ON AN EXPRESSION FOR THE NUMBERS OF BERNOULLI, BY MEANS OF A DEFINITE INTEGRAL, AND ON SOME CONNECTED PROCESSES OF SUMMATION AND INTEGRATION

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On an Expression for the Numbers of Bernoulli, *by means of a Definite Integral; and on some connected Processes of Summation and Integration. By* Sir William Rowan Hamilton, *LL.D., P.R.I.A., Member of several Scientific Societies at Home and Abroad, Andrews' Professor of Astronomy in the University of Dublin, and Royal Astronomer of Ireland.*

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The following analysis, extracted from a paper which has been in part communicated to the Royal Irish Academy, but has not yet been printed, may interest some readers of the Philosophical Magazine.

1. Let us consider the function of two real variables, m and n, represented by the definite integral

$$y_{m,n} = \int_0^\infty dx \left(\frac{\sin x}{x}\right)^m \cos nx; \tag{1.}$$

in which we shall suppose that m is greater than zero; and which gives evidently the general relation

$$y_{m,-n} = y_{m,n}.$$

By changing m to $m+1$; integrating first the factor $x^{-m-1}\,dx$, and observing that

$$x^{-m} \sin x^{m+1} \cos nx$$

vanishes both when $x = 0$, and when $x = \infty$; and then putting the differential coefficient $\dfrac{d}{dx}(\sin x^{m+1} \cos nx)$ under the form

$$\tfrac{1}{2} \sin x^m \{(m+1+n)\cos(nx+x) + (m+1-n)\cos(nx-x)\};$$

we are conducted to the following equation, in finite and partial differences,

$$2my_{m+1,n} = (m+1+n)y_{m,n+1} + (m+1-n)y_{m,m-1}; \tag{2.}$$

and if we suppose that the difference between the two variables on which y depends is an even integer number, this equation takes the form

$$my_{m+1,m+1-2k} = (m+1-k)y_{m,m+2-2k} + ky_{m,m-2k}. \tag{3.}$$

The same equation in differences (2.) shows easily that

$$\begin{aligned}&y_{m+1,n} = 0,\ \text{when}\ n =\ \text{or}\ > m+1,\\ &\text{if}\ y_{m,n-1} = 0,\ \text{when}\ n-1 > m;\end{aligned}$$

but, by a well-known theorem, which in the present notation becomes

$$y_{1,0} = \frac{\pi}{2}, \tag{4.}$$

it is easy to prove, not only that

$$y_{1,1} = y_{1,-1} = \frac{\pi}{4}, \tag{5.}$$

but also that

$$y_{1,n} = 0,\ \text{if}\ n^2 > 1; \tag{6.}$$

we have therefore, generally, for all whole values of $m > 1$, and for all real values of n,

$$y_{m,n} = 0,\ \text{unless}\ n^2 < m^2. \tag{7.}$$

2. If then we make

$$\mathrm{T}_m = \Sigma y_{m,m-2k}(-t)^k, \tag{8.}$$

the sign Σ indicating a summation which may be extended from as large a negative to as large a positive whole value of k as we think fit, but which extends at least from $k = 0$ to $k = m$, m being here a positive whole number; this sum will in general (namely when $m > 1$) include only $m-1$ terms different from 0, namely those which correspond to $k = 1, 2, \dots m-1$; but in the particular case $m = 1$, the sum will have two such terms, instead of none, namely those answering to $k = 0$ and $k = 1$, so that we shall have

$$\mathrm{T}_1 = y_{1,1} - y_{1,-1}t = \frac{\pi}{4}(1-t). \tag{9.}$$

Multiplying the first member of the equation in differences (3.) by $(-t)^k$, and summing with respect to k, we obtain $m\mathrm{T}_{m+1}$, m being here any whole number > 0. Multiplying and summing in like manner the second member of the same equation (3.), the term $my_{m,m+2-2k}$ of that member gives $-mt\mathrm{T}_m$, because we may change k to $k+1$ before effecting the indefinite summation; $ky_{m,m-2k}$ gives $t\frac{d}{dt}\mathrm{T}_m$; and $(1-k)y_{m,m+2-2k}$ gives $t^2\frac{d}{dt}\mathrm{T}_m$; but

$$-mt\mathrm{T}_m + (t+t^2)\frac{d}{dt}\mathrm{T}_m = (1+t)^{m+1}\frac{td}{dt}(1+t)^{-m}\mathrm{T}_m;$$

therefore

$$m(1+t)^{-m-1}\mathrm{T}_{m+1} = \frac{d}{d\log t}(1+t)^{-m}\mathrm{T}_m. \tag{10.}$$

