational problems may be monotone. For these facts, see Gilbarg–Trudinger [2], Heinonen–Kilpeläinen–Martio [3], Reshetnyak [18], Serrin [19], and Vuorinen [25], [26].

It will be seen that the existence of lower limit in Theorem 1 is derived as a consequence of fine limit argument on the line $\mathbf{R}^1$. Next we show that the exceptional sets disappear for monotone functions.

Theorem 2. *Let u be a monotone function on $\mathbf{B}$ satisfying* (1)*. If $n-1 < p < n+\alpha$, $p < q < \infty$ and*

$$\frac{1}{q} < \frac{n-p+\alpha}{p(n-1)},$$

then

$$\lim_{r\to 1}(1-r)^{(n-p+\alpha)/p-(n-1)/q} S_q(u,r) = 0.$$

Corollary 1. *Let u be a coordinate function of a quasiregular mapping on $\mathbf{B}$ satisfying* (1)*. If $n-1 < p < n+\alpha$, $p < q < \infty$ and*

$$\frac{1}{q} < \frac{n-p+\alpha}{p(n-1)},$$

then

$$\lim_{r\to 1}(1-r)^{(n-p+\alpha)/p-(n-1)/q} S_q(u,r) = 0.$$

For the definition and basic properties of quasiregular mappings, we refer to [3], [18] and [25]. In particular, a coordinate function $u = f_i$ of a quasiregular mapping $f = (f_1, \dots, f_n)\colon \mathbf{B} \to \mathbf{R}^n$ is $\mathscr{A}$-harmonic (see [3, Theorem 14.39] and monotone in $\mathbf{B}$, so that Theorem 2 gives the present corollary.

In case $1/q = (n-p+\alpha)/p(n-1) > 0$, one might expect that $S_q(u,r)$ is bounded. In fact, we can show that this is true only in case $0 \leqq \alpha < p-1$ without assuming the monotonicity; see Remark 3 given below in the final section. We refer the reader to the result by Yamashita [27] who showed affirmatively the case $p = 2$ and $\alpha = 1$ for harmonic functions. The case $\alpha = p-1$ remains open.

Finally we treat the case $q = \infty$. In order to give a general result, we consider a nondecreasing positive function φ on the interval $[0,\infty)$ such that φ is log-type, that is, there exists a positive constant M satisfying

$$\varphi(r^2) \leqq M\varphi(r) \qquad \text{for all } r \geqq 0.$$

Set $\Phi_p(r) = r^p \varphi(r)$ for $p > 1$. Our final aim is to study the existence of weighted boundary limits of monotone BLD functions u on $\mathbf{B}$, which satisfy

$$\int_{\mathbf{B}} \Phi_p\big(|\nabla u(x)|\big)\varrho(x)^\alpha\, dx < \infty, \tag{3}$$

where ϱ is as in (1). Consider the function

$$\kappa(r) = \left[\int_r^1 \left(t^{n-p+\alpha} \varphi(t^{-1}) \right)^{-1/(p-1)} \frac{dt}{t} \right]^{1-1/p}$$

for $0 \leqq r \leqq 2^{-1}$; set $\kappa(r) = \kappa(2^{-1})$ for $r > 2^{-1}$. We see (cf. [20, Lemma 2.4]) that if $n - p + \alpha > 0$, then

$$\kappa(r) \sim \left[r^{n-p+\alpha} \varphi(r^{-1}) \right]^{-1/p} \qquad \text{as } r \to 0$$

and if $n - p + \alpha = 0$ and $\varphi(r) = \big(\log(e + r)\big)^{\sigma}$ with $0 \leqq \sigma < p - 1$, then

$$\kappa(r) \sim \left[\log(1/r) \right]^{(p-1-\sigma)/p} \qquad \text{as } r \to 0.$$

Theorem 3. *Let u be a monotone function on $\mathbf{B}$ satisfying (3). If $n - 1 < p \leqq n + \alpha$ and $\kappa(0) = \infty$, then*

$$\lim_{|x| \to 1} \left[\kappa\big(\varrho(x)\big) \right]^{-1} u(x) = 0.$$

In case $\varphi \equiv 1$, $p = n$ and $\alpha = 0$, Theorem 3 was proved by Herron–Koskela [4, Theorem 7.3, Corollary 7.5]. In view of [11, Theorem 1] and [16, Theorem 4.1], we see that if u is harmonic in $\mathbf{B}$, then the conclusions of Theorems 2 and 3 remain true for p smaller than $n - 1$.

