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An extremal result for paths. (In English)

Capobianco, Michael F. (ed.) et al., Graph theory and its applications: East and West. Proceedings of the first China-USA international conference, held in Jinan, China, June 9-20, 1986. New York: New York Academy of Sciences, Ann. N. Y. Acad. Sci. 576, 155-162 (1989). [ISBN 0-89766-579-1/hbk; ISBN 0- 89766-580-5/pbk]

One of the best known extremal results involving paths is the following one proved more than 25 years ago.

Theorem [*P. Erdős* and *T. Gallai*, Acta Math. Acad. Sci. Hung. 10, 337-356 (1959; Zbl 090.39401)]: A graph G_m on m vertices with at least $[m(k-1)+1]/2$ edges contains a path P_{k+1} on $k+1$ vertices. Furthermore, when $m = kt$ the graph tK_k contains the maximal number of edges in an m vertex graph with no P_{k+1} and is the unique such graph.

There are many other results in the literature that use that a graph with many edges or with high-degree vertices has a long path.

The problem we address here is of a similar nature. Let m, n and k be fixed positive integers with $m > n \geq k$. We wish to determine the minimum value l such that each graph on m vertices with l vertices of degree at least n contains a P_{k+1}.

A plausible minimum value for l is suggested by the following graph. Let $m = t(n+1)+r$, $0 \leq r < n+1$, with $k < 2n+1$. Then the graph consisting of t vertex disjoint copies of $H = \overline{K}_{n+1-\lfloor (k-1)/2 \rfloor} + K_{\lfloor (k-1)/2 \rfloor}$ contains $t\lfloor (k-1)/2 \rfloor$ vertices of degree n and no P_{k+1}. When k is even and $r + \lfloor (k-1)/2 \rfloor \geq n$, the number of vertices of degree $\geq n$ in this graph can be increased by 1 to $t\lfloor (k-1)/2 \rfloor + 1$ without forcing the graphs to contain a P_{k+1}. Simply take one of the vertices of degree $\lfloor (k-1)/2 \rfloor$ and make it adjacent to the r vertices in no copy of H. Thus we have the following conjecture:

Let m, n, and k be fixed positive integers with $m > n \geq k$ and set $\delta = 2$ when k is even and $\delta = 1$ when k is odd. If G_m is a graph on m vertices and at least $l = \lfloor (k-1)/2 \rfloor \lfloor m/(n+1) \rfloor + \delta$ vertices of degree $\geq n$, then G_m contains a P_{k+1}.

Although we do not prove the conjectured result, we do show that the value of l given in the conjecture is "essentially" correct. Much attention is given to the special case when $n+1 \leq m \leq 2n+1$. In this case we show that approximately $k/2$ vertices of degree $\geq n$ is enough to guarantee that G_m contains a P_{k+1}. Unfortunately, even for this interval of values we are not able to prove the exact statement of the conjecture.

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05C38 Paths and cycles

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