This equation in mixed differences gives, by (9.),

$$\mathrm{T}_m = \frac{\pi}{4}\frac{(1+t)^m}{1\,.\,2\,.\,3\,\dots\,(m-1)}\left(\frac{d}{d\log t}\right)^{m-1}\frac{1-t}{1+t}; \tag{11.}$$

the factorial denominator being considered as $= 1$, when $m = 1$, as well as when $m = 2$. If $m > 1$, we may change $\dfrac{1-t}{1+t}$ to $\dfrac{2}{1+t}$, from which it only differs by a constant; and then by changing also t to e^h, and multiplying by $\dfrac{2}{\pi}$, we obtain the formula:

$$\frac{(e^h+1)^m}{1\,.\,2\,.\,3\,\ldots\,(m-1)}\left(\frac{d}{dh}\right)^{m-1}(e^h+1)^{-1}$$
$$= \frac{2}{\pi}\Sigma_{(k)}\,{}_1^{m-1}\int_0^\infty dx\left(\frac{\sin x}{x}\right)^m(-e^h)^k\cos(mx-2kx); \qquad (12.)$$

which conducts to many interesting consequences. A few of them shall be here mentioned.

3. The summation indicated in the second member of this formula can easily be effected in general; but we shall here consider only the two cases in which m is an odd or an even whole number greater than unity, while h becomes $= 0$ after the $m-1$ differentiations of $(e^h+1)^{-1}$, which are directed in the first member.

When m is odd (and greater than one), each power, such as $(-e^h)^k$ in the second member, is accompanied by another, namely $(-e^h)^{m-k}$, which is multiplied by the cosine of the same multiple of x; and these two powers destroy each other, when added, if $h = 0$: we arrive therefore in this manner at the known result, that

$$\left(\frac{d}{dh}\right)^{2p}(e^h+1)^{-1} = 0, \quad \text{when} \quad h = 0, \quad \text{if} \quad p > 0. \qquad (13.)$$

On the contrary, when m is even, and $h = 0$, the powers $(-e^h)^k$ and $(-e^h)^{m-k}$ are equal, and their sum is double of either; and because

$$(-1)^p\{1 - 2\cos 2x + 2\cos 4x - \cdots + (-1)^{p-1}2\cos(2px-2x)\} = -\frac{\cos(2px-x)}{\cos x},$$

by making $m = 2p$ we arrive at this other result, which perhaps is new, that (if $p > 0$ and $h = 0$)

$$\left(\frac{d}{dh}\right)^{2p-1}(e^h+1)^{-1} = \frac{-1\,.\,2\,.\,3\,\ldots\,(2p-1)}{2^{2p-1}\pi}\int_0^\infty dx\left(\frac{\sin x}{x}\right)^{2p}\frac{\cos(2px-x)}{\cos x}. \qquad (14.)$$

Developing therefore $(e^h+1)^{-1}$ according to ascending powers of h; subtracting the development from $\frac{1}{2}$, multiplying by h, and changing h to $2h$; we obtain

$$h\frac{e^h-e^{-h}}{e^h+e^{-h}} = \frac{2}{\pi}\int_0^\infty \frac{dx}{\cos x}\Sigma_{(p)}\,{}_1^\infty\left(\frac{h\sin x}{x}\right)^{2p}\cos(2px-x); \qquad (15.)$$

that is, effecting the summation, and dividing by h^2,

$$\frac{1}{h}\frac{e^h-e^{-h}}{e^h+e^{-h}} = \frac{2}{\pi}\int_0^\infty \frac{dx\,x^{-2}\sin x^2(1-h^2x^{-2}\sin x^2)}{1-2h^2x^{-2}\sin x^2\cos 2x + h^4x^{-4}\sin x^4}; \qquad (16.)$$

or, integrating both members with respect to h,

$$\int_0^h \frac{dh}{h}\frac{e^h - e^{-h}}{e^h + e^{-h}} = \frac{1}{\pi}\int_0^\infty \frac{dx}{x}\tan x \log\sqrt{\frac{1 + hx^{-1}\sin 2x + h^2x^{-2}\sin x^2}{1 - hx^{-1}\sin 2x + h^2x^{-2}\sin x^2}}. \tag{17.}$$