Corollary 2. *Let u be a coordinate function of a quasiregular mapping on $\mathbf{B}$ satisfying (3). If $n - 1 < p \leqq n + \alpha$ and $\kappa(0) = \infty$, then*

$$\lim_{|x| \to 1} \left[\kappa\big(\varrho(x)\big) \right]^{-1} u(x) = 0.$$

3. Preliminary lemmas

Throughout this paper, let $\varrho(x)$ denote the distance of $x \in \mathbf{R}^n$ from the unit spherical surface $S(0,1)$, that is,

$$\varrho(x) = \big| |x| - 1 \big|.$$

Further, let M denote various constants independent of the variables in question.

Recall the definition of relative p-capacity in the previous section. We write $C_p(E) = 0$ if

$$C_p(E \cap G; G) = 0 \qquad \text{for every bounded open set } G.$$

We say that a property holds p-q.e. on G if the property holds for every $x \in G$ except that in a set of p-capacity zero. In view of [13, Lemma 2.2], if $E \subset \mathbf{B}$ and $C_p(E) = 0$, then we can find a nonnegative measurable function f on $\mathbf{B}$ such that

$$\int_{\mathbf{B}} f(y)^p \varrho(y)^{\alpha} \, dy < \infty$$

and

$$\int_{\mathbf{B}} |x - y|^{1-n} f(y) \, dy = \infty \qquad \text{for every } x \in E.$$

Now we give several results which are used for the proof of Theorem 1.

Lemma 1. *If u is a locally p-precise function on $\mathbf{B}$ satisfying* (1) *with $-1<\alpha<p-1$, then it has an extension $\overline{u}$ with compact support in $\mathbf{R}^n$ which is q-precise in $\mathbf{R}^n$ for $1<q<\min\{p,p/(1+\alpha)\}$ and satisfies*

$$\int_{\mathbf{R}^n} |\nabla\overline{u}(x)|^p \varrho(x)^\alpha\,dx<\infty.$$

Proof. If $1<q<p$ and $q<p/(1+\alpha)$, then Hölder's inequality gives

$$\int_{\mathbf{B}} |\nabla u(x)|^q\,dx \leqq \left(\int_{\mathbf{B}} \varrho(x)^{-\alpha q/(p-q)}\,dx\right)^{1-q/p} \left(\int_{\mathbf{B}} |\nabla u(x)|^p \varrho(x)^\alpha\,dx\right)^{q/p} <\infty.$$

Hence we can find a q-precise extension $\overline{u}$ to $\mathbf{R}^n$ by Stein [21, Chapter 5], or we may consider the inversion to define

$$\overline{u}(x)=u(x/|x|^2) \qquad \text{for } |x|>1.$$

We may further assume that the extension $\overline{u}$ vanishes outside $B(0,2)$, by considering $\chi\overline{u}$, where χ is an infinitely differentiable function on $\mathbf{R}^n$ with compact support in $B(0,2)$.

We introduce Sobolev's integral representation.

Lemma 2 (cf. [9]). *Let $1<q<\infty$ and v be a q-precise function on $\mathbf{R}^n$ with compact support. Then*

$$v(x)=c\sum_{j=1}^n \int_{\mathbf{R}^n} \frac{x_j-y_j}{|x-y|^n}\frac{\partial v}{\partial y_j}(y)\,dy$$

holds for q-q.e. on $\mathbf{R}^n$, where $c=|S(0,1)|^{-1}$.

Corollary 3. *Let u be a locally p-precise function on $\mathbf{B}$ satisfying* (1) *with $-1<\alpha<p-1$. Then*

$$u(x)=c\sum_{j=1}^n \int_{\mathbf{R}^n} \frac{x_j-y_j}{|x-y|^n}\frac{\partial \overline{u}}{\partial y_j}(y)\,dy$$

holds for p-q.e. on $\mathbf{B}$, where $\overline{u}$ is an extension of u as in Lemma 1.

