Zbl 228.10028
Erdős, Paul; Selfridge, J.L.
Some problems on the prime factors of consecutive integers. II. (In English)
Proc. Washington State Univ. Conf. Number Theory 1971, 13-21 (1971).

[For the entire collection see Zbl 225.00004.]
[Part I, Ill. J. Math. 11, 428-430 (1967; Zbl 149.28901).]

The authors develop a number of interesting results centering round the following interesting conjecture of C. A. Grimm: Let $n+1,\dots,n+k$ be consecutive composite numbers. Then for each $i, 1 \le i \le k$ there is a p_i, $p_i \mid n+i$, $p_{i_1} \ne p_{i_2}$ for $i_1 \ne i_2$. Grimm also stated the following weaker conjecture: The product of k consecutive composite numbers need to have at least k prime factors. The interest in Grimm's conjectures is that even the weaker conjecture is enough to imply $p_{i+1} - p_i \ll (p_i/\log p_i)^{1/2}$. Actually in view of a result of the reviewer the weaker conjecture implies that $p_{i+1} - p_i \ll p_i^{1/2-c_1}$ where c_1 is a certain positive constant. This is known to the authors. These results show that there is not much hope to prove Grimm's conjectures in the "near future".
The authors prove a number of interesting results independent of any hypothesis. Let $\nu(n,k)$ be the number of distinct prime factors of $(n+1)\dots(n+k)$; $f_1(k)$ be the smallest integer k so that for every $1 \le l \le k$, $\nu(n,1) \ge 1$ but $\nu(n,k+1) = k$; $f_0(n)$ be the largest integer k for which $\nu(n,k) \ge k$. Let $f_2(n)$ be the largest integer k so that for each $1 \le i \le k$ there is a $p_i \mid n+i$, $p_{i_1} \ne p_{i_2}$ if $i_1 \ne i_2$. Let $p(m)$ be the greatest prime factor of m; $f_3(n)$ the largest integer so that all the primes $p(n+i)$, $1 \le i \le k$, are distinct; $f_4(n)$ be the largest integer k so that $p(n+i) \ge i$, $1 \le i \le k$ and $f_5(n)$ be the largest integer k so that $p(n+i) \ge k$ for every $1 \le i \le k$. The main object of the paper is the study of the functions $f_i(n)$, $0 \le i \le 5$. We content by stating two theorems on $f_2(n)$.
Theorem 3: $f_2(n) > (1+0(1))\log n$ (this theorem has been improved by the reviewer to $f_2(n) \gg \log n(\log_2 n/\log_3 n)^{1/2}$. Recently the reviewer received a letter from Tijdeman who says that he can improve this further to $f_2(n) \gg (\log n)^2(\log_2 n)^{-8}$; $\log_r n$ denotes the r-th iterated logarithm). Theorem (stated without proof): $f_2(n) < \exp(c\log n\log_3 n/\log_2 n)$ for infinitely many n.

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Classification:
11N99 Multiplicative number theory
11N56 Rate of growth of arithmetic functions