ON THE PROPAGATION OF LIGHT IN VACUO AND IN CRYSTALS

By

William Rowan Hamilton

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On the Propagation of Light in vacuo. *By Professor* Sir W. R. Hamilton.

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The object of this communication was to advance the state of our knowledge respecting the law which regulates the attractions or repulsions of the particles of the ether on each other. The general differential equations of motion of any system of attracting or repelling points being reducible to the form

$$\frac{d^2x}{dt^2} = \mathrm{S} \,.\, m_{\prime}\, \Delta x f(r), \tag{1.}$$

the equations of minute vibration are of the form

$$\frac{d^2\delta x}{dt^2} = \mathrm{S} \,.\, m_{\prime} \left(\Delta\delta x \,.\, f(r) + \Delta x \,.\, \delta f(r)\right), \tag{2.}$$

in which

$$\delta f(r) = f'(r)\, \delta r, \tag{3.}$$

and

$$\delta r = \frac{\Delta x}{r}\, \Delta\delta x + \frac{\Delta y}{r}\, \Delta\delta y + \frac{\Delta z}{r}\, \Delta\delta z. \tag{4.}$$

A mode of satisfying the differential equations (2), and at the same time of representing a large class of the phenomena of light, is to assume

$$\frac{\delta x}{\xi^{\backprime}} = \frac{\delta y}{\eta^{\backprime}} = \frac{\delta z}{\zeta^{\backprime}} = \text{const.} + \cos.\, \frac{2\pi(vy - ax - by - cz)}{\lambda}, \tag{5.}$$

in which $\xi^{\backprime}$, $\eta^{\backprime}$, $\zeta^{\backprime}$ are constants, depending on the extent and direction of vibration: a, b, c, are the cosines of the inclinations of the direction of propagation of a plane wave to the positive semi-axes of x, y, z; v is the velocity of propagation of that wave, and λ is the length of an undulation; and π is the semicircumference of a circle, of which the radius is unity. With this assumption (5.), and with a natural and obvious supposition respecting a certain symmetry of arrangement in the ether, causing the sums of odd powers to vanish, it is permitted to substitute in (2.) the expressions

$$\frac{d^2\delta x}{\delta t^2} = -\left(\frac{2\pi v}{\lambda}\right)^2 .\, \delta x, \tag{6.}$$

$$\Delta\delta x = -\,\text{vers.}\, \Delta\theta \,.\, \delta x, \tag{7.}$$

in which

$$\Delta\theta = -\frac{2\pi}{\lambda}(a\,\Delta x + b\,\Delta y + c\,\Delta z); \tag{8.}$$

and thus arises a system of conditions of the form

$$\begin{aligned}\xi' \left(\frac{2\pi v}{\lambda}\right)^2 &= \xi' m \mathrm{S}\,.\left\{f(r) + \frac{\Delta x^2}{r} f'(r)\right\} \text{vers.}\,\Delta\theta \\ &+\eta' m\mathrm{S}\,.\,\frac{\Delta x\,\Delta y}{r} f'(r)\,\text{vers.}\,\Delta\theta \\ &+\zeta' m\mathrm{S}\,.\,\frac{\Delta x\,\Delta z}{r} f'(r)\,\text{vers.}\,\Delta\theta \end{aligned} \tag{9.}$$

the masses m' of the etherial particles, being supposed each $= m$. Three conditions of this form (9.) exist for every particle, and determine, in general, for any given values of a, b, c, λ, that is, for any given direction of propagation, and any given length of wave, the value of v, and the ratios of ξ', η', ζ', that is, the velocity of propagation of the wave and the direction of vibration of the particle. Accordingly, with some slight differences of notation, they have been proposed for this purpose by Cauchy, and adopted by other mathematicians. Suppose now, for simplicity, that the plane wave is vertical, so that $c = 0$, and let, at first, the direction of its propagation coincide with the positive semi-axis of x, so that b also vanishes, and a is $= 1$. Then, for transversal vibrations, the expression for the square of the velocity of propagation is

