Zbl 146.27103
Erdős, Pál; Sarközi, A.; Szemeredi, E.
On divisibility properties of sequences of integers (In English)
Stud. Sci. Math. Hungar. 1, 431-435 (1966).

H.Davenport and *P.Erdős* (Zbl 015.10001) proved that if the sequence $A = \{a_i\}$ is of positive upper logarithmic density, i. e. if

$$(^*)\ \limsup_{x\to\infty}(\log x)^{-1}\sum_{a_i<x} a_i^{-1} > 0$$

holds, then there exists an infinite increasing subsequence $\{a_{n_j}\} \subset A$, satisfying $a_{n_i} \mid a_{n_j}$. Such a subsequence is called a chain. Using the notations $\log_2 x = \log\log x$ and $\overline{\lim}$ for $\limsup_{x\to\infty}$, and with $c_i > 0$, the following sharper results are now proved; (1) If A satisfies $(^*)$, then it contains a chain, satisfying for infinitely many $y : \sum_{a_i<y} 1 > c_1(\log_2 y)^{1/2}$. (2) If satisfies $\overline{\lim}(\log_2 x)^{-1}\sum_{a_n<x}(a_n \log a_n)^{-1} = c_2$ then it contains a chain, satisfying for infinitely many $x : \sum_{a_{n_i}<x} 1 > c_3 \log_2 x$. It is shown that (1) is best possible, by exhibiting a class of sequences, for which all chains are such that $\sum_{a_{n_i}<x} 1 < 3(\log_2 x)^{1/2}$; and also, that, in general, (2) will not hold for all x. The following two conjectures are stated: (a) For every sequence A, there is a chain satisfying

$$\overline{\lim}(\log_2 y)^{-1}\sum_{a_{n_i}<y} \geq \overline{\lim}(\log_2 x)^{-1}\sum_{a_n<x}(a_n \log a_n)^{-1}.$$

(b) For every sequence A,

$$\overline{\lim}(\log x)^{-1}\sum_{a_i<x, a_k|a_i}(a_k/a_i) \geq \overline{\lim}(\log x)^{-1}\sum_{a_k\leq x} a_k^{-1}.$$

(Remark: On the first line the word ”lower” sems to stand for ”upper”.)

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Classification:
11B83 Special sequences of integers and polynomials
11B05 Topology etc. of sets of numbers