Zbl 132.34902

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Large and small subspaces of Hilbert space (In English)

Mich. Math. J. 12, 169-178 (1965). [0026-2285]

This paper is concerned with the properties of closed subspaces V of the sequential Hilbert space l_2 and of $L_2(0,1)$. We shall suffice by quoting the following interesting results of this paper which speak for themselves.
Theorem 1. Let V be a closed linear subspace of l_2, and let $\{\varrho_n\}$ be given with $\varrho_n \geq 0$ and $\sum \varrho_n^2 < \infty$. If $|x(n)| = O(\varrho_n)$ for all $x \in V$, then V is finite-dimensional.
Theorem 3. If V is a closed subspace of l_2 and $V \subset I_p$ for some $1 \leq p < 2$, then V is finite-dimensional.
Theorem 4. If $\varrho_n \geq 0$ and $\sum \varrho_n^2 = \infty$, then there exists an infinite-dimensional subspace V of I_2 such that $\sum |x(n)|\varrho(n) = \infty$ for all $x \neq 0$ in V. In the case of $L_2(0,1)$ the situation is different.
The authors quote the well-known result from the theory of Fourier series that there exists an infinite-dimensional closed subspace V of $L_2(0,1)$ such that $V \subset L_q$ for all $1 \leq q < \infty$ and in fact satisfies the condition that $\int \exp\{c|f(x)|^2\}dx < \infty$ for all $c > 0$ and all $f \in V$. Then it is shown that if φ is convex, continuous and strictly increasing on $[0,\infty)$ with $\varphi(0) = 0$ and $\varphi(x)e^{-cx^2} \to \infty$ as $x \to \infty$ for all $c > 0$, then $\int \varphi(|f|) < \infty$ for all $f \in V$ implies that V is finite dimensional. Let V be a closed linear subspace of I_2. Then there exist elements λ_n $(n = 1, 2, ...)$ in V such that $(x, \lambda_n) = x(n)$ for all $x \in V$ and dim $V = \sum ||\lambda_n||^2$. This result is used to prove the following theorem. Theorem 9. Let $\varphi(z) = \sum a_n z^n$ be an inner function. Then $\sum n|a_n|^2 = \dim(\varphi H_2)\perp$. Thus the Dirichlet integral of φ is finite (and is then an integral multiple of π) if and only if φ is a finite Blaschke product. The paper finishes with the following question: Does H_2 contain an infinite dimensional closed subspace. V with $|f(z)| = O(1/(1-|z|)^{1/4})$ $(|z| < 1)$.

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Classification:
46C05 Geometry and topology of inner product spaces