$$v^2 = \left(\frac{\lambda}{2\pi}\right)^2 m\mathrm{S}\left\{f(r) + \frac{r^2 - \Delta x^2}{2r} f'(r)\right\} \text{vers.}\,\frac{2\pi\,\Delta x}{\lambda}; \tag{10.}$$

which appears to extend not only to interplanetary spaces, but also to all ordinary transparent media, and contains, for them, the theoretical law of dispersion, which was first discovered by Cauchy, namely, the expression

$$v^2 = A_0 + A_1\lambda^{-2} + A_2\lambda^{-4} \ \ \&\text{c.} \tag{11.}$$

in which

$$A_i = \frac{(2\pi)^{2i} m}{1\,.\,2\,.\,3\,.\,4\,\ldots\,(2i+2)} \mathrm{S}\left\{f(r) + \frac{r^2 - \Delta x^2}{2r} f'(r)\right\} \Delta x^{2i+2}. \tag{12.}$$

But, in order that this law may agree with the phenomena, it is essential that the series (11.) should be convergent, even in its earliest terms; and this consideration enables us to exclude the supposition which has occurred to some mathematicians, that the particles of the ether attract each other with forces which are inversely as the squares of the distances between them. For if we suppose $rf(r) = r^{-2}$, and therefore $f(r) = r^{-3}$, $f'(r) = -3r^{-4}$, we shall have

$$A_i = \tfrac{1}{2}\frac{(2\pi)^{2i} m}{1\,.\,2\,.\,3\,.\,4\,\ldots\,(2i+2)} \mathrm{S}\left\{-r^{-3} + 3r^{-5}\Delta x^2\right\} \Delta x^{2i+2}; \tag{13.}$$

and by extending the summation to particles, distant by several times the length of an undulation from the particle which they are supposed to attract, these sums (13.) become extremely large and the terms of the series (11.) diverge very rapidly at first, though they always finish by converging. In fact, if we conceive a sphere, whose radius $= n\lambda = n$ times the length of an undulation (n being a very large multiplier), and whose centre is the attracted particle; and if we consider only the combined effect of the actions of all the particles within this sphere, we may, as a good approximation, convert each sum (13.) into a triple definite integral, and thus obtain, for the general term of the series (11.), the expression

$$(-1)^i A_i \lambda^{-2i} = \frac{(-1)^i \, 4\pi m n^2 \lambda^2}{(2i+5)\epsilon^3} \cdot \frac{(2\pi n)^{2i}}{1 \,.\, 2 \,.\, 3 \,\ldots\, (2i+3)}, \tag{14.}$$

ϵ being the mean interval between any two adjacent particles of the ether, so that the number of such particles contained in any sphere of radius r, is nearly $= \dfrac{4\pi r^3}{3\epsilon^3}$, if r be a large multiple of ϵ. And hence we find, by taking the sum of all these terms (14.), the expression

$$v^2 = \frac{\lambda^2 m}{\pi \epsilon^3} \left\{ \tfrac{1}{3} + \frac{\cos. 2\pi n}{(2\pi n)^2} - \frac{\sin. 2\pi n}{(2\pi n)^3}; \right\}; \tag{15.}$$

so that, by taking the limit to which v^2 tends, when n is taken greater and greater, we get at last as a near approximation

$$v^2 = \frac{\lambda^2 m}{3\pi\epsilon^3}, \tag{16.}$$

and

$$\frac{\lambda}{v} = \sqrt{\frac{3\pi\epsilon^3}{m}}. \tag{17.}$$

But $\dfrac{\lambda}{v}$ expresses the time of oscillation of any one vibrating particle; this time would therefore be nearly constant, if the particles attracted each other according to the law of the inverse square of the distance; and consequently this law is inadmissible, as being incompatible with the law of dispersion. It had appeared to Sir William Hamilton important to reproduce these results, though he remarked that they agree substantially with those of Cauchy, because the law of the inverse square was one which naturally offered itself to the mind, and had, in fact, been proposed by at least one mathematician of high talent. There was, however, another law which had great claims on the attention of mathematicians, as having been proposed by Cauchy to represent the phenomena of the propagation of light *in vacuo*, namely the law of a repulsive action, proportional inversely to the fourth power, or to the square of the square of the distance. M. Cauchy had, indeed, supposed that this law might hold good only for small distances, but in examining into its admissibility, it appeared fair to treat it as extending to all the neighbouring particles which act on any one. But against this law also, Sir William Hamilton brought forward objections, which were founded partly on algebraical, and partly on numerical calculations, and which appeared to him decisive.