Lemma 3 (cf. [12, Lemma 2.1] and [13, Lemma 5.1]). *If we set $k_y(x)=|x-y|^{\delta(1-n)}$ for fixed y and $\delta>0$, then*

$$S_q(k_y,r)\leqq M\begin{cases} |y|^{-\delta(n-1)} & \text{if } |y|\geqq 2r,\\ r^{-\delta(n-1)} & \text{if } \frac12 r<|y|<2r \text{ and } 1/q>\delta,\\ r^{-(n-1)/q}\big||y|-r\big|^{(1/q-\delta)(n-1)} & \text{if } \frac12 r<|y|<2r \text{ and } 1/q<\delta,\\ r^{-\delta(n-1)}\left[\log\left(2r/\big||y|-r\big|\right)\right]^{1/q} & \text{if } \frac12 r<|y|<2r \text{ and } 1/q=\delta,\\ r^{-\delta(n-1)} & \text{if } |y|\leqq \frac12 r. \end{cases}$$

Corollary 4. *If $1<q<\infty$, then*

$$\int_{S(0,r)} |x-y|^{q(1-n)}\,dS(y)\leqq M\big||x|-r\big|^{-(n-1)(q-1)}$$

for every $x\in\mathbf{R}^n$.

Lemma 4. *If* $-1 < \beta < 0$ *and* $0 < (1-n)q + n < -\beta$, *then*

$$\int_{\mathbf{R}^n} |x-y|^{q(1-n)} \varrho(y)^\beta \, dy \leqq M \varrho(x)^{q(1-n)+n+\beta}$$

for every $x \in \mathbf{B}$.

Proof. In view of Corollary 4, we have

$$\begin{aligned}\int_{\mathbf{R}^n} |x-y|^{q(1-n)} \varrho(y)^\beta \, dy &= \int_0^\infty \Big(\int_{S(0,r)} |x-y|^{q(1-n)} \, dS(y) \Big) |1-r|^\beta \, dr \\ &\leqq M \int_{\mathbf{R}^1} \big| r - |x| \big|^{-(n-1)(q-1)} |1-r|^\beta \, dr.\end{aligned}$$

Since $0 < -(n-1)(q-1) + 1 < 1$ and $0 < \beta + 1 < 1$ by our assumptions, the Riesz composition theorem yields

$$\int_{\mathbf{R}^n} |x-y|^{q(1-n)} \varrho(y)^\beta \, dy \leqq M \varrho(x)^{-(n-1)(q-1)+\beta+1},$$

as required.

Lemma 5 (cf. [13, Corollary 5.1]). *If* μ *is a finite measure on the real line* $\mathbf{R}^1$ *and* $0 < d < 1$, *then*

$$\liminf_{r \to 0} |r|^d \int_{\mathbf{R}^1} |r-t|^{-d} \, d\mu(t) = \mu(\{0\}).$$

4. Proof of Theorem 1

Under the assumptions on p, α and q in Theorem 1, we can take (β, γ) such that

$$\alpha < \beta < p-1, \qquad 0 < \gamma < 1,$$

$$p(n-1)\gamma + p - n > 0,$$

$$p(n-1)\gamma + p - n < \beta < p(n-1)\gamma + \alpha - p(n-1)/q$$

and

$$\frac{1}{q} < \gamma < \frac{1}{q} + \frac{1}{p(n-1)}.$$

In view of Lemma 1 and Corollary 3, we may assume that

$$|u(x)| \leqq \int_{\mathbf{R}^n} |x-y|^{1-n} f(y) \, dy$$

for every $x \in \mathbf{B}$, where f is a nonnegative function on $\mathbf{R}^n$ which vanishes outside a bounded set and satisfies

$$\int_{\mathbf{R}^n} f(y)^p \varrho(y)^\alpha \, dy < \infty;$$

recall that $\varrho(y) = \big||y|-1\big|$. Using Hölder's inequality, we have with $1/p+1/p' = 1$

