
Zbl 298.10012
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The number of distinct subsums of $\sum_1^N 1/i$. (In English)
Math. Comput. 29, 29-42 (1975). [0025-5718]

In this paper we improve the lower bounds given in "Denominators of Egyptian fractions. II", Ill. J. Math 20, 598-613 (1976; Zbl 336.10007) ans Notices Am. Math. Soc. 20, # 706-10-3 (1973)] for the number, $S(N)$, of distinct values obtained as subsums of the first N terms of the harmonic series. The estimates in J. Number Theory 8, 157-168 (1976; Zbl 328.10010) and the before mentioned articles were derived because the upper bound was needed for lower estimates of the denominators of Egyptian fractions. In this paper we concentrate on the lower bounds. We obtain a bound of the form

$$S(N) \geq e\left(\frac{N\log 2}{\log N}\prod_{3}^{k+1}\log_j N\right)$$

whenever $\log_{k+1} N \geq k+1$, for $k \geq 3$. Slight modifications are needed for $k = 1, 2$; see Corollaries 1, 2, 3 and 4 for more details. In order to do this we begin by discussing the number $Q_k(N)$ of integers $n \leq N$, $n = p_1, p_2 \dots p_k$ where $p_i > e^{\alpha p_{i-1}}$, $i = 2, \dots, k$. We first prove that

$$\frac{N}{\log N}\prod_{i=3}^{k+1}\log_i N \leq Q_k(N) \leq \left(1+\frac{k}{\log_{k+1} N}\right)\frac{N}{\log N}\prod_{i=3}^{k=1}\log_i N.$$

This bound is valid for $\log_{k+1} N \geq k+1$ and for $1 \leq \alpha \leq 2(1-e_2(4)/e_3(4))$. The bounds on N and α are for convenience in evaluating the range of validity and the constants in the inequality, not for essential reasons. The symbols $\log_i x$ and $e_i(x)$ are defined by $e_o(x) = x$, $e_{i+1}(x) = e^{e_i(x)}$, $\log_o x = x$, $\log_{i+1} x = \log(\log_i x)$, where $\log x$ denotes the logarithm to the base e.

Classification:
11A99 Elementary number theory
11A41 Elemementary prime number theory
11D61 Exponential diophantine equations