The spirit of these objections consisted in showing that the law in question would give too great a preponderance to the effect of the immediately adjacent particles, and would thereby produce irregularities which are not observed to exist. In particular, if it be supposed that

$$\mathrm{S} \,.\, r^i \, \Delta x^2 = \mathrm{S} \,.\, r^i \, \Delta y^2 = \mathrm{S} \,.\, r^i \, \Delta z^2,$$
$$\mathrm{S} \,.\, r^i \, \Delta x^4 = \mathrm{S} \,.\, r^i \, \Delta y^4 = \mathrm{S} \,.\, r^i \, \Delta z^4,$$
$$\mathrm{S} \,.\, r^i \, \Delta x^2 \, \Delta y^2 = \mathrm{S} \,.\, r^i \, \Delta y^2 \, \Delta z^2 = \mathrm{S} \,.\, r^i \, \Delta z^2 \, \Delta x^2,$$

and also, in (5.), that $c = 0$, $a = b$, and that λ is much greater than ϵ, it is found that the two values v^2 and $v_{\prime}^2$ of the square of the velocity v, corresponding to vertical and to horizontal but transversal vibrations, are connected by the relation

$$v_{\prime}^2 = -\frac{2}{3} v^2,$$

being expressed as follows:

$$v_{\prime}^2 = \frac{m}{4} \mathrm{S} \left(5 r^{-7} \, \Delta x^4 - r^{-3} \right),$$
$$v^2 = \frac{3m}{8} \mathrm{S} \left(r^{-3} - 5 r^{-7} \, \Delta x^4 \right);$$

In conclusion, he offered reasons for believing that the law of action of the particles of the ether on each other resembles more the law which Poisson has in one of his memoirs proposed as likely to express the mutual action of the particles of ordinary and solid bodies, being perhaps of some such form as the following:—

$$r f(r) = -a \, . \, b^{-\left(\frac{r}{g \epsilon}\right)^h} + a_{\prime} \, . \, b_{\prime}^{-\left(\frac{r}{g_{\prime} \epsilon}\right)_{\prime}^{h}}; \tag{18.}$$

b and $b_{\prime}$ being each greater than unity, and g, $g_{\prime}$, h, $h_{\prime}$ being some large positive numbers, while a and $a_{\prime}$ are constant and positive multipliers, and ϵ is, as before, the mean or average interval between two adjacent particles. With such a law there would be a nearly constant repulsion, if a be greater than $a_{\prime}$, and if g be less than $g_{\prime}$, as long as $\dfrac{r}{g \epsilon}$ is sensibly less than unity; but the force would rapidly change, as the distance r approached to $g\epsilon$, and would then become a nearly constant attraction, until r became nearly $= g_{\prime}\epsilon$; it would then diminish rapidly, and soon become insensible. Sir William Hamilton did not, however, intend to exclude the hypothesis, that the function $r f(r)$ may contain several alternations of such repulsive and attractive terms,—much less did he deny that at great distances it may reduce itself to the law of the inverse square.

On the Propagation of Light in Crystals. By Professor Sir W. R. Hamilton.

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By continuing to modify the analysis of M. Cauchy in the manner already explained, he had succeeded in deducing, more satisfactorily than had in his opinion been done before, from dynamical principles, a large and important class of the phenomena of light in crystals; though much still remained to be done before it could be said that a perfect theory of light was obtained. He had employed, for the purposes of calculation, the supposition that the arrangement of the particles of the ether in a crystal differs from an exactly cubical arrangement only by very small displacements, caused by the action of the particles of the crystalline body; and had attended only to those indirect or reflex effects of the latter particles which are owing to the disturbances which they produce in the arrangement of the former particles: but he did not mean to assert that he had established any strong physical probability for this being the true *modus operandi* in crystals, though he thought the hypothesis had explained so much already that it deserved to be still further developed.