$$|u(x)| \leqq \left(\int_{\mathbf{R}^n} |x-y|^{a(1-n)} \varrho(y)^b \, dy \right)^{1/p'} \left(\int_{\mathbf{R}^n} |x-y|^{\gamma(1-n)p} f(y)^p \varrho(y)^\beta \, dy \right)^{1/p},$$

where $a = (1-\gamma)p'$ and $b = -\beta p'/p$. Since $-1 < b < 0$ and

$$\frac{b}{a} < \frac{n}{a'} - 1 < 0, \qquad a' = \frac{a}{a-1},$$

Lemma 4 yields

$$|u(x)| \leqq M \varrho(x)^{(1-\gamma)(1-n)+n/p'-\beta/p} \left(\int_{\mathbf{R}^n} |x-y|^{\gamma(1-n)p} f(y)^p \varrho(y)^\beta \, dy \right)^{1/p}.$$

In view of Minkowski's inequality for integral we have

$$S_q(u,r) \leqq M(1-r)^{(1-\gamma)(1-n)+n/p'-\beta/p} \times \left[\int_{\mathbf{R}^n} \left(\int_{S(0,r)} |x-y|^{\gamma(1-n)q} \, dS(x) \right)^{p/q} f(y)^p \varrho(y)^\beta \, dy \right]^{1/p}$$

for $2^{-1} < r < 1$. Since $\gamma q > 1$, Corollary 4 gives

$$S_q(u,r) \leqq M(1-r)^{(1-\gamma)(1-n)+n/p'-\beta/p} \times \left(\int_{\mathbf{R}^n} \big||y|-r\big|^{-(n-1)(\gamma q-1)p/q} f(y)^p \varrho(y)^\beta \, dy \right)^{1/p}.$$

For simplicity, set $d = (n-1)(\gamma q-1)p/q$. Then we see that $0 < d < 1$. Consider the function

$$K(s,t) = s^{p\omega} s^{p[(1-\gamma)(1-n)+n/p'-\beta/p]} |t-s|^{-d} t^{\beta-\alpha}$$

for $0 \leqq s < 1$ and $0 \leqq t < \infty$, where we set

$$\omega = (n-p+\alpha)/p - (n-1)/q.$$

Here note that

$$(1-r)^{\omega} S_q(u,r) \leqq M\biggl(\int_{\mathbf{R}^n} K\bigl(1-r,\varrho(y)\bigr) f(y)^p \varrho(y)^{\alpha}\,dy\biggr)^{1/p}.$$

Since $\omega + [(1-\gamma)(1-n) + n/p' - \beta/p] > 0$, we see that

$$\lim_{s\to 0} K(s,t) = 0$$

for all fixed $t > 0$. If $t \geqq \frac{3}{2}s$, then

$$K(s,t) \leqq M(s/t)^{(n-1)\gamma p + \alpha - \beta - p(n-1)/q} \leqq M,$$

if $0 \leqq t \leqq \frac{1}{2}s$, then

$$K(s,t) \leqq M(s/t)^{\alpha-\beta} \leqq M$$

and if $\frac{1}{2}s < t < \frac{3}{2}s$, then

$$K(s,t) \leqq M s^d |s-t|^{-d}.$$

Consequently, applying Lemma 5, we conclude that

$$\liminf_{r\to 1} (1-r)^{\omega} S_q(u,r) = 0.$$

Now the proof of Theorem 1 is completed.

5. Proof of Theorem 2

For a proof of Theorem 2, we need the following result, which gives an essential tool in treating monotone functions.

Lemma 6 (cf. [4, Lemma 7.1], [6, Remark, p. 9], [16, Section 16]). *Let $p > n-1$. If u is a monotone p-precise function on $B(x_0, 2r)$, then*

$$(4)\quad |u(x) - u(y)|^p \leqq M r^{p-n} \int_{B(x_0,2r)} |\nabla u(z)|^p\,dz \qquad \text{whenever } x,\ y \in B(x_0, r).$$

Lemma 6 is a consequence of Sobolev's theorem, so that the restriction $p > n-1$ is needed; for a proof of Lemma 6, see for example [4, Lemma 7.1] or [15, Theorem 5.2, Chapter 8].

Now we are ready to prove Theorem 2.