It might seem, at first sight, from this equation, that the integral in the first member ought to vanish, when taken from $h = 0$ to $h = \infty$; because, if we set about to develope the second member, according to the descending powers of h, we see that the coefficient of h^0 vanishes; but when we find that, on the same plan, the coefficient of h^{-1} is infinite, being $= \dfrac{2}{\pi}\displaystyle\int_0^\infty dx$, we perceive that this mode of development is here inappropriate: and in fact, it is clear that the first member of the formula (17.) increases continually with h, while h increases from 0.

4. Again, since

$$\frac{-h}{e^h + 1} = \psi(2h) - \psi(h), \quad \text{if} \quad \psi(h) = \frac{h}{e^h - 1}, \tag{18.}$$

we shall have (for $p > 0$) the expression

$$\mathrm{A}_{2p} = \frac{2^{1-2p}\pi^{-1}}{2^{2p} - 1}\int_0^\infty dx \left(\frac{\sin x}{x}\right)^{2p} \frac{\cos(2px - x)}{\cos x}, \tag{19.}$$

if, according to a known form of development, which the foregoing reasonings would suffice to justify, we write

$$\frac{h}{e^h - 1} + \frac{h}{2} = 1 + \mathrm{A}_2h^2 + \mathrm{A}_4h^4 + \mathrm{A}_6h^6 + \&\text{c}. \tag{20.}$$

If p be a large number, the rapid and repeated changes of sign of the numerator of the fraction $\dfrac{\cos(2px - x)}{\cos x}$ produce nearly a mutual destruction of the successive elements of the integral (19.), except in the neighbourhood of those values of x which cause the denominator of the same fraction to vanish; namely those values which are odd positive multiples of $\frac{\pi}{2}$ (the integral itself being not extended so as to include any negative values of x). Making therefore

$$x = (2i - 1)\frac{\pi}{2} + \omega, \tag{21.}$$

in which i is a whole number > 0, and ω is positive or negative, but nearly equal to 0; we shall have

$$\cos(2px - x) = (-1)^{p+i-1}\sin(2p\omega - \omega),$$

exactly, and $\cos x = (-1)^i\omega$, nearly; changing also $\left(\dfrac{\sin x}{x}\right)^{2p}$ to the value which it has when $\omega = 0$, namely $\left(\dfrac{2}{\pi}\right)^{2p}(2i - 1)^{-2p}$; and observing that

$$\int_{-\omega}^{\omega} d\omega\,\frac{\sin(2p\omega - \omega)}{\omega} = \pi, \quad \text{nearly}, \tag{22.}$$

even though the extreme values of ω may be small, if p be very large; we find that the part of A_{2p}, corresponding to any one value of the number i, is, at least nearly, represented by the expression

$$\frac{(-1)^{p-1}2(2i-1)^{-2p}}{(2^{2p}-1)\pi^{2p}}; \tag{23.}$$

which is now to be summed, with reference to i, from $i = 1$ to $i = \infty$. But this summation gives rigorously the relation

$$\Sigma_{(i)}{}_1^\infty (2i-1)^{-2p} = (1 - 2^{-2p})\Sigma_{(i)}{}_1^\infty i^{-2p}; \tag{24.}$$

we are conducted, therefore, to the expression

$$\mathrm{A}_{2p} = (-1)^{p-1}2(2\pi)^{-2p}\Sigma_{(i)}{}_1^\infty i^{-2p}, \tag{25.}$$

as *at least* approximately true, when the number p is large. But in fact the expression (25.) is *rigorous* for all whole values of p greater than 0; as we shall see by deducing from it an analogous expression for a Bernoullian number, and comparing this with known results.

