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Erdős, Paul; Richards, Ian
Density functions for prime and relatively prime numbers. (In English)
Monatsh. Math. 83, 99-112 (1977). [0026-9255]

This is a sequel to papers by *P.Erdős* and *J.L.Selfridge* [Proc. Manitoba Conf. numer. Math. 1971, 1-14 (1971; Zbl 267.10054)] and *D.Hensley* and *I.Richards* [Acta Arith. 25, 375-391 (1974; Zbl 285.10004)]. Let $\rho^*(x)$ be the maximum number of primes in any interval beyond x of length x. Let $r^*(x)$ be the maximum number of pairwise coprime integers in any interval of length x. A finite set S of integers is "$\rho^* - admissible$" if for each prime p some residue class $(modp)$ excludes all elements of S. S is "$r^* - admissible$" if for each prime p some residue class $(modp)$ excludes all but at most one element of S. The prime k-tuples hypothesis asserts that if $\{b_1 < b_2 < \cdots < b_k\}$ is $\rho^* - admissible$ then there are infinitely many positive integers n for which all of $n+b_1, n+b_2, \dots, n+b_k$ are prime. Under the prime k-tuples hypothesis it is proved that $\rho^*(x)$ is the number of elements in a maximal $\rho^* - admissible$ set in any interval of length x (proposition 4). With no hypothesis (proposition 5) $r^*(x)$ is the maximum number of elements in any r^*-admissible set in any interval of length x.
Sieve methods are used to get upper an lower bounds on $r^*(x)-\rho^*(x)$. Namely, theorem 1: There is an effectively computable $c > 0$ for which $r^*(x)-\rho^*(x) > x^c$ for all sufficiently large x. Theorem 2: Under the prime k- tuples hypothesis,

$$r^*(x) - \rho^*(x) = o(x/\log^2 x) \quad \text{as} \quad x \to \infty.$$

The previously known lower bound was $logx$. Since Hensley and Richards have proved under the prime k-tuples hypothesis that $\rho^*(x) < \pi(x) + Kx/\log^2 x$, then it appears that $r^*(x) \sim \rho^*(x)$. This is not surprising, however, under the prime k-tuples hypothesis we have the even stronger fact that $r^*(X)-\rho^*(X) = o(x/\log^2 x)$ where as $\rho^*(x) - \pi(x) > Kx/\log^2 x$. Thus it seems that $\rho^*(x)$ is much closer to $r^*(x)$ than to $\pi(x)$. Of course, the prime k-tuples hypothesis is a rather strong assumption which has as yet not been verified even for $k = 2$.

J.P.Tull

Classification:
11N05 Distribution of primes
11N35 Sieves