Proof of Theorem 2. Let u be a monotone function on $\mathbf{B}$ satisfying (1) with $n-1<p<n+\alpha$. If $|s-t| \leqq r < \frac{1}{2}(1-t)$, then Lemma 6 yields

$$|S_q(u,s)-S_q(u,t)| \leqq \left(\frac{1}{\sigma_n}\int_{S(0,1)} |u(s\xi)-u(t\xi)|^q \, dS(\xi)\right)^{1/q}$$
$$\leqq Mr^{(p-n)/p}\left(\int_{S(0,1)}\left(\int_{B(t\xi,2r)} |\nabla u(z)|^p \, dz\right)^{q/p} dS(\xi)\right)^{1/q},$$

so that Minkowski's inequality for integral yields

$$|S_q(u,s)-S_q(u,t)| \leqq Mr^{(p-n)/p}(2r/t)^{(n-1)/q} \times \left(\int_{B(0,t+2r)-B(0,t-2r)} |\nabla u(z)|^p \, dz\right)^{1/p}.$$

Let $r_j = 2^{-j-1}$, $t_j = 1-r_{j-1}$ and $A_j = B(0,1-r_j)-B(0,1-3r_j)$ for $j=1,2,\ldots$. As before, set

$$\omega = (n-p+\alpha)/p-(n-1)/q > 0.$$

Then we find

$$|S_q(u,t_j)-S_q(u,r)| \leqq Mr_{j+1}^{-\omega}\left(\int_{A_j} |\nabla u(z)|^p \varrho(z)^\alpha \, dz\right)^{1/p}$$

for $t_j \leqq r < t_j + r_{j+1}$,

$$|S_q(u,t_j+r_{j+1})-S_q(u,r)| \leqq Mr_{j+2}^{-\omega}\left(\int_{A_j} |\nabla u(z)|^p \varrho(z)^\alpha \, dz\right)^{1/p}$$

for $t_j + r_{j+1} \leqq r < t_j + r_{j+1} + r_{j+2}$ and

$$|S_q(u,r)-S_q(u,t_{j+1})| \leqq Mr_{j+2}^{-\omega}\left(\int_{A_{j+1}} |\nabla u(z)|^p \varrho(z)^\alpha \, dz\right)^{1/p}$$

for $t_j + r_{j+1} + r_{j+2} \leqq r < t_{j+1}$. Collecting these results, we have

$$|S_q(u,t_j)-S_q(u,r)| \leqq Mr_j^{-\omega}\left(\int_{A_j} |\nabla u(z)|^p \varrho(z)^\alpha \, dz\right)^{1/p} + Mr_{j+1}^{-\omega}\left(\int_{A_{j+1}} |\nabla u(z)|^p \varrho(z)^\alpha \, dz\right)^{1/p}$$

for $t_j \leqq r < t_{j+1}$. Hence it follows that

$$|S_q(u,t_j) - S_q(u,t_{j+m})| \leqq M \sum_{l=j}^{j+m} r_l^{-\omega} \left(\int_{A_l} |\nabla u(z)|^p \varrho(z)^\alpha \, dz \right)^{1/p}.$$

Since $A_l \cap A_k = \emptyset$ when $l \geqq k+2$, Hölder's inequality gives

$$\begin{aligned} |S_q(u,t_j) - S_q(u,t_{j+m})| &\leqq M \left(\sum_{l=j}^{j+m} r_l^{-p'\omega} \right)^{1/p'} \left(\sum_{l=j}^{j+m} \int_{A_l} |\nabla u(z)|^p \varrho(z)^\alpha \, dz \right)^{1/p} \\ &\leqq M r_{j+m}^{-\omega} \left(\int_{B(0,1-r_{j+m}) - B(0,1-3r_j)} |\nabla u(z)|^p \varrho(z)^\alpha \, dz \right)^{1/p}. \end{aligned}$$

More generally, if $t_j \leqq r < 1$, then we take m such that $t_{j+m-1} \leqq r < t_{j+m}$, and establish

$$|S_q(u,t_j) - S_q(u,r)| \leqq M(1-r)^{-\omega} \left(\int_{\mathbf{B} - B(0,1-3r_j)} |\nabla u(z)|^p \varrho(z)^\alpha \, dz \right)^{1/p},$$

which implies that

$$\limsup_{r \to 1} (1-r)^\omega S_q(u,r) \leqq M \left(\int_{\mathbf{B} - B(0,1-3r_j)} |\nabla u(z)|^p \varrho(z)^\alpha \, dz \right)^{1/p}$$

for all j. Therefore it follows that

$$\lim_{r \to 1} (1-r)^\omega S_q(u,r) = 0,$$

as required.

