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On set-systems having large chromatic number and not containing prescribed subsystems. (In English)

Infinite finite Sets, Colloq. Honour Paul Erdős, Keszthely 1973, Colloq. Math. Soc. Janos Bolyai 10, 425-513 (1975).

[For the entire collection see Zbl 293.00009.]

This lengthy paper is devoted to a variety of problems concerning the chromatic number of set systems. A "set system" is defined to be a family of S of sets each of at least two elements. The "chromatic number" of S is the smallest cardinal κ for which there is a partition of length κ of $\cup S$, $\cup S = \cup\{P_\nu; \nu < \kappa\}$, such that $A \not\subseteq P_\nu$ whenever $A \in S$, $\nu < \kappa$. The family S is said to be an (n,i)-system if it is a family of n-tuples every two of which have at most i elements in common. — In an earlier paper [Cambridge Summer School Math. Logic, Cambridge 1971, Lecture Notes Math. 337, 531-538 (1973; Zbl 289.04002)], *P. Erdős, A. Hajnal* and *B. Rothchild* have shown that if $1 \le i \le n < \omega \le \kappa$ then there is an (n,i)-system with chromatic number $> \kappa$, and that the cardinality of such a system must be large. One of the aims of the present paper is to determine just how large. The following are proved: (a) Assume $mi + 2 \le n < \aleph_0$. Let S be a system of n- tuples with $|S| = \aleph_{\alpha+m}$ such that every $\aleph_{\alpha+1}$- size subset of S has at most i elements in its intersection. Then S has chromatic number at most $\aleph_\alpha$. (b) Assume $2 \le n < mi + 2 < \aleph_0$ and that G.C.H. holds. Then there is an (n,i)-system of cardinality $\aleph_{\alpha+m}$ and chromatic number $> \aleph_\alpha$. — The G.C.H. is needed in (b), for it is shown that if MA_κ holds then every (3,1)-system of cardinality κ has chromatic number at $\{u \in P | u \sim x, u \sim y, u \sim z\}$. Bose has proved that $|tr(x,y,z)| = q+1$. The triple (x,y,z), $x \not\sim$, $y \not\sim z$, $z \not\sim x$, is said to be regular provided each point collinear with at least three points of $tr(x,y,z)$ is actually collinear with all points of $tr(x,y,z)$. If for a point x each triple (x,y,z), with $x \not\sim y$, $y \not\sim z$, $z \not\sim x$, is regular, x is said to be regular. The following theorems are proved. a) If the 4-gonal configuration S with parameters $r = q^2+1$ and $k = q+1$, where q is even and $q > 2$, possesses a regular point, then S is isomorphic to a 4- gonal configuration $T(0)$ (i.e. a 4-gonal configuration arising from an ovoid 0 of $PG(3,q)$) of *J. Tits.* b) If each point of the 4-gonal configuration S with parameters $r = q^2+1$ and $k = q+1$ ($q > 2$) is regular, then S is isomorphic to the 4-gonal configuration $Q(5,q)$ arising from a non-singular hyperquadric Q of index 2 of $PG(5,q)$. c) Suppose that the 4-gonal configuration $S = (P,B,I)$, with parameters $r = q^2+1$ and $k = q+1$ ($q > 1$), has a 4-gonal subconfiguration $S' = (P',B',I)$, with parameters $k' = r' = k$, for which the following condition is satisfied: if $x,y,z \in P'$ with $x \not\sim y$, $y \not\sim z$, $z \not\sim x$, then the triple (x,y,z) is regular and moreover $sp(x,y,z) \subset P'$. Then we have (i) S has an involution with fixes P' pointwise (ii) S' is isomorphic to the 4-gonal configuration $Q(4,q)$ arising from a non-singular hyperquadric in $PG(4,q)$. — Remark: Recently the author has proved a) for q odd.

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Classification:

04A20 Combinatorial set theory
04A10 Ordinal and cardinal numbers; generalizations
05C15 Chromatic theory of graphs and maps