5. The development

$$\frac{1}{e^h - 1} + \frac{1}{2} = h^{-1} + \mathrm{B}_1 \frac{h}{1\,.\,2} - \mathrm{B}_3 \frac{h^3}{1\,.\,2\,.\,3\,.\,4} + \&\text{c.}, \tag{26.}$$

being compared with that marked (20.), gives, for the pth Bernoullian number, the known expression

$$\mathrm{B}_{2p-1} = (-1)^{p-1} 1\,.\,2\,.\,3\,.\,4\,\ldots\,2p\mathrm{A}_{2p}; \tag{27.}$$

and therefore, rigorously, by the equation (19.) of the present paper,

$$\mathrm{B}_{2p-1} = \frac{(-1)^{p-1}1\,.\,2\,.\,\ldots\,2p}{2^{2p-1}(2^{2p}-1)\pi} \int_0^\infty dx \left(\frac{\sin x}{x}\right)^{2p} \frac{\cos(2px - x)}{\cos x}; \tag{28.}$$

a formula which is believed to be new. Treating the definite integral which it involves by the process just now used, we necessarily obtain the same result as if we combine at once the equations (25.) and (27.). We find, therefore, in this manner, that the equation

$$\frac{2^{2p-1}\pi^{2p}\mathrm{B}_{2p-1}}{1\,.\,2\,.\,3\,.\,4\,\ldots\,2p} = \Sigma_{(i)}{}_1^\infty i^{-2p}, \tag{29.}$$

(in which, by the notation here employed,

$$\Sigma_{(i)}{}_1^\infty i^{-2p} = 1^{-2p} + 2^{-2p} + 3^{-2p} + \&\text{c.})$$

is *at least nearly true*, when p is a large number, but Euler has shown, in his *Institutiones Calculi Differentialis* (vol. i. cap. v. p. 339. ed. 1787), that this equation (29.) is *rigorous*, each member being the coefficient of u^{2p} in the development of $\frac{1}{2}(1 - \pi i \cot \pi u)$. [See also

Professor De Morgan's Treatise on the Diff. and Int. Calc., 'Library of Useful Knowledge,' part xix. p. 581.] The analysis of the present paper is therefore not only verified generally, but also the modifications which were made in the form of that definite integral which entered into our rigorous expressions (19.) and (28.) for A_{2p} and B_{2p-1}, by the process of the last article, (on the ground that the parts omitted or introduced thereby must at least nearly destroy each other, through what may be called the "principle of fluctuation,") are now seen to have produced no ultimate error at all, their mutual compensation being perfect; a result which may tend to give increased confidence in applying a similar process of approximation, or transformation, to the treatment of other similar integrals; although the logic of this process may deserve to be more closely scrutinized. Some assistance towards such a scrutiny may be derived from the essay on "Fluctuating Functions," which has been published by the present writer in the second part of the nineteenth volume of the Transactions of the Royal Irish Academy.

6. It may be worth while to notice, in conclusion, that the property marked (7.) of the definite integral (1.), enables us to change $\dfrac{\cos(2px - x)}{\cos x}$ to $\sin 2px \,\tan x$, in the equations (14.), (15.), (19.), (28.); so that the pth Bernoullian number may rigorously be expressed as follows:—

$$\mathrm{B}_{2p-1} = \frac{(-1)^{p-1} \,.\, 1 \,.\, 2 \,\ldots\, 2p}{2^{2p-1}(2^{2p}-1)\pi} \int_0^\infty dx \left(\frac{\sin x}{x}\right)^{2p} \sin 2px \,\tan x; \tag{30.}$$

under which form the preceding deduction of its transformation (29.) admits of being slightly simplified. The same modification of the foregoing expressions conducts easily to the equation

$$\log \frac{e^h + e^{-h}}{2} = \frac{1}{\pi} \int_0^\infty dx \,\tan x \tan^{-1} \frac{h^2 \sin x^2 \sin 2x}{x^2 - h^2 \sin x^2 \cos 2x}; \tag{31.}$$

in which $\tan^{-1}$ is a characteristic equivalent to arc tang., and which may be made an expression for $\log \sec x$, by merely changing the sign of h^2 in the last denominator; and from this equation (31.) it would be easy to return to an expression for the coefficients in the development of $\dfrac{e^h - e^{-h}}{e^h + e^{-h}}$, or in that of $\tan h$, and therefore to the numbers of Bernoulli. Those numbers might thus be deduced from the following very simple equation:

$$\pi \log \sec h = \int_0^\infty dx \, y \tan x; \tag{32.}$$

in which y is connected with x and h by the relation

$$\frac{\sin y}{\sin(2x - y)} = \left(\frac{h \sin x}{x}\right)^2. \tag{33.}$$

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