6. Proof of Theorem 3

Let u be a monotone function on $\mathbf{B}$ satisfying (3) with $n-1 < p < n+\alpha$. If $B(x,2r) \subset \mathbf{B}$ and $0 < \delta < 1$, then, applying Lemma 6 and dividing the domain of integration into two parts

$$\begin{aligned} E_1 &= \{ z \in B(x,2r) : |\nabla u(z)| > r^{-\delta} \}, \\ E_2 &= B(x,2r) - E_1, \end{aligned}$$

we have

$$|u(x) - u(y)|^p \leqq M r^{p-n-\delta p} \int_{E_2} dz + M r^{p-n} [\varphi(r^{-\delta})]^{-1} \int_{E_1} \Phi_p(|\nabla u(z)|) \, dz.$$

Since $\varphi(r^{-\delta}) \geqq M\varphi(r^{-1})$ for $r > 0$, it follows that

$$|u(x) - u(y)|^p \leqq Mr^{(1-\delta)p} + Mr^{p-n}[\varphi(r^{-1})]^{-1} \int_{B(x,2r)} \Phi_p\big(|\nabla u(z)|\big)\, dz \tag{5}$$

for $y \in B(x,r)$.

Let $x_0 = 0$ and $r_j = 2^{-j-1}$, $j = 0, 1, \ldots$. For $\xi \in S(0,1)$, let $x_j = (1-2r_j)\xi$. Then we find with the aid of (5)

$$|u(x_j) - u(y_1)|^p \leqq Mr_j^{(1-\delta)p} + Mr_j^{p-n}[\varphi(r_j^{-1})]^{-1} \int_{B(x_j,r_j)} \Phi_p\big(|\nabla u(z)|\big)\, dz$$

for $y_1 \in S(x_j, \frac{1}{2}r_j)$,

$$|u(y_1) - u(y_2)|^p \leqq Mr_j^{(1-\delta)p} + Mr_j^{p-n}[\varphi(r_j^{-1})]^{-1} \int_{B(x_j,r_j)} \Phi_p\big(|\nabla u(z)|\big)\, dz$$

for $y_2 \in S(y_1, \frac{1}{4}r_j)$ and

$$|u(y_2) - u(x_{j+1})|^p \leqq Mr_{j+1}^{(1-\delta)p} + Mr_{j+1}^{p-n}[\varphi(r_{j+1}^{-1})]^{-1} \int_{B(x_{j+1},r_{j+1})} \Phi_p\big(|\nabla u(z)|\big)\, dz$$

for $y_2 \in S(x_{j+1}, \frac{1}{2}r_{j+1})$. Thus it follows that

$$\begin{aligned}
|u(x_j) - u(x_{j+1})| &\leqq Mr_j^{1-\delta} + Mr_{j+1}^{1-\delta} \\
&\quad + Mr_j^{(p-n)/p}[\varphi(r_j^{-1})]^{-1/p}\left(\int_{B(x_j,r_j)} \Phi_p\big(|\nabla u(z)|\big)\, dz\right)^{1/p} \\
&\quad + Mr_{j+1}^{(p-n)/p}[\varphi(r_{j+1}^{-1})]^{-1/p}\left(\int_{B(x_{j+1},r_{j+1})} \Phi_p\big(|\nabla u(z)|\big)\, dz\right)^{1/p} \\
&\leqq Mr_j^{1-\delta} + Mr_{j+1}^{1-\delta} + Mr_j^{(p-n-\alpha)/p}[\varphi(r_j^{-1})]^{-1/p} \\
&\qquad \times \left(\int_{B(x_j,r_j)} \Phi_p\big(|\nabla u(z)|\big)\varrho(z)^\alpha\, dz\right)^{1/p} \\
&\quad + Mr_{j+1}^{(p-n-\alpha)/p}[\varphi(r_{j+1}^{-1})]^{-1/p}\left(\int_{B(x_{j+1},r_{j+1})} \Phi_p\big(|\nabla u(z)|\big)\varrho(z)^\alpha\, dz\right)^{1/p},
\end{aligned}$$

