
Zbl 010.10303
Erdős, Paul
On the density of the abundant numbers. (In English)
J. London Math. Soc. 9, 278-282 (1934).

This paper gives a new and elementary proof that $A(n)/n$ tends to a limit as $n \to \infty$ where $A(n)$ is the number of abundant numbers $\leq n$ (see *H.Davenport* Zbl 008.19701; *S.Chowla* Zbl 010.00803). It is proved that the series of reciprocals of the primitive abundant numbers converges, and from this the result follows immediately. It is first shown that there are only $o(n/(\log n)^2)$ primitve abundant numbers $m \leq n$ which do not (a) have a simple prime divisor p_m between $(\log n)^{10}$ and $n^{\frac{1}{40 \log \log n}}$, (b) satisfy

$$2 \leq \frac{\sigma(m)}{m} \leq 2 + \frac{2}{n^{\frac{1}{20 \log \log n}}}, \tag{1}$$

$[\sigma(m) =$ sum of divisors of $m]$. Further, the number of integers $m \leq n$ which do have properties (a) and (b) is $O(n/(\log n)^{10})$. For let $m = p_m l_m$, then $l_m < n/(\log n)^{10}$, and it suffices to prove that $l_m \neq l_{m'}$, when $m \neq m'$. Now $l_m = l_{m'}$, $p_m < p_{m'}$ would imply

$$\frac{\sigma(m)}{m}\frac{m'}{\sigma(m')} = \frac{(p_m+1)}{p_m}\frac{p_{m'}}{(p_{m'}+1)} \geq 1 + \frac{1}{p_m(p_{m'}+1)} > 1 + \frac{1}{n^{\frac{1}{20 \log \log n}}},$$

which would contradict (1).

Davenport (Cambridge)

Classification:
11A25 Arithmetic functions, etc.
11N25 Distribution of integers with specified multiplicative constraints