ON THEOREMS OF HODOGRAPHIC AND ANTHODOGRAPHIC ISOCHRONISM

By

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On a Theorem of Hodographic Isochronism.

William Rowan Hamilton

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The following note by Sir W. R. Hamilton, announcing a theorem of hodographic isochronism, was read:

If two circular hodographs, having a common chord, which passes through or tends towards a common centre of force, be cut perpendicularly by a third circle, the times of hodographically describing the intercepted arcs will be equal.

On a Theorem of Anthodographic Isochronism.

William Rowan Hamilton

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Sir William Hamilton stated and illustrated a theorem of anthodographic (or anthodic) isochronism, namely, that if two circular anthodes, having a common chord, which passes through, or tends towards a common centre of force, be both cut perpendicularly by any third circle, the times of anthodically describing the intercepted arcs will be equal:—the *anthode* of a planet being the circular locus of the extremities of its vectors of slowness, or of straight lines representing, in length and direction, the reciprocals of its velocities, and drawn from a common origin.

The theorem is intimately connected with the analogous theorem respecting hodographic isochronism (or synchronism), which was communicated to the Academy by Sir William Hamilton, in a note read at the Meeting in last March. He had been led to perceive that former theorem by combining the principles of his first paper on a General Method in Dynamics, published in the second part of the Philosophical Transactions for 1834, with those of his communication of last December, since published in the Proceedings of the Academy, respecting the Law of the Circular Hodograph. This *Hodograph* was, for a planet or comet, the circular locus of the extremities of its *vectors of velocity*, as the Anthode is the locus of the extremities of the vectors of slowness; so that the rectangular coordinates of the Hodograph are x', y', z', if

$$x' = \frac{dx}{dt}, \quad y' = \frac{dy}{dt}, \quad z' = \frac{dz}{dt};$$

while those of the Anthode may be denoted as follows:

$$x_{\prime} = -v^{-2}x', \quad y_{\prime} = -v^{-2}y', \quad z_{\prime} = -v^{-2}z',$$

where $v^2 = x'^2 + y'^2 + z'^2$.

He had effected the passage from the theorem respecting hodographic to that respecting anthodic isochronism, by the help of his calculus of quaternions; but had since been able to prove both theorems by means of certain elementary properties of the circle.

For a hyperbolic comet, the Anthode is a circular arc *convex* to the sun; for a parabolic comet, the Anthode is a *straight line.* And for comets of this latter class the theorem of isochronism takes this curiously simple form: "Any two diameters of any one circle (or sphere) in space, are anthodically described in equal times, with reference to any one point, regarded as a common centre of force." By this last theorem, the general problem of determining the time of *orbital* description of a finite arc of a *parabola*, is reduced to that of determining the time of *anthodical* description of a finite *straight line* directed to the sun; and thus it is found that "the interval of time between any two positions of a parabolic comet, divided by the mass of the sun, is equal to the sixth part of the difference of the cubes of the sum and difference of the diagonals of the parallelogram, constructed with the initial and final vectors of slowness as two adjacent sides." Another very simple extension for the time of description of a parabolic arc, to which Sir William Hamilton is conducted by his own method, but which he sees to admit of easy proof from known principles (though he does not remember meeting the expression itself), is given by the following formula:

$$t = \tfrac{1}{2}\mathrm{T} \tan(\theta - \tan^{-1} \tfrac{1}{2} \tan \tfrac{1}{2}\theta);$$

where θ is the true anomaly, and t is the time from perihelion, while T is the time of describing the first quadrant of true anomaly.