so that

$$\begin{aligned}
|u(x_{j+m}) - u(x_j)| &\leqq M\sum_{l=j}^{j+m} r_l^{1-\delta} \\
&\quad + M\sum_{l=j}^{j+m} r_l^{(p-n-\alpha)/p}[\varphi(r_l^{-1})]^{-1/p}\left(\int_{B(x_l,r_l)} \Phi_p\big(|\nabla u(z)|\big)\varrho(z)^\alpha\, dz\right)^{1/p}.
\end{aligned}$$

Since $B(x_l, r_l) \cap B(x_k, r_k) = \emptyset$ when $l \geqq k+2$, Hölder's inequality gives

$$
\begin{aligned}
|u(x_j) - u(x_{j+m})| &\leqq M r_j^{1-\delta} \\
&+ M\left(\sum_{l=j}^{j+m} r_l^{p'(p-n-\alpha)/p}[\varphi(r_l^{-1})]^{-p'/p}\right)^{1/p'} \left(\sum_{l=j}^{j+m}\int_{B(x_l,r_l)} \Phi_p\big(|\nabla u(z)|\big)\varrho(z)^\alpha\, dz\right)^{1/p} \\
&\leqq M r_j^{1-\delta} + M\kappa(r_{j+m})\left(\int_{B(0,1-r_{j+m})-B(0,1-3r_j)} \Phi_p\big(|\nabla u(z)|\big)\varrho(z)^\alpha\, dz\right)^{1/p}.
\end{aligned}
$$

If $x \in B(x_{j+m}, r_{j+m})$ with $x_j = (1-2r_j)x/|x|$, then

$$
|u(x) - u(x_j)| \leqq M r_j^{1-\delta} + M\kappa\big(\varrho(x)\big)\left(\int_{\mathbf{B}-B(0,1-3r_j)} \Phi_p\big(|\nabla u(z)|\big)\varrho(z)^\alpha\, dz\right)^{1/p},
$$

which implies that

$$
\limsup_{|x|\to 1}\big[\kappa\big(\varrho(x)\big)\big]^{-1}|u(x)| \leqq M\left(\int_{\mathbf{B}-B(0,1-3r_j)} \Phi_p\big(|\nabla u(z)|\big)\varrho(z)^\alpha\, dz\right)^{1/p}
$$

for all j. Therefore it follows that

$$
\lim_{|x|\to 1}\big[\kappa\big(\varrho(x)\big)\big]^{-1}u(x) = 0,
$$

as required.

7. Remarks

Remark 1. Let $\{e_j\}$ be a sequence in $\mathbf{B}$ which tends to a boundary point. For a number $a > 0$ and a sequence $\{\varepsilon_j\}$ of positive numbers, consider the function

$$
u(x) = \sum_j \varepsilon_j |x - e_j|^{-a}.
$$

If $a < (n-p)/p$, then we can choose $\{\varepsilon_j\}$ such that

$$
\int_{\mathbf{B}} |\nabla u(x)|^p\, dx < \infty.
$$

Further, if $a > (n-1)/q$, then we have

$$
S_q(u, |e_j|) = \infty.
$$

This implies that the lower limit in Theorem 1 can not be replaced by the upper limit.

Remark 2. Let $-1 < \alpha < p-1$. For $\delta > 0$, consider the function

$$f(y) = \big||y|-1\big|^a |y-e|^{-b},$$

where $a = \delta - (\alpha+1)/p$, $b = (n-1)/p$ and $e = (1, 0, \dots, 0)$. Then

$$\int_{B(2e,1)} f(y)^p \varrho(y)^\alpha \, dy < \infty.$$

We consider the harmonic function u on $\mathbf{B}$ defined by

$$u(x) = \int_{B(2e,1)} (y_1 - x_1)|x-y|^{-n} f(y) \, dy.$$

Then we apply [13, Lemmas 12.1 and 12.2] to establish

$$\int_{\mathbf{R}^n} |\nabla u(x)|^p \varrho(x)^\alpha \, dx < \infty$$

by considering Lipschitz transformations from neighborhoods of boundary points of $\mathbf{B}$ to half spaces. If $x \in \mathbf{B}$, then

$$u(x) > \int_{B(x^*, |x-e|/4)} (y_1 - x_1)|x-y|^{-n} f(y) \, dy > M|x-e|^{1+a-b},$$

where $x^* = (1 + \frac{1}{2}|x-e|)e$. Hence, if $k(x) = |x-e|^{1+a-b}$ and $\delta < (n-p+\alpha)/p - (n-1)/q$, then

$$S_q(u,r) \geqq M S_q(k,r) \geqq M(1-r)^{(p-n-\alpha)/p+(n-1)/q+\delta}.$$

This implies that the exponent $(n-p+\alpha)/p - (n-1)/q$ is sharp in Theorems 1 and 2.

Remark 3. Let u be a locally p-precise function on $\mathbf{B}$ satisfying

(6) $$\int_{\mathbf{B}} |\nabla u(x)|^p \varrho(x)^\alpha \, dx < \infty.$$

We see that if $0 \leqq \alpha < p-1$ and

$$\frac{1}{q} = \frac{n-p+\alpha}{p(n-1)} > 0,$$

then

(7) $$S_q(u,r) \leqq M \left(\int_{\mathbf{R}^n} |\nabla u(x)|^p \varrho(x)^\alpha \, dx \right)^{1/p}.$$

Yamashita [27] derived the above inequality for harmonic functions u on $\mathbf{B}$ satisfying (6) with $p = 2$ and $0 \leqq \alpha \leqq 1$. In the hyperplane case, we refer to [16, Theorem 2.2], and the present result will be proved similarly. In fact, to prove (7), we apply Sobolev's integral representation (Lemma 2 and Corollary 3) and write

$$u(x) = c \sum_{j=1}^{n} \int_{\mathbf{R}^n} \frac{x_j - y_j}{|x-y|^n} \frac{\partial \overline{u}}{\partial y_j}(y)\, dy.$$

Here we may assume that the extension $\overline{u}$ vanishes outside $B(0,2)$. As in the proof of Theorem 2.2 of [16], we have by Hölder's inequality

$$\begin{aligned} |u(x)| &\leqq M \int_{S(0,1)} \left(\int_0^2 |x - ty^*|^{(1-n)p'} \big| |t| - 1 \big|^{-\alpha p'/p} t^{n-1}\, dt \right)^{1/p'} \\ &\quad \times \left(\int_{\mathbf{R}^1} |\nabla \overline{u}(ty^*)|^p \big| |t| - 1 \big|^{\alpha} t^{n-1}\, dt \right)^{1/p} dS(y^*) \\ &\leqq M \int_{S(0,1)} |x^* - y^*|^{1-n+1/p'-\alpha/p} \\ &\quad \times \left(\int_{\mathbf{R}^1} |\nabla \overline{u}(ty^*)|^p \big| |t| - 1 \big|^{\alpha} t^{n-1}\, dt \right)^{1/p} dS(y^*), \end{aligned}$$

where $x^* = x/|x|$ and $y^* = y/|y|$. Now it suffices to apply Sobolev's inequality.

The case $\alpha = p - 1$ remains open.

Remark 4. Let u be a locally p-precise function on $\mathbf{B}$ satisfying (3). Note here that if

$$\int_0^1 \left[r^{n-p} \varphi(r^{-1}) \right]^{-1/(p-1)} \frac{dr}{r} < \infty, \tag{8}$$

then u is continuous on $\mathbf{B}$ and satisfies (5) on the basis of [10, Lemma 3], so that the conclusions of Theorems 2 and 3 are also valid for u. If in addition

$$\int_0^1 \left[r^{n-p+\alpha} \varphi(r^{-1}) \right]^{-1/(p-1)} \frac{dr}{r} < \infty, \tag{9}$$

then u has a continuous extension to $\mathbf{R}^n$, according to [10, Theorem 2]. For these facts, see also [13], [